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$\phi(0) = 0$. The data for the curve of equation (8) is deduced from that for Eq. (12) similarly, i.e., $g(t) = g(0) = -117$ BTU/(sq. ft.)(hr.). Since only two terms of the solution (29) have been determined, all curves are drawn using only the first two terms of each series. The straight line shown in Figure 2 represents the time it would take a

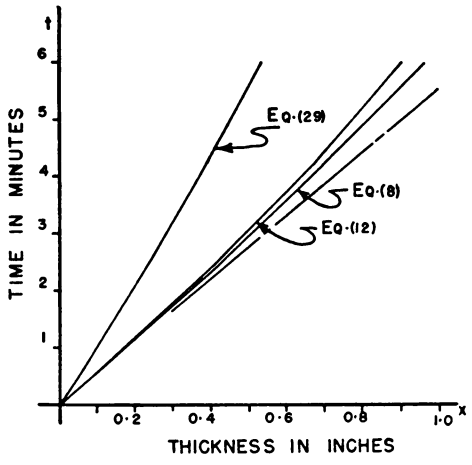

FIG. 2.

slab of thickness x at the critical temperature to recrystallize uniformly with a constant heat input of -117 BTU/(sq. ft.)(hr.).

According to (29), if t can be taken so small that all terms above the first are neglected, recrystallization cannot take place if $k_1\phi'(0) > k_2g(0)$. In fact, this condition would represent decrystallization and the curve would be in the negative x half plane.

BOOK REVIEWS

Electromagnetic theory. Proceedings of Symposia in Applied Mathematics, Volume II. American Mathematical Society, New York, 1950. iii + 91 pp. $3.00.

This volume contains the proceedings of the Second Symposium in Applied Mathematics, held at the Massachusetts Institute of Technology in July 1948. Of the seventeen papers included, ten are presented in the volume by abstract.

The subject matter covers a wide range under the general topic of Electromagnetic Theory including related quantum theory and statistics. The papers in full are: H. Feshbach, *The new quantum electrodynamics.* J. L. Synge, *Electromagnetism without metric.* W. H. Watson, *Discontinuity in electromagnetism.* L. Infeld, *The factorization method and its application to differential equations in theoretical physics.* R. J. Duffin, *Nonlinear electrical networks.* C. L. Pekeris, *Ray theory vs. normal mode theory in wave propagation problems.* A. E. Heins, *Systems of Wiener-Hopf integral equations and their application to some boundary value problems in electromagnetic theory.*

W. D. HAYES

Deformation and flow. An elementary introduction to theoretical rheology. By Markus Reiner. H. K. Lewis & Co., Ltd., London, 1949. xix + 342 pp. 32s. 6d. (Sold in the U.S.A. by Interscience Publishers, New York, at $6.50).

This treatment of the theory of Rheology is primarily concerned with the fundamental behavior of the various rheological media and not with the general formulations of the typical problems of the mechanics of continua. Thus, the kinematic considerations have been limited to deformations which are essentially pure shear but the range of media discussed is very broad. There are chapters dealing with viscous fluids, plastic media, the generalized Newtonian fluid, and complex media such as concrete; and there are chapters devoted to discussions of work hardening, creep, strength and failure, and the behavior near solid boundaries. The book contains a wealth of information for those interested in the behavior of various media under comparatively simple loading or simple flow conditions.

G. F. Carrier

Experimental designs. By William G. Cochran and Gertrude M. Cox. John Wiley & Sons, Inc., New York and Chapman & Hall, Ltd., London. ix + 454 pp. $5.75.

In this book the authors present a large number of different designs covering a great variety of experimental situations. The application of these designs and their analysis is illustrated by well chosen examples. The book being intended as an elementary text and as a guide for research workers with little or no mathematical training, the authors do not fully discuss the basic assumptions underlying the analysis and the principles of statistical inference that lead from the assumptions to the analysis. However, the book distinguishes itself from many other elementary books in Statistics by giving a correct, although incomplete discussion of the necessary fundamental statistical principles.

The greatest value of the book in the reviewer's opinion is the extensive collection of experimental designs presented. A disadvantage for the reader with more mathematical background is the absence of closed expressions for the computation of test functions. Such closed expressions, which would inform the mathematically trained reader at a glance, are replaced by numerical examples, probably because formulae involving multiple summation signs and other formidable mathematical symbols would deter the average reader. The reviewer believes, however, that the book will be very profitable reading also for the student of Mathematical Statistics, particularly when used as complementary reading to a treatise presenting the mathematical theory of the design of experiments and the analysis of variance.

