TRANSIENTS IN MULTIPLY PERIODIC NON-LINEAR SYSTEMS*

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1. Introduction. The transient behavior of the general dynamical system of n degrees of freedom, subject to certain restrictions, will be treated. The system must be (a) nearly linear, (b) nearly conservative, (c) nearly autonomous; the deviation from the case of complete linearity, conservativeness, and autonomy being indicated by the magnitude of a parameter μ. It is further assumed that the explicit dependence of the non-autonomous terms on time is in the form of sinusoids of frequencies $\Omega_1, \Omega_2, \cdots, \Omega_m$.

The general system is transformed to normal coordinates and thence to elliptical coordinates, the resulting form being convenient for the development of the solution for small values of μ. An auxiliary set of differential equations is so constructed that its singular points determine the limit cycles of the original system for the limiting case, $\mu \to 0$. The stability of each limit cycle in the original system is the same as the stability of the corresponding singular point in the auxiliary system. Furthermore, the coordinates of the auxiliary set are the amplitudes of the n fundamental frequency components of the original system, so that the motion of the representative point in the phase space of the auxiliary system describes approximately the transient behavior of the amplitudes of the fundamental frequency components of the original system for small values of the parameter μ.

The general cases of internal and external resonances, in which two or more fundamental frequency components are commensurate, are treated. Finally, the method is applied to a problem of practical interest.

2. The method. Under restrictions (a), (b), and (c), the equations which describe the general system are

$$\sum_{j=1}^{n} \left(a_{ij} \frac{d^2 y_j}{dt^2} - b_{ij} y_j \right) + \mu g_i = 0, \qquad i = 1, 2, \cdots, n. \tag{1}$$

The quantities g_i are functions of the parameter μ, the independent variable t, and the n dependent variables y_i and their various derivatives. The final restriction on the system (1) is that there exist a closed region D, including the origin, in which each g_i is bounded for all values of t and for sufficiently small values of μ.

By means of a suitable linear transformation [1]

$$y_j = \sum_{k=1}^{n} c_{jk} x_k, \qquad j = 1, 2, \cdots, n,$$

Eqs. (1) may be transformed to normal coordinates. Thus

$$\frac{d^2 x_i}{dt^2} + \omega_i^2 x_i + \mu f_i = 0, \qquad i = 1, 2, \cdots, n. \tag{2}$$

The constants ω_i^2 are the roots of the characteristic equation of the linear system which

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occurs when μ is set equal to zero in (1). Since the kinetic energy of the linear system is a positive definite form, the constants ω_i^2 are positive [1].

The classical approach to equations (2) involves the expansion of the normal modes x_i and their fundamental frequencies in power series in μ. This procedure is an extension of the method presented by Kryloff and Bogoliuboff for a system of a single degree of freedom [2]. Concerning this extension, only one remark will be made. In the system of a single degree of freedom, secular terms in the series solution may be avoided by satisfying two equations at each stage of the process. One of the two equations determines the amplitude of the corresponding term in the series, and the other equation determines the contribution of the corresponding term to the fundamental frequency of oscillation. In the case of a multiply periodic system, each single equation is replaced by a set of n simultaneous non-linear equations.

For the purpose of this note, it is convenient to use elliptical coordinates defined by the equations

$$\begin{aligned} x_i &= r_i \cos \theta_i \\ \frac{dx_i}{dt} &= \omega_i r_i \sin \theta_i \end{aligned} \qquad i = 1, 2, \cdots, n. \tag{3}$$

In these coordinates, Eqs. (2) become

$$\begin{aligned} \frac{dr_i}{dt} &= -\frac{\mu}{\omega_i} f_i \sin \theta_i \\ \frac{d\theta_i}{dt} &= -\omega_i - \frac{\mu}{\omega_i r_i} f_i \cos \theta_i \end{aligned} \qquad i = 1, 2, \cdots, n. \tag{4}$$

By means of the change of variable,

$$T = \mu t,$$

Eqs. (4) may be written

$$\begin{aligned} \frac{dr_i}{dT} &= -\frac{1}{\omega_i} f_i \sin \theta_i \\ \frac{d\theta_i}{dT} &= -\frac{\omega_i}{\mu} - \frac{1}{\omega_i r_i} f_i \cos \theta_i \end{aligned} \qquad i = 1, 2, \cdots, n. \tag{5}$$

