NOTE ON AERODYNAMIC HEATING*

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The acceleration of an initially stationary semifinite viscous fluid by the sudden starting from rest of the infinite bounding plane at velocity U parallel to the plane has been derived many times. See for example, Aerodynamic Theory.[1] The equations of motion for this case assuming constant physical properties reduce to the well known heat conduction equation,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{1}$$

yielding as the solution for the velocity distribution

$$u = U \frac{2}{\pi^{1/2}} \int_{y/2(\nu t)^{1/2}}^{\infty} \exp[-\beta^2]\, d\beta. \tag{2}$$

The question of the temperature distribution resulting from the energy dissipated has not been previously discussed to the author's knowledge and seems worthy of a note.

The energy equation reduces for the present case to

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

where the physical constants a, ν, the thermal diffusivity and kinematic viscosity, show the rate of spread of the energy and the momentum respectively. Their ratio is the Prandtl Number

$$\Pr = \frac{c_p \mu}{k} = \frac{\nu}{a}. \tag{4}$$

A particular integral T_p of (3) may be written down by the method of sources

$$T_p = \int_0^t \int_{-\infty}^{\infty} \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 \exp\left\{-\frac{(y-\lambda)^2}{4a(t-\tau)}\right\} \frac{d\tau\, d\lambda}{[4\pi a(t-\tau)]^{1/2}}. \tag{5}$$

On substituting the velocity distribution from (2)

$$T_p = \frac{U^2}{\pi c_p} \int_0^t \frac{d\tau}{\tau} \int_{-\infty}^{\infty} \exp\left\{-\frac{(y-\lambda)^2}{4a(t-\tau)} - \frac{\lambda^2}{2\nu\tau}\right\} \frac{d\lambda}{[4\pi a(t-\tau)]^{1/2}}. \tag{6}$$

This result may be simplified as follows:

Let $\xi = -(y-\lambda)/[4a(t-\tau)]^{1/2}$ be used to replace λ. One integration then yields

$$T_p = \frac{U^2}{\pi c_p} \int_0^t \exp\left\{-\frac{y^2}{4at - 4a\tau + 2\nu\tau}\right\}\left[\frac{2\nu}{\tau(4at - 4a\tau + 2\nu\tau)}\right]^{1/2} d\tau. \tag{7}$$

Now let $\zeta = y^2/(4at - 4a\tau + 2\nu\tau)$

$$T_p = \frac{U^2}{\pi c_p}[\Pr/(2-\Pr)]^{1/2} \int_{y^2/4at}^{(2/\Pr)(y^2/4at)} \frac{e^{\zeta}\, d\zeta}{\zeta[(4at\zeta/y^2) - 1]^{1/2}}. \tag{8}$$

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[1] W. F. Durand, *Aerodynamic Theory*, vol. III, p. 64.

A trigonometric substitution $\zeta = y^2/4at \sin^2 \theta$ gives finally

$$T_p = \frac{U^2}{c_p} [\Pr/(2 - \Pr)]^{1/2} \frac{2}{\pi} \int_{\sin^{-1}[\Pr/2]^{1/2}}^{\pi/2} \exp \{-y^2/4at \sin^2 \theta\} \, d\theta. \tag{9}$$

Since there were no sources except those of dissipation, this becomes the solution for the adiabatic plate if we add Eq. (9) to the initial temperature T_0. We take then as the particular integral the solution for an adiabatic plate

$$T_p = T_0 + \frac{U^2}{c_p} [\Pr/(2 - \Pr)]^{1/2} \frac{2}{\pi} \int_{\sin^{-1}[\Pr/2]^{1/2}}^{\pi/2} \exp \{-y^2/4at \sin^2 \theta\} \, d\theta. \tag{10}$$

To satisfy any other boundary condition at the plate, it is only necessary to add the appropriate complementary function chosen from the many known solutions of the heat