Henry B. Mann

An introduction to the LaPlace transformation. By J. C. Jaeger. John Wiley & Sons, Inc., New York and Methuen & Co., Ltd., London. viii + 132 pp. $1.50.

This member of the Methuen series of "Monographs on physical subjects" is concerned with the use of the LaPlace transform in solving differential equations with constant coefficients. The first chapter introduces the theorems needed; the second deals with the analysis of problems associated with electric circuit analysis; the third introduces a few more theorems and the final chapter applies the operational technique to a partial differential equation. The concise presentation seems well suited to those who wish to become familiar with an operational tool without wading through a careful analysis of the pertinent mathematics.

G. F. Carrier

Elements of aerodynamics of supersonic flows. By Antonio Ferri. The Macmillan Company, New York, 1949. x + 434 pp. $10.00.

This is a text designed for the practicing supersonic aerodynamicist. By limiting his field of coverage to purely supersonic flow, excepting incidentally subsonic flow where encountered behind a shock, the author has produced a treatment which may be termed reasonably comprehensive. The mathematical requirements for the reader are not too high—he needs a thorough knowledge of basic analysis through differential equations, but partial differential equations theory is not prerequisite and complex variables do not appear in the book. Although linearized theory is presented, the principal emphasis is on the non-linear, as befits a realistic approach. Test results are included wherever feasible for comparison and illustration and the entire presentation is made very graphic by a satisfying number of figures. The index is completely inadequate.

This is definitely not a text for workers in fundamental research. The main reason for this is that the main fundamental problems of high speed flow lie outside the field of coverage of the book; they involve subsonic flow regions, viscous effects, and stability questions, and much of the mathematics involved is more advanced. Also, in designing his book for the engineer, the author has emphasized the specific at the expense of the fundamental, omitting such basic concepts in supersonic flow as those of the Euler momentum integral and the linearized and hypersonic similitudes.

W. D. Hayes

Methoden der praktischen analysis. By F. A. Willers. 2nd ed. Walter de Gruyter & Co., Berlin, 1950. 410 pp. DM 24.-.

The first edition of this well-known text appeared in 1928. In preparing the present revised edition, the author has included many new methods developed since the publication of the first edition. To keep the size of the volume approximately the same, some special graphical methods discussed were omitted. While there are frequent minor revisions throughout the book, the major additions are sections on the approximate integration of integral equations of the second kind, evaluation of improper integrals, trigonometric interpolation, and the solution of systems of linear equations by iteration. The last chapter on approximate integration of differential equations has been completely rewritten. According to a statement in the preface, the manuscript of this revised edition was completed in 1943. This explains the fact that useful techniques developed during the war in the United States and Great Britain have not been included.

W. Prager

Tables of the generalized exponential-integral functions. By The Staff of the Computation Laboratory. Harvard University Press, Cambridge, Massachusetts, 1949. xxv + 416 pp. $8.00.

The twenty-five introductory pages of this volume are concerned with the definitions of the functions to be tabulated, a discussion of the computation methods, the interpolation technique and a brief application discussion with bibliography. Three functions are computed: They are $\int_0^x F_n(a, s)\, ds$ where the three integrands used are $u^{-1}(1 - e^{-u})$, $u^{-1}e^{-u} \sin u$, and $u^{-1}(1 - e^{-u} \cos u)$, with $u(a, s) = (a^2 + s^2)^{1/2}$.

These functions are tabulated in the range $0 \leq a \leq .049$ in steps $\Delta a = .001$ with $0 \leq x \leq .049$ in steps of .001; in $0 \leq a \leq .098$ and $0 \leq x \leq .098$ with increments in x and a of .002; in $0 \leq a \leq .245$ and $0 \leq x \leq .245$ with increments of .005; for $0 \leq a \leq .49$, $0 \leq x \leq .49$ with increments .01; in $0 \leq a, x \leq .98$ with increments .02; in $0 \leq a, x \leq 2.45$ with increments .05; for $0 \leq a, x \leq .49$ with increments .1; and for $0 \leq a, x \leq .98$ with increments .2.

G. F. Carrier

Tables of the function $\sin \phi/\phi$ *and of its first eleven derivatives.* By The Staff of the Computation Laboratory. Harvard University Press, Cambridge, Massachusetts, 1949. sviii + 241 pp. $8.00.