Inasmuch as each g_i is bounded in D, each f_i must be bounded in D. Since the quantities f_i are functions of the variables x_i and their derivatives, and since the latter are periodic functions of the variables θ_i, the right sides of (5) are periodic functions of each variable θ_i. Furthermore, by hypothesis, each f_i is a periodic function of each variable ϕ_i where

$$\phi_i = -\frac{\Omega_i}{\mu} T + \phi_{i0}, \qquad i = 1, 2, \cdots, m.$$

As the parameter μ is decreased, the right sides of (5) take the form of certain slowly varying functions of T(due to the variation of r_i with time) modulated by rapidly varying quasi-periodic functions of T(due to the variations of θ_i and ϕ_i with time). Furthermore, the quasi-periods of the modulation approach zero with μ.

Let $K_i(r_1, r_2, \cdots r_n, \mu)$ represent the unmodulated portion of $f_i \sin \theta_i$, defined by

$$K_i(\mu) = \lim_{P\to\infty} \frac{1}{P} \int_0^P f_i \sin \theta_i \, dt, \qquad i = 1, 2, \cdots, n, \tag{6}$$

in which the variables r_i are considered constant in the integration. When evaluating the integral (6) for the limiting case $\mu \to 0$, the integrand is a function of each r_i (considered constant in the integration), a sinusoidal function of each θ_i (equal to $-\omega_i t + \theta_{i0}$ in the limiting case), and a sinusoidal function of each ϕ_i (equal to $-\Omega_i t + \phi_{i0}$).

Now the quantities

$$G_i = f_i \sin \theta_i - K_i(\mu), \qquad i = 1, 2, \cdots, n,$$

are quasi-periodic functions of T, the quasi-periods of which approach zero with μ. Furthermore, each G_i is bounded in D, so that the limit as $\mu \to 0$ of the indefinite integral of G_i is identically zero. Therefore, for the limiting case, $\mu \to 0$, Eqs. (5) may be written

$$\frac{dr_i}{dT} = -\frac{K_{i(0)}}{\omega_i}, \qquad i = 1, 2, \cdots, n,$$

and for small values of μ, (4) may be approximated by

$$\frac{dr_i}{dt} = -\frac{\mu}{\omega_i} K_{i(0)}, \qquad i = 1, 2, \cdots, n. \tag{7}$$

The system of n simultaneous non-linear differential equations (7) comprises the set which is auxiliary to the system (4). In fact, the variables of (7) are the amplitudes of the set (4). Therefore, the singular points of the former, defined by the equations

$$K_i(r_{10}, r_{20}, \cdots, r_{n0}, 0) = 0, \qquad i = 1, 2, \cdots, n, \tag{8}$$

fix the amplitudes of oscillation on the limit cycles of (4). Furthermore, the stability of each singular point of (7) is the same as the stability of the corresponding limit cycle of (4).

Let us consider first the non-resonant case, that is, the case in which no two of the set of frequencies ω_i and Ω_i are commensurate. In this case, (6) may be replaced by the expressions

$$K_i(0) = \frac{1}{(2\pi)^{n+m}} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} f_i \sin \theta_i \, d\theta_1 \, d\theta_2 \cdots d\theta_n \, d\phi_1 \, d\phi_2 \cdots d\phi_m, \tag{9}$$

$$i = 1, 2, \cdots, n.$$

Thus the values of K_i, the singular points of the auxiliary system (7), and hence the limit cycles of (4) for small μ are independent of the phases θ_{i0} and ϕ_{i0} of the autoperiodic and heteroperiodic oscillations.

3. Internal and external resonances. If two frequences, say ω_1 and ω_2 are commensurate, that is, if there exist two positive integers n_1 and n_2 such that

$$n_1\omega_1 = n_2\omega_2,$$

the system will be said to have a simple resonance of order n_1/n_2 or n_2/n_1, whichever is smaller. On the other hand, if there exist k positive integers, $n_1, n_2, \cdots, n_k$, such that

$$n_1\omega_1 = n_2\omega_2 = \cdots = n_k\omega_k, \tag{10}$$

the system will be said to have a complex resonance of degree k, there being associated with the resonance $k - 1$ orders, n_2/n_1, n_3/n_1, $\cdots$, n_k/n_1, where n_1 is the largest of the integers.

