From this follows the convergence of (13) for $s < s_0$, the interior of the singularity circle, and the convergence of (14) for $s > s_0$. The reader may wish to compare this with reference [1], pp. 46-49.

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ON THE METHOD OF INVERSION IN THE TWO-DIMENSIONAL THEORY OF ELASTICITY*

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1. Introduction. The method of inversion, originally introduced by J. H. Michell [1], has led to a variety of technically significant solutions to "plane" problems in the theory of elasticity [2], [3], [4], [5]. The usefulness of Michell's stress-field transformation stems from its invariant properties which assure the preservation of an important class of boundary conditions. In the present note we show that any conformal stress-field transformation which preserves the principal-stress trajectories for every choice of the antecedent Airy function, is essentially a Michell transformation.

2. The Michell transformation. The inversion theorem of Michell may be stated as follows. Let $U(z, \bar{z})$ be real and biharmonic in a region R of the z-plane, and let R^* be the image of R with respect to the mapping[1]

$$\zeta = w(z) = \frac{az + b}{cz + d}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1. \tag{1}$$

Then the function

$$U^*(\zeta, \bar{\zeta}) = hU[g(\zeta), \overline{g(\zeta)}], \tag{2}$$

where $h^2 = |w'|^2$ and g is the inverse of w, is biharmonic in R^*. The stress fields generated by U and U^*, considered as Airy functions in R and R^* respectively, are related according to [2]

$$\left.\begin{aligned} \sigma^* + i\tau^* &= \lambda(\sigma + i\tau) + p, \\ \lambda = \frac{1}{h}, \qquad p &= 2(\lambda_{z\bar{z}}U - \lambda_{\bar{z}}U_z - \lambda_z U_{\bar{z}}). \end{aligned}\right\} \tag{3}$$

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[1]Michell actually used $\zeta = 1/z$.

[2]Subscripts attached to functions which originally bear no subscripts denote partial differentiation.

Here $[\sigma, \tau]$ and $[\sigma^*, \tau^*]$ are the normal and shearing stresses on any arc Γ and on its image Γ^* with respect to the mapping $\zeta = w(z)$.

Moreover, the stress-field transformation characterized by the mapping (1) and the law of Airy function association (2) has the invariant properties:

(A) The images of the principal stress trajectories of R are the principal stress trajectories of R^*.

(B) If any arc Γ of R is acted on by constant normal tractions only, so is its image Γ^* of R^*.

(C) A concentrated load acting at a point z_0 of a boundary arc Γ of R, and including a certain angle with Γ, is carried into a concentrated load of the same magnitude acting at $w(z_0)$ and including the same angle with the image arc Γ^* of R^*.

Properties (A) and (B) were also established by V. P. Jensen and D. L. Holl [6] by aid of derivatives of non-analytic (polygenic) functions [7], [8].

3. A converse of the inversion theorem. We now prove the following theorem. Let R be a region of the z-plane and let R^* be the image of R with respect to the conformal mapping

$$\zeta = w(z), \qquad w'(z) = \frac{dw}{dz} = he^{i\delta} \neq 0. \tag{4}$$

Moreover, for every Airy function $U(z, \bar{z})$, bi-harmonic in R, let there exist an Airy function $U^*(\zeta, \bar{\zeta})$, bi-harmonic in R^*, such that the corresponding stress-field transformation preserves principal stress trajectories, and the image field of stress is purely hydrostatic only if the antecedent field has the same property.[3] Then $w(z)$ is given by (1) and U^* is given by (2), i.e., the transformation is a Michell transformation.

To establish the theorem, we recall a result of Jensen and Holl [6] who showed that

$$\left.\begin{aligned} \sigma + i\tau &= \gamma_H(z, \bar{z}, \theta) \\ &= 2U_{z\bar{z}} + 2U_{\bar{z}\bar{z}}e^{-2i\theta}, \end{aligned}\right\} \tag{5}$$

where

$$H_{(z,\bar{z})} = 2U_{\bar{z}} \tag{6}$$

and γ_H is the directional derivative of H along Γ, θ being the inclination of Γ. In view of (5), property (A) is equivalent to the statement

$$\gamma_{H^*}[w(z), \overline{w(z)}, \theta + \delta] = \bar{\gamma}_{H^*} \tag{7}$$

whenever

$$\gamma_H(z, \bar{z}, \theta) = \bar{\gamma}_H\ , \tag{8}$$

provided

$$H^*(\zeta, \bar{\zeta}) = 2U^*_{\bar{\zeta}}. \tag{9}$$

Equations (7), (8) by aid of (5) become[4]

$$\left.\begin{aligned} U_{\bar{z}\bar{z}} &= U_{zz}e^{4i\theta}, \\ U^*_{\bar{\zeta}\bar{\zeta}} &= U^*_{\zeta\zeta}e^{4i(\theta+\delta)}. \end{aligned}\right\} \tag{10}$$

[3]This restriction is essential in order to rule out the trivial transformation which carries all antecedent stress distributions into hydrostatic fields of stress.