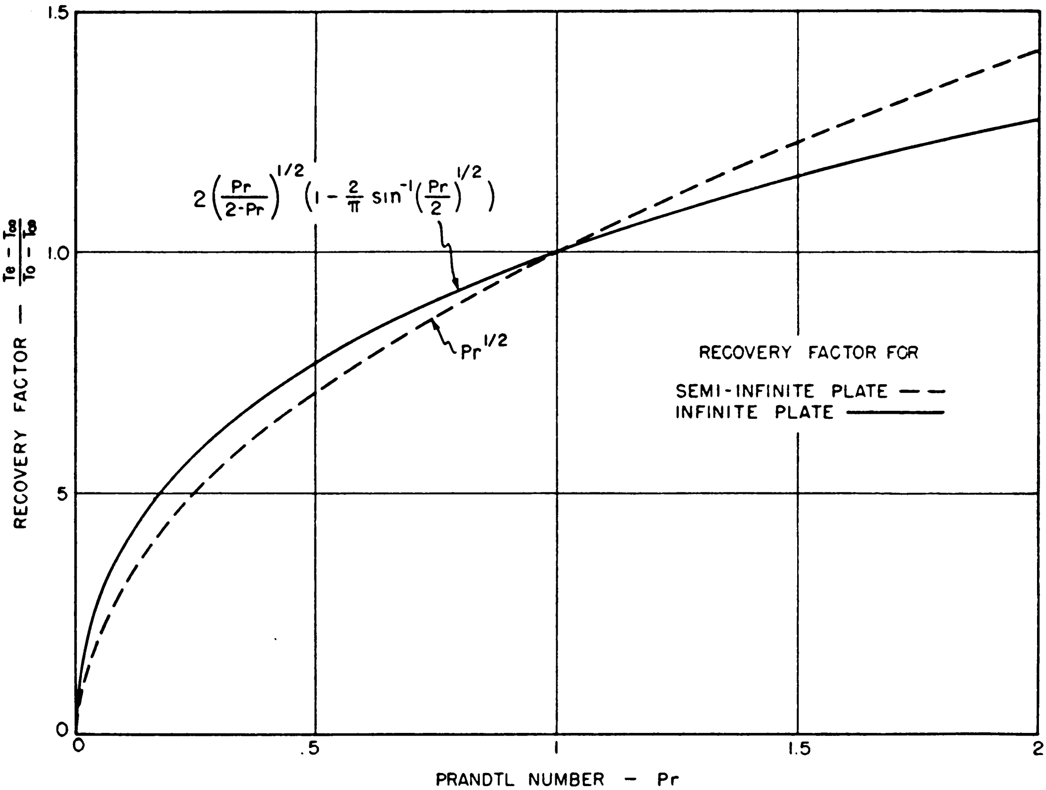

FIG. 1.

conduction equation. In particular, for a rate of heat transfer q we add the complementary function

$$T_c = \frac{2q}{k} \left([at/\pi]^{1/2} \exp \{-y^2/4at\} - \frac{y}{\pi^{1/2}} \int_{y/2[at]^{1/2}}^{\infty} \exp [-\beta^2] \, d\beta \right). \tag{11}$$

Returning now to a consideration of the solution for an adiabatic plate, we note that while the integration must in general be carried out numerically, the most important question, that of the plate temperature, can be answered without difficulty. If $y = 0$ Eq. (10) gives for the plate temperature

$$T = T_0 + 2[\mathrm{Pr}/(2 - \mathrm{Pr})]^{1/2}\left(1 - \frac{2}{\pi}\sin^{-1}[\mathrm{Pr}/2]^{1/2}\right)\frac{U^2}{2c_p}\,. \tag{12}$$

Thus for the infinite plate the recovery factor is

$$r = 2[\mathrm{Pr}/(2 - \mathrm{Pr})]^{1/2}\left(1 - \frac{2}{\pi}\sin^{-1}[\mathrm{Pr}/2]^{1/2}\right),$$

which is to be compared to the corresponding recovery factor for a finite plate with a steady boundary layer

$$r = \mathrm{Pr}^{1/2}. \tag{13}$$

For this latter result see for example Emmons and Bainerd[2]. Figure 1 compares these two results. The equilibrium temperature of an accelerated plate is independent of time and is very nearly the same as that of the plate in steady flow of the same velocity. The temperature distribution of Eq. (10) has been obtained by graphical integration and is shown in Fig. 2.