In this case, the integral (6) may be replaced by

$$K_i(0) = \frac{1}{2\pi^{n+m-k+1}} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} f_i \sin \theta_i \, dx \, d\theta_{k+1} \cdots d\theta_n \, d\phi_1 \cdots d\phi_m , \qquad (11)$$

$$i = 1, 2, \cdots, n,$$

in which

$$\theta_i = \frac{N}{n_i} x + \theta_{i0} , \qquad i = 1, 2, \cdots, k,$$

where

$$N = \prod_{j=1}^{k} n_j .$$

Thus the integration on x from zero to 2π insures that θ_1, θ_2, $\cdots$ θ_k range over integral numbers of their respective periods.

If only frequencies ω_i and not Ω_i are involved in (10), the resonance is said to be internal. If only frequencies Ω_i and not ω_i are involved, the resonance will be called external. If both ω_i and Ω_i are involved, the resonance will be called compound.

In the general case, one may have a system with several complex compound resonances. In any case, it is always possible to find a set of integers (sufficiently large) such that (10) is approximately true. Therefore, the resonance case makes sense only provided the integers are all small. The values of the integrals (11) in many cases depend upon the phases θ_{i0} and ϕ_{i0}.

4. Example. Consider the circuit of Fig. 1, containing the non-linear resistor

$$R = -\mu(1 - \gamma^2 I^2). \qquad (12)$$

The resistance R is composed of a passive resistor in series with the negative resistance of a vacuum tube. The passive resistor is so chosen that the total resistance of the circuit at zero current (that is, $-\mu$) is small and negative. The term $\gamma^2 I^2$ is due to saturation of the tube.

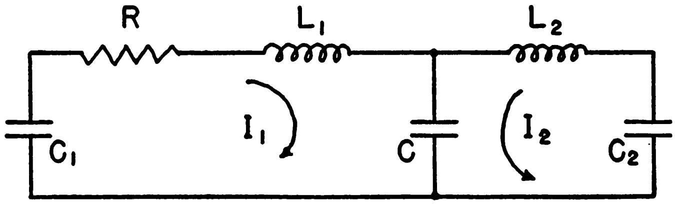

Fig. 1.

The two degrees of freedom of the circuit of Fig. 1 may be uncoupled by the transformation to normal coordinates

$$I_1 = I_1' + I_2'$$

$$I_2 = \alpha I_1' + \beta I_2' ,$$

in which

$$\alpha = \frac{1}{2\omega_{12}^2}\,[\omega_{22}^2 - \omega_{11}^2 + \{(\omega_{22}^2 - \omega_{11}^2)^2 + 4\omega_{12}^2\omega_{21}^2\}^{1/2}],$$

$$\beta = \frac{1}{2\omega_{12}^2}\,[\omega_{22}^2 - \omega_{11}^2 - \{(\omega_{22}^2 - \omega_{11}^2)^2 + 4\omega_{12}^2\omega_{21}^2\}^{1/2}],$$

where

$$\omega_{11}^2 = \frac{1}{L_1}\left(\frac{1}{c_1} + \frac{1}{c}\right), \qquad \omega_{12}^2 = \frac{1}{L_1 c},$$

$$\omega_{22}^2 = \frac{1}{L_2}\left(\frac{1}{c_2} + \frac{1}{c}\right), \qquad \omega_{21}^2 = \frac{1}{L_2 c}.$$

The resulting circuit is shown in Fig. 2.

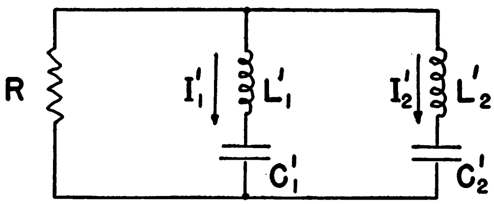

FIG. 2.