[4]The partial derivatives of U^* with respect to ζ and $\bar{\zeta}$ are to be evaluated at $\zeta = w(z)$

Thus

$$U_{zz}U^*_{\bar{\zeta}\bar{\zeta}}(\bar{w}')^2 = U_{\bar{z}\bar{z}}U^*_{\zeta\zeta}(w')^2, \tag{11}$$

which implies that the function

$$\phi(z, \bar{z}) \equiv U_{\bar{z}\bar{z}}U^*_{\zeta\zeta}(w')^2 \tag{12}$$

is real-valued. Furthermore, assuming that the original stress distribution is not hydrostatic, so that $U_{zz} \neq 0$, it follows by hypothesis that $U^*_{\zeta\zeta} \neq 0$ and hence ϕ does not vanish identically. For convenience, let

$$\varphi(z, \bar{z}) = U_{zz}U_{\bar{z}\bar{z}}/\phi. \tag{13}$$

Equation (12) then appears as

$$U_{zz} = \varphi(w')^2U^*_{\zeta\zeta}\ , \tag{14}$$

where φ is again real-valued. Equation (14) constitutes a necessary and sufficient condition for the preservation of the principal-stress trajectories.

We next apply to (14) the condition that $U(z, \bar{z})$ and $U^*(\zeta, \bar{\zeta})$ are both biharmonic, i.e., $U_{zz\bar{z}\bar{z}} = 0$ and $U^*_{\zeta\zeta\bar{\zeta}\bar{\zeta}} = 0$. This leads to

$$\left[\varphi\frac{w''}{w'} + 2\varphi_{\bar{z}}\right]U_{zz\bar{z}} + \left[\varphi_{\bar{z}\bar{z}} - \varphi_{\bar{z}}\frac{w''}{w'} - \frac{(2\varphi_{\bar{z}})^2}{\varphi}\right]U_{zz} = 0, \tag{15}$$

or, since (15) must hold for every bi-harmonic U,

$$\varphi\frac{w''}{w'} + 2\varphi_z = 0, \tag{16}$$

$$\varphi\varphi_{zz} - \varphi\varphi_z\frac{w''}{w'} - 2(\varphi_z)^2 = 0. \tag{17}$$

The complete solution of (16), subject to the requirement $\varphi = \bar{\varphi}$, is

$$\varphi(z, \bar{z}) = \kappa(w'\bar{w}')^{-1/2} = \kappa/h \tag{18}$$

with κ an arbitrary real constant. Noting that (16), (17) require $\varphi_{zz} = 0$, we conclude from (18) that

$$\left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = 0, \tag{19}$$

i.e., the Schwarzian derivative of $w(z)$ vanishes. The complete solution of (19) is given by

$$w(z) = \frac{az + b}{cz + d}, \tag{20}$$

and since the mapping is to be (1, 1) we may put $ad - bc = 1$. In order to arrive at the law of Airy-function association we now integrate (14) by use of (18). This integration yields,

$$U^* = \frac{h}{\kappa}[U + Az\bar{z} + \alpha\bar{z} + \bar{\alpha}z + B], \tag{21}$$

where A, B are arbitrary real constants, and α is an arbitrary complex number. It is readily confirmed by direct computation that the constants A, B, and α give rise to an arbitrary, uniform hydrostatic stress-field in the image domain R^*. The constant κ, on the other hand, affects merely the scale of the image stress-distribution. We may therefore put $A = B = \alpha = 0$, $\kappa = 1$. This completes the proof.

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A MINIMUM PRINCIPLE FOR STRUCTURAL STABILITY*

By H. J. WEISS and G. H. HANDELMAN (*Carnegie Institute of Technology*)

1. Statement of the problem. In a recent paper,[1] W. Prager has discussed the problem of structural stability from the following point of view. Consider a given configuration of a deformable body, referred to a fixed system of rectangular axes, $x_i (i = 1, 2, 3)$, under a set of stresses $\lambda\sigma_{ij}$ which are in equilibrium with given surface tractions. These stresses are prescribed only to within the arbitrary constant factor λ. The configuration is assumed to be stable if λ is sufficiently small. A system of infinitesimal perturbation displacements u_i is then applied, and the question is asked for what values of the factor λ will the equilibrium become indifferent.

The solution to the problem leads to the following system of linear, homogeneous, second order, partial differential equations

$$J_{ij,i} = 0, \tag{1}$$

subject to the homogeneous boundary conditions on the surface

$$J_{ij}n_i = 0, \tag{2}$$

where

$$J_{ij} = [\tau_{ij} + \tfrac{1}{2}\lambda(\sigma_{i\nu}\epsilon_{\nu j} - \sigma_{j\nu}\epsilon_{\nu i}) - \lambda\sigma_{i\nu}\omega_{\nu j}].$$

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