STEFAN-LIKE PROBLEMS*[1]

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Introduction. Consider the recrystallization of an infinite metal slab, as for example, the change from α to β crystals when iron is heated through the temperature 1643°F. Accompanying this change of crystalline form is a latent heat of recrystallization. If the slab is originally of uniform crystalline structure and is heated from the front face through the critical temperature, new crystals appear at the heated face. If the front face is planar and is heated uniformly and the back face is insulated, the interface between the two types of crystals is planar. For the determination of stresses occurring in the metal, it would be necessary to determine the history of the interface as it moves inward. To determine the history (a space-time curve) of this interface, it is necessary to solve the heat equation, in this case, a linear homogeneous parabolic equation, with appropriate boundary conditions and with a suitable discontinuity in the first order space derivatives across the unknown interface.

In each of the several problems to be discussed here, we will consider the recrystallization of a metal slab of uniform thickness, L, bounded by the planes $x = 0$ and $x = L$. We assume that a uniform heat source covers the front face, $x = 0$, and that a perfect insulator covers the back face, $x = L$, if L is not infinite. Thus, the problems will involve only one space dimension. Furthermore, the coefficient of thermal diffusivity, α^2, is assumed to be a constant for a given type of crystal, and recrystallization is assumed to take place at a temperature $u = 0$. Since in each problem the object is to find the propagation of the interface through the slab, the choice of the critical temperature as zero in no way affects the results.

To determine the type of discontinuity across the interface, assume that the recrystallization has already occurred to a distance x where $0 < x < L$. The subscript 1 after a symbol will be used to indicate that the symbol refers to the region of the original crystalline form and the subscript 2 to indicate that the symbol refers to the recrystallized region. The equation of heat balance across the interface as it moves a distance dx then is, after division by dt,

$$\rho_1 H \dot{x}(t) = (k_2 u_{2_x} - k_1 u_{1_x}) \big|_x = x(t), \tag{1}$$

where $x = x(t)$ is the space-time curve along which the interface moves, $\dot{x}(t) = dx/dt$, ρ is the density, H is the latent heat of recrystallization, and the k_i are the coefficients of thermal conductivity. The notation used is $\partial u_i/\partial x = u_{i_x}$ for $i = 1, 2$. The temperature distribution in each region is given by the differential equation

$$u_{i_t} = \alpha_i^2 u_{i_{xx}} \tag{2}$$

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[1]a) The term "Problem of Stefan" has recently been used for problems similar to the ones treated here by L. I. Rubinstein, A. B. Datsev, and others.

b) The authors express their indebtedness to the members of the Institute of Mathematics and Mechanics and especially to Professors R. Courant, B. Friedman, and J. Keller.

[2]Deceased. The practical results of this paper were obtained prior to the untimely death of our inspiring co-author J. K.L. MacDonald. The surviving authors shall endeavor to complete, in the mathematical sense, the material covered in this note.

1. The first problem to be considered will illustrate a method of solution for all. In this problem the slab has any thickness L, $0 < L \leq \infty$. The slab is considered to have been heated uniformly to the critical temperature, 0, and then a constant heat source is applied to the front face of the slab. This may be represented by the following mathematical problem:

Find $x = x(t)$ where the temperature distribution satisfies the equations[3]

$$u_t = \alpha^2 u_{xx} \qquad \text{for } 0 < x < x(t), \tag{3}$$

$$u = 0 \qquad \text{for all } x \geq x(t) \tag{4}$$

with the boundary conditions

$$A\, \dot{x}(t) = u_x[x(t),\, t] \tag{5}$$

$$x(0) = 0 \tag{6}$$

$$u_x(0,\, t) = g, \tag{7}$$

where g is a constant. The condition (5) is a consequence of equation (1) where $A = \rho H/k$.

We obtain a power series representation for $x = x(t)$ about $t = 0$ by assuming u expressible in a power series about $x = 0$, $t = 0$ for all u in the region $0 \leq x \leq x(t)$, i.e., any discontinuities in the derivatives of u are assumed to occur only on crossing the curve $x = x(t)$.

By differentiation of (4) and (5) along the curve $x = x(t)$ and of (3) in the region $0 \leq x \leq x(t)$ and use of (7), all derivatives of $x(t)$ may be determined at the origin. $x(t)$ may then be written as the following Taylor series:

$$\begin{aligned} x(t) = \frac{g}{A}\, t - \frac{1}{2!}\frac{g^3}{\alpha^2 A^3}\, t^2 + \frac{5}{3!}\frac{g^5}{\alpha^4 A^5}\, t^3 \\ - \frac{51}{4!}\frac{g^7}{\alpha^6 A^7}\, t^4 + \frac{827}{5!}\frac{g^9}{\alpha^8 A^9}\, t^5 - \cdots \end{aligned} \tag{8}$$

The coefficients of the series have not been calculated beyond those given. They may be found from the following relations:

Let $u = \sum_{i,j=0}^{\infty} a_{ij} x^i t^j$ and $x = \sum_{i=0}^{\infty} C_i t^i$, then,

$$\sum_{i,j=0}^{\infty} (i+1) a_{i+1,j} \left(\sum_{n=0}^{\infty} C_n t^n \right)^i t^j = A \sum_{n=0}^{\infty} (n+1) C_{n+1} t^n, \tag{9}$$

$$\sum_{i,j=0}^{\infty} a_{ij} \left(\sum_{n=0}^{\infty} C_n t^n \right)^i t^j = 0, \tag{10}$$

$$a_{i+2,j} = \frac{1}{\alpha^2} \frac{(j+1)}{(i+1)(i+2)}\, a_{i,j+1} \tag{11}$$

with $a_{1,0} = g$ and $C_0 = 0$.