The equivalent circuit elements are given by the expressions

$$L_1' = \frac{\beta - \alpha}{\beta}\,L_1\,, \qquad C_1' = \frac{1}{L_1'\Omega_1^2},$$

$$L_2' = \frac{\alpha - \beta}{\alpha}\,L_1\,, \qquad C_2' = \frac{1}{L_2'\Omega_2^2}, \tag{13}$$

where

$$\Omega_1^2 = \frac{1}{2}\,[\omega_{11}^2 + \omega_{22}^2 + \{(\omega_{11}^2 - \omega_{22}^2)^2 + 4\omega_{12}^2\omega_{21}^2\}^{1/2}],$$

$$\Omega_2^2 = \frac{1}{2}\,[\omega_{11}^2 + \omega_{22}^2 - \{(\omega_{11}^2 - \omega_{22}^2)^2 + 4\omega_{12}^2\omega_{21}^2\}^{1/2}].$$

In the circuit of Fig. 2, the inductances and capacitances are positive according to Eqs. (13).

It is convenient to make the substitution

$$x_i = \gamma I_i'\,, \qquad i = 1, 2.$$

The Kirchhoff equations for Fig. 2 are

$$\frac{d^2x_i}{dt^2} + \Omega_i^2 x_i + \mu f_i\left(x_1\,, x_2\,, \frac{dx_1}{dt}, \frac{dx_2}{dt}\right) = 0, \qquad i = 1, 2, \tag{14}$$

in which

$$L_1'f_1 = [(x_1 + x_2)^2 - 1]\left(\frac{dx_1}{dt} + \frac{dx_2}{dt}\right) = L_2'f_2 . \tag{15}$$

Equations (14) together with (15) are an extension of Van der Pol's equation.

For the case of μ very small, one may proceed directly to equations (7) after evaluating the integrals (6). If Ω_1 and Ω_2 are not commensurate (non-resonant case), the corresponding integrals (9) are

$$K_i(0) = \frac{1}{4\pi^2 L_i'} \int_0^{2\pi} \int_0^{2\pi} [(r_1 \cos\theta_1 + r_2 \cos\theta_2)^2 - 1](\omega_1 r_1 \sin\theta_1 + \omega_2 r_2 \sin\theta_2) \sin\theta_i \, d\theta_1 \, d\theta_2 , \qquad i = 1, 2 \tag{16}$$

so that equations (7) take the form

$$\begin{aligned} \frac{dr_1}{dt} &= \frac{\mu}{8L_1'} r_1(4 - r_1^2 - 2r_2^2) \\ \frac{dr_2}{dt} &= \frac{\mu}{8L_2'} r_2(4 - r_2^2 - 2r_1^2). \end{aligned} \tag{17}$$

The singular points of (17) occur at (0,0), (2,0), (0,2), and $(2/3^{1/2}, 2/3^{1/2})$. They are respectively an unstable star point, a stable nodal point, a stable nodal point, and a saddle point. The corresponding limit cycles of (14) have the sets of amplitudes (0,0), (2,0), (0,2), and $(2/3^{1/2}, 2/3^{1/2})$ and are respectively unstable, stable, stable, and unstable.

The transient behavior of the amplitudes of the normal oscillations of (14) can be shown in the r_1r_2 phase plane of the system (17). The differential equation of the phase plane trajectories is obtained by dividing the second Eq. (17) by the first. Thus

$$\frac{dr_2}{dr_1} = \frac{L_1'}{L_2'} \frac{r_2}{r_1} \frac{4 - r_2^2 - 2r_1^2}{4 - r_1^2 - 2r_2^2}.$$

For the case $L_1' = L_2'$, the solution is

$$(r_2^2 - r_1^2)^3 = 4\left[r_2^4 - r_1^4 + r_1^2 r_2^2\left(c - 2 \ln \frac{r_2^2}{r_1}\right)\right], \tag{18}$$

c being the constant of integration. The trajectories are shown in Fig. 3.

If Ω_1 and Ω_2 are commensurate, that is, if there exist two integers, n_1 and n_2 , such that

$$n_1\Omega_1 = n_2\Omega_2 ,$$

equation (11) must be used to evaluate K_1 and K_2 . The integrals are easily evaluated and yield four cases depending upon the ratio n_2/n_1 . If the ratio is unity, the auxiliary set of differential Eqs. (7) contains the synchronous phase $\theta_{10} - \theta_{20}$. If the ratio is 1/2 or 3, the auxiliary set contains the synchronous phase $3\theta_{10} - \theta_{20}$ or $3\theta_{20} - \theta_{10}$. Any other ratio yields equations identical with Eqs. (17) describing the non-resonant case.