The function $\sin \phi/\phi$ and its first eleven derivatives are tabulated for values of ϕ in the range $0 \leq \phi < 20\pi$. The increments in ϕ are taken to be .5 degrees for computation convenience. Remarks about computational procedure and a brief discussion of the applications in which these functions are useful make up the introductory pages.

G. F. CARRIER

Tables of the Bessel functions of the first kind of orders sixty-four through seventy-eight. By The Staff of the Computation Laboratory. Harvard University Press, Cambridge, Massachusetts, 1949. 566 pp. $8.00.

The Bessel Functions $J_{64}(x)$, $J_{65}(x)$, $\cdots$ $J_{78}(x)$ are tabulated to ten decimal places. The argument varies in steps of .01 for $0 \leq x < 100$.

G. F. CARRIER

The variational principles of mechanics. By Cornelius Lanczos. University of Toronto Press, Toronto, 1949. xxv + 307 pp. $5.75.

This is an excellent book covering one of the most brilliant achievements of scientific theory, and the text will certainly be welcomed by teachers and students of analytical mechanics.

The author leads the reader through the historical development, starting from the basic concepts of mechanics and discussing a number of specific examples to familiarize the student with the general principles. Short, clear summaries present the fundamental results obtained in each section and make it possible to follow step by step the general line of thought. The chapter on Liouville's theorem may be taken as a typical illustration of the author's method, with a general discussion of phase-space, then a few simple examples, and finally the general theorem about integral-invariants.

One may only regret that the author did not include Ehrenfest's adiabatic invariants, which play such an important role in quantum theories. The analytical methods used in the old quantum mechanics are very completely and clearly discussed in the last chapter, which will be found especially valuable by many students in theoretical physics.

This is a brilliant presentation of a great chapter of science, and the text will be of great help to students who have studied mechanics in Synge and Griffith's excellent treatise where these problems were barely sketched.

L. BRILLOUIN

The meaning of relativity. By Albert Einstein. Princeton University Press, Princeton, New Jersey, 1950. 150 pp. $2.50.

The first part of the book is a general explanation of the physical background of Relativity. It begins with a discussion of the fundamental notions of space and time in classical physics and the role of relativity in classical mechanics, where all physical laws are assumed to be invariant for Cartesian systems of space coordinates, while the time variable is not supposed to be affected by these Cartesian changes of coordinates; at this stage, time and distance are absolute.

This set of definitions fails when applied to the Maxwell-Lorentz equations of electromagnetism. The discussion of this problem led Einstein to the discovery of the theory of Special Relativity, and the trend of thought is very clearly explained in the second chapter of the book, where the fusion of space and time in a comprehensive system of 4 variables is introduced.

The famous relation $E = mc^2$ is obtained, but there is just a short footnote to tell of the experimental proofs of this most important prediction.

General relativity was developed in order to include gravitation in this general description. The point of departure is the identity of inert mass and gravitational mass. This leads directly to the principle of equivalence, which states that gravitation plays a similar role to acceleration, and explains the need for Riemannian geometry. The connection between the General Relativity and Mach's ideas makes for a very interesting discussion of the fundamental assumptions and of the new results to be expected. The chapter concludes in favor of a space-bounded universe, as representing the most satisfactory type of general equation.

In the appendix to the second edition, this cosmologic problem is discussed, in connection with the experimental results about the red shift of spectral lines from distant nebulae and the expanding universe. The author does not believe in the addition of a "cosmologic term" in the equations of gravity, and presents the theory of Friedman as the best solution, with a four-dimensional universe that is isotropic with respect to three dimensions (space). This yields a value of 10^9 years for the age of the universe, a very small value indeed and difficult to reconcile with other astronomical data.

The second appendix is a new addition to the present third edition, and contains the new generalized theory of gravitation, recently developed by A. Einstein. This is an attempt to include Electromagnetism in the general geometry of the universe, instead of leaving Maxwell's equations as a separate group of equations, unconnected with the geometry. The previous general Relativity was a frame in which Electromagnetism was included as a separate picture. The aim is now to rebuild frame and picture together as a single unit. For this problem, our general knowledge of physics does not permit an unequivocal choice as did the principle of equivalence for gravitation. The only clue is that something similar to Maxwell's equation must be contained within the total field.

Many such attempts have been presented in the last decades, and they can be classified into two main groups. The electromagnetic field representing a skew-symmetrical tensor, one may try to add such skew-symmetrical terms to either

A—the $g_{\mu\nu}$ fundamental metric tensor

or B—the $\Gamma^i_{\mu\nu}$ coefficients of parallel displacement (the $\Gamma^i_{\mu\nu}$ do not constitute a tensor)

Both quantities were usually defined as symmetrical in $\mu\nu$, and can be completed with skew-symmetrical terms. Einstein uses both generalizations simultaneously and builds up the corresponding Riemann Tensor for curvature. He then looks for field equations, and in order to be certain of the compatibility of these equations he derives them from a variational principle according to the well-known Hamilton procedure. After some simplification, the result is a system of equations very similar to Maxwell's equations (with no free magnetic charges).

The author does not give any discussion of the practical value of these new equations and does not tell the new facts that they may include. One must remember that the theoretical problem was a formidable one, and that A. Einstein has been working on it for two decades, trying all possible lines of attack and attempting to use all sorts of new geometrical structures. He finally makes a choice, and gives us a set of formulas that seem to him to represent the best logical solution, leaving to others the trouble to discuss the practical and physical meaning of the new theory. Many questions remain still obscure, and first of all: how is it possible to obtain the equations of motion for different particles with different masses, charges and magnetic moments? One single geodesic can not do, since all trajectories are different from each other. This problem is not discussed in Einstein's paper, and we hope he shall give us his ideas about the type of logical solution to be expected.

The reader will refer with great interest to the paper published by Einstein in the "Scientific American", p. 13, vol. 182, no. 4, April 1950, where the philosophical background of the theory is discussed with great care and remarkable clarity. Einstein indicates in this paper a very curious feature of his theory: the initial conditions cannot be freely chosen. He points out that this fact does not seem to lead to serious trouble, provided one may prove that the system is extensive enough to cover all the possible physical problems, and this is still an open question.

Let us admire the genius of a great thinker and an exceptional scientist, who fulfills now, in his ripe age, an audacious plan of research that he started as a young man. A wonderful achievement and a rare one.

L. BRILLOUIN

Physical principles of oil production. By Morris Muskat. First Edition. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1949. xv + 922 pp. $15.00.

As stated in the preface this work is neither a textbook nor a reorganized version of well established theories for pedagogical purposes, but an exposition of material which is in an active state of flux, with the purpose of formulating and correlating the physical principles and facts underlying the mechanisms of oil production.

The physics of oil production has been a rapidly expanding branch of applied physics in the last fifteen years and one cannot help but admire the art and ingenuity with which the scientific approach has been introduced in a field where rule of thumb used to prevail and where the object of study, the oil reservoir, offers an infinite variety of individual aspects which are susceptible to only limited observation.

Chapters 1 to 3 describe and discuss the general physical properties of petroleum fluids and oil bearing rocks with particular reference to the thermodynamic behavior of multiphase hydrocarbon systems and their interaction with porous rock. The following three chapters deal with the motion of single phase fluids through porous rock. Incompressible fluid flows form the most important class for practical applications and such flows in homogeneous and heterogeneous media are treated by solving the Laplace equation. Many types of flow fields are described corresponding to various physical conditions of stratification and well penetration. The effect of compressibility is also introduced and the corresponding diffusion-type equation is applied to a few idealized cases. The more difficult problems of mult phase fluid flow phenomena are treated in Chapters 7 to 11. The physical concepts and characteristics of this type of flow are reviewed and applied to the solution of several simple steady-state flow equations. This is supplemented by examples of actual performance records of fields. The last three chapters are devoted to problems of secondary recovery, condensate reservoirs, well spacing and the estimation of recoverable reserves. The mathematical and analogue computer methods developed for the solution of these problems will be of interest also to those who are not directly concerned with oil production.

In view of the authoritative nature of this work it is unfortunate that serious criticism may be expressed concerning the presentation of the general laws of fluid flow through porous media. The treatment lacks the clarity and generality which one expects to find in a treatise of this scope. In the opinion of the reviewer the book would have gained appreciably if it had followed a more systematic presentation as developed, e.g., by M. K. Hubbert (cf. Theory of Ground Water Motion, The Journal of Geology Vol. 48, No. 8, November-December 1940).

A great deal of interest has been attracted lately to the question of the rapid depletion of the world oil reserves. The challenge to the applied mathematician and physicist is well illustrated by this most thoroughly documented treatment of the field.

M. A. BIOT