[3]We have adopted the convention of using the letter u without subscripts when the temperature is identically 0 in the region 1 (i.e., to the right of the discontinuity).

In the event that the heat input, g, is no longer a constant but an analytic function of time, $g = g(t)$, we find in a similar way the interface curve to be:

$$x(t) = C_1\, t - \frac{C_2}{2!}\, t^2 + \frac{C_3}{3!}\, t^3 - \frac{C_4}{4!}\, t^4 + \frac{C_5}{5!}\, t^5 - \cdots, \tag{12}$$

where

$$C_1 = \frac{g(0)}{A},$$

$$C_2 = \frac{g^3(0)}{\alpha^2 A^3} - \frac{\dot g(0)}{A},$$

$$C_3 = \frac{5g^5(0)}{\alpha^4 A^5} - \frac{6\, \dot g(0) g^2(0)}{\alpha^2\, A^3} + \frac{\ddot g(0)}{A},$$

$$C_4 = \frac{51 g^7(0)}{\alpha^6\, A^7} - \frac{75\, \dot g(0)\, g^4(0)}{\alpha^4\, A^5} + \frac{10\, \ddot g(0) g^2(0)}{\alpha^2\, A^3} + \frac{15\, [\dot g(0)]^2\, g(0)}{\alpha^2\, A^3} - \frac{\dddot g(0)}{A},$$

$$C_5 = \frac{827\, g^9(0)}{\alpha^8\, A^9} - \frac{1438\, \dot g(0) g^6(0)}{\alpha^6\, A^7} + \frac{175\, \ddot g(0) g^4(0)}{\alpha^4\, A^5} + \frac{525\, [\dot g(0)]^2 g^3(0)}{\alpha^4\, A^5} - \frac{15\, \dddot g(0)\, g^2(0)}{\alpha^2\, A^3} - \frac{15\, [\dot g(0)]^3}{\alpha^2\, A^3} - \frac{60\, \dot g(0)\, \ddot g(0)\, g(0)}{\alpha^2\, A^3} + \frac{\ddddot g(0)}{A}.$$

The series of Eq. (12) reduces to that of Eq. (8) for $g(t)$ a constant, $g(0) = g$.

2. To generalize the treatment of Sec. 1 to cover the case of non-analytic boundary and initial conditions, we employ Laplace transformations. Furthermore, we believe that the questions of uniqueness and existence will be answered by these methods. In this part, we derive the integral equation which is satisfied by the interface curve $x = x(t)$. To this end, the discontinuity curve is considered in the form $t = f(x)$, and for greater generality we consider the heat input, $g = g(t)$, to be a function of time. The problem is to solve the differential equation

$$u_t(x, t) = \alpha^2 u_{xx}(x, t) \qquad \text{for} \qquad t > f(x) \tag{13}$$

with the boundary conditions

$$u(x, t) = 0, \qquad \text{for } t \le f(x) \tag{14}$$

$$u_x(0, t) = g(t), \tag{15}$$

$$f(0) = 0, \tag{16}$$

$$u_x[x, f(x)] = \frac{A}{f'(x)}, \tag{17}$$

where $f' = df/dx$.
The Laplace transform for $u(x, t)$ is defined as

$$u^{\square}(x, s) = \int_0^{\infty} u(x, t)\, e^{-st}\, dt = \int_{f(x)}^{\infty} u(x, t)\, e^{-st}\, dt.$$

Application of the transform to (13) furnishes the transformed differential equation

$$\left[\frac{d^2}{dx^2} - \frac{s}{\alpha^2}\right] u^{\square} = -A\, e^{-sf(x)} \tag{18}$$

with the boundary conditions:

$$u_x^{\square}(0, s) = g^{\square}(s), \qquad \lim_{x\to\infty}\{u^{\square}(x, s)\} = 0.$$

The solution of Eq. (18) is

$$u^{\square}(x, s) = \frac{A\alpha}{2(s)^{1/2}} \int_0^{\infty} \exp\{-sf(\xi)\}\left[\exp\left\{-\frac{s^{1/2}}{\alpha}\mid x - \xi \mid\right\} + \exp\left\{-\frac{s^{1/2}}{\alpha}(x + \xi)\right\}\right] d\xi$$
$$- \frac{\alpha g^{\square}(s)}{s^{1/2}} \exp\left\{-\frac{s^{1/2}}{\alpha}x\right\},$$

and the inverse transform of this equation is

$$\begin{aligned} u(x, t) = \frac{A\alpha}{2} \int_0^{\infty} \frac{1}{\{\pi[t - f(\xi)]\}^{1/2}} &\left[\exp\left\{\frac{-(x - \xi)^2}{4\alpha^2[t - f(\xi)]}\right\}\right. \\ &\left. + \exp\left\{\frac{-(x + \xi)^2}{4\alpha^2[t - f(\xi)]}\right\}\right] d\xi_{t>f(\xi)} \\ &- \alpha \int_0^t \frac{g(t - \tau)}{(\pi\tau)^{1/2}} \exp\left\{\frac{-x^2}{4\alpha^2\tau}\right\} d\tau. \end{aligned} \tag{19}$$

Once $t = f(x)$ is known, Eq. (19) gives the temperature distribution in that region which has been recrystallized. Now, applying the condition (14) in the form $u[x, f(x)] = 0$, Eq. (19) becomes

$$\begin{aligned} 2\int_0^{f(x)} \frac{g(t - \tau)}{\tau^{1/2}} \exp\left\{\frac{-x^2}{4\alpha^2\tau}\right\} d\tau = A \int_0^{\infty} \frac{1}{[f(x) - f(\xi)]^{1/2}} &\left[\exp\left\{\frac{-(x - \xi)^2}{4\alpha^2[f(x) - f(\xi)]}\right\}\right. \\ \