The location and stability of the limit cycles of the original system, as well as the transient behavior of the amplitudes of the normal modes, are strongly affected by the order of resonance. In the above example for the case of internal resonance of order

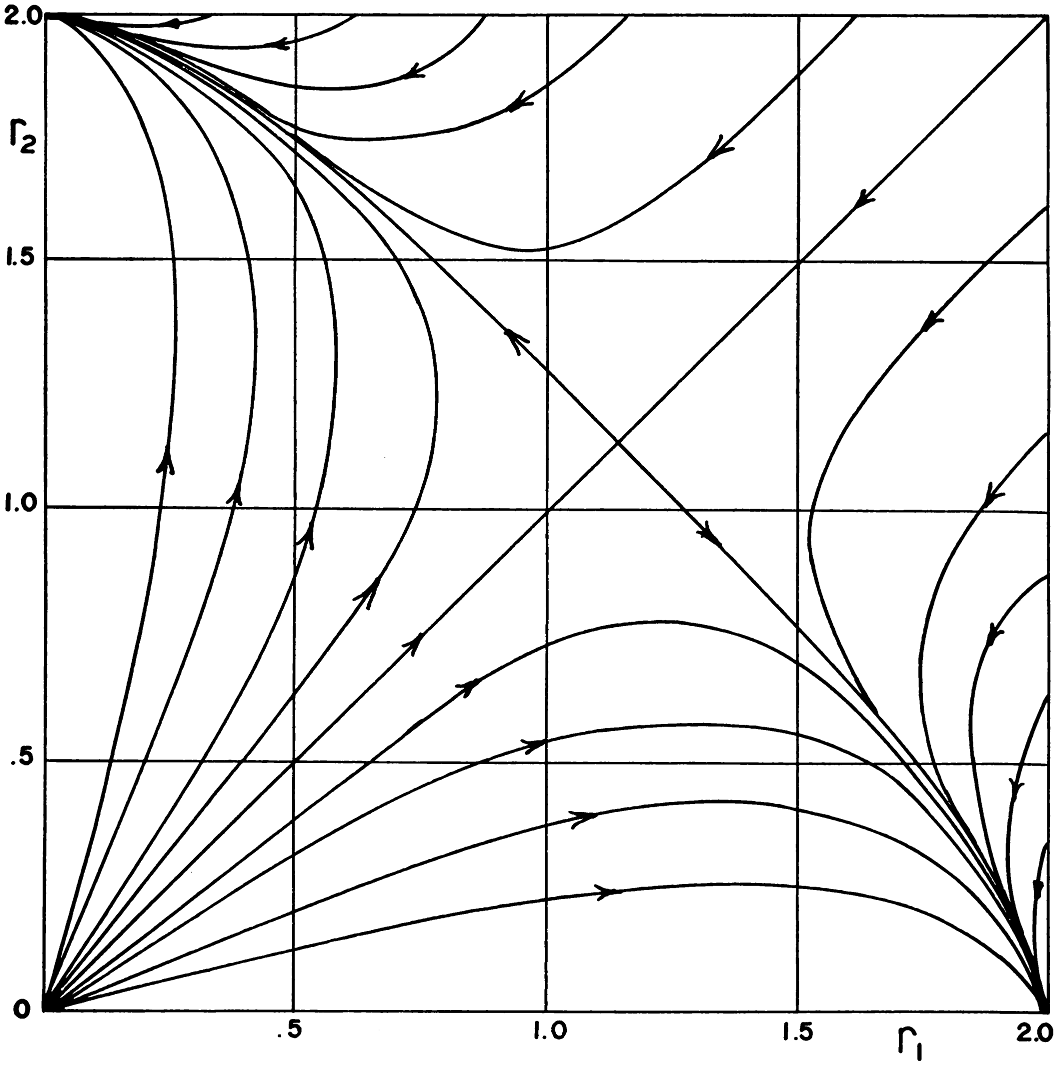

FIG. 3. Transient variation of the amplitudes of the normal modes of a doubly periodic system.

one-third ($n_1 = 3n_2$), there are four limit cycles. Three of the limit cycles have the respective amplitudes (0,0), (2,0), and (0,2), and correspond respectively to an unstable star point, a stable nodal point, and a stable nodal point in the auxiliary system. The remaining singular point is a saddle point; its location is a function of the synchronous phase

$$\theta_0 = 3\theta_{10} - \theta_{20} ,$$

and is shown in Figure 4.