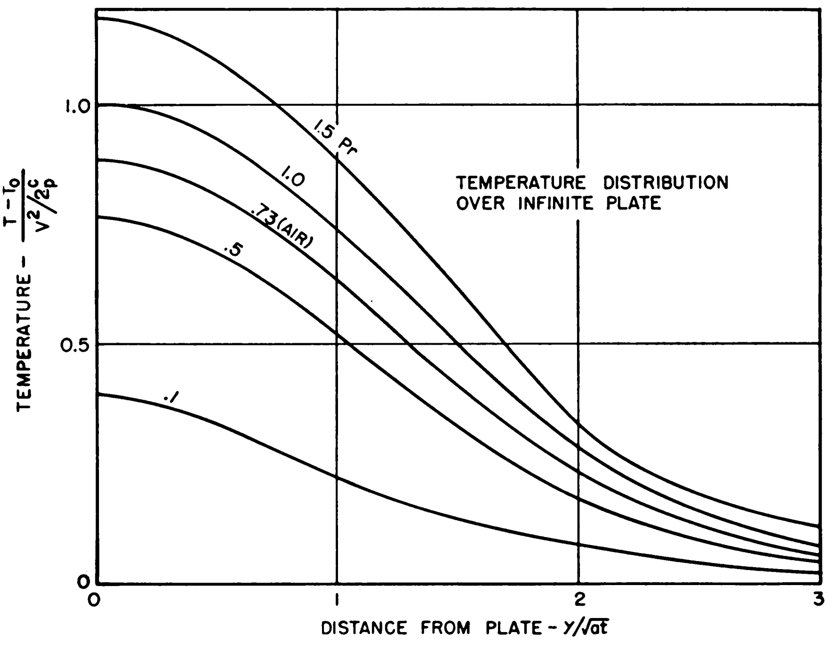

FIG. 2.

By comparing the thickness of the boundary layer of Eq. (2), and Fig. 2 with that of a point on a steadily moving plate distant x from the leading edge it is found that the

[2]Emmons and Bainerd, *Temperature effects in a laminar compressible fluid boundary layer along a flat plate*, J. Appl. Mech., p. 4 (1941).

non-steady layer would equal the steady layer in times given approximately by

$$t \approx \frac{5x}{U} \quad \text{for velocity,}$$

$$t \approx \frac{6x}{U} \quad \text{for temperature.}$$

Thus to a reasonable approximation it can be said that by the time a point on a suddenly accelerated plate moves 5 times its distance from the leading edge, its boundary layers will have become steady state ones.

PLASTIC WAVE PROPAGATION IN A BAR OF MATERIAL EXHIBITING A STRAIN RATE EFFECT[1]

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1. Introduction. The propagation of a transient wave of plastic deformation due to longitudinal impact on a bar has been treated by Donnell,[2] and White and Griffis,[3] by a non-linear superposition method. The partial differential equations governing the wave propagation were derived independently by Taylor[4] and von Kármán[5] under the assumption of a relation between stress and strain independent of strain rate. Constant velocity tension impact tests at the California Institute of Technology[6,7] gave fair agreement with the theory. Some systematic discrepancies were, however, observed. In the tension impact tests the maximum residual strain was smaller than predicted by the theory, and the observed force-time variation at the fixed end during impact showed that the stress there was greater than the theory predicted. It has been suggested[6] that these discrepancies were due to the use in the theory of an invariant relation between stress and strain independent of strain rate. At the high strain rates involved in deformation under impact a considerable deviation from the static stress-strain relation may be expected. The present work extends the theory to apply to materials in which the stress is a function of the instantaneous plastic strain and strain rate.

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[2]L. H. Donnell, *Longitudinal wave transmission and impact*, A.S.M.E., Trans. **52** (1), APM 153-167 (1930).

[3]M. P. White and L. Griffis, *The permanent strain in a uniform bar due to longitudinal impact*, J. Appl. Mech., A.S.M.E., Trans. **69**, A-337-A-343 (1947).

[4]G. I. Taylor, *Propagation of earth waves from an explosion*, British Official Report R.C. 70 (1940).

[5]Th. v. Kármán, *On the propagation of plastic deformation in solids*, N.D.R.C. Report No. A-29 (O.S.R.D. No. 365) (1942).

[6]P. E. Duwez, D. S. Wood, D. S. Clark, and J. V. Charyk, *The effect of stopped impact and reflection on the propagation of plastic strain in tension*, N.D.R.C. Report No. A-108, (O.S.R.D. No. 988) (1942).

[7]P. E. Duwez and D. S. Clark, *An experimental study of the propagation of plastic deformation under conditions of longitudinal impact*, A.S.T.M., Proc. **47**, 502-532 (1947).