Except for internal resonance of order one or one-third, the auxiliary equations are the same as for the non-resonant case, and hence the transient motion is not affected. The effect of resonance on the location and stability of limit cycles is complicated even

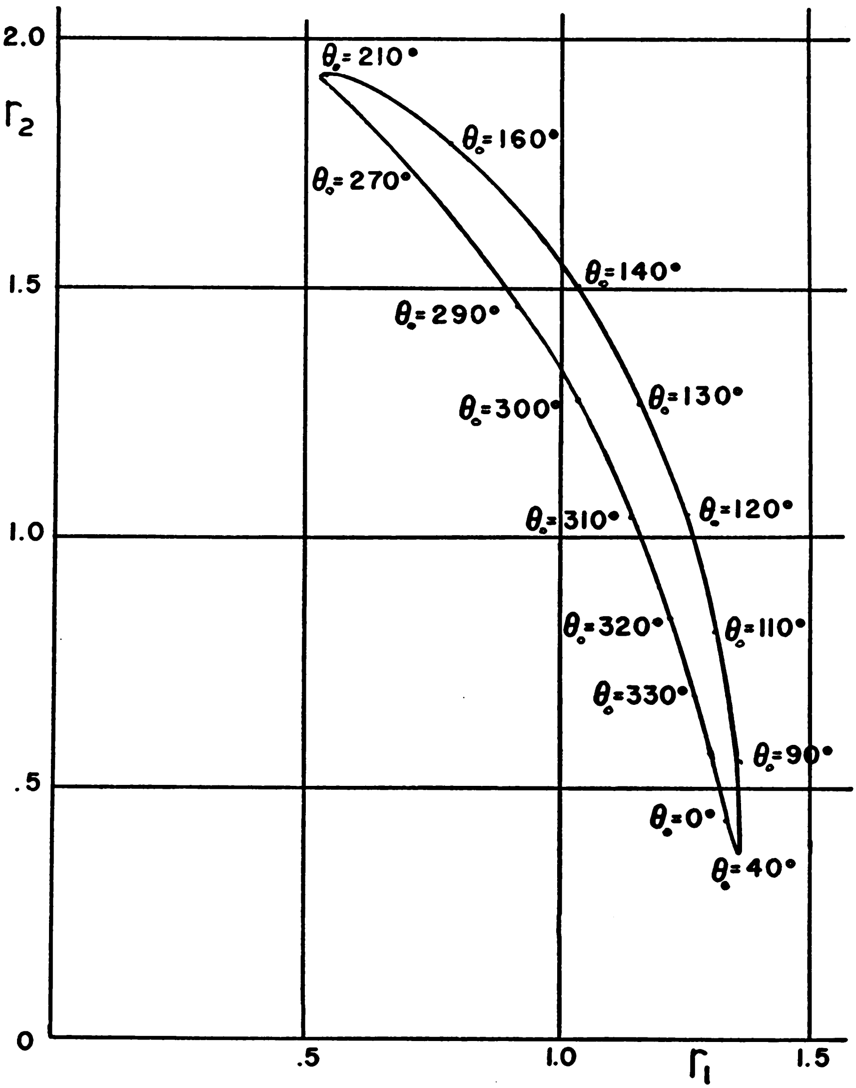

FIG. 4. Effect of synchronous phase, θ_0 , on the amplitudes of oscillation of the normal modes of a doubly periodic system.

in this simple case of a doubly periodic system, and will not be discussed in more detail here.

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2. N. Kryloff and N. Bogoliuboff, *Introduction to non-linear mechanics*, translation by S. Lefschetz, pp. 40-55, Princeton University Press, 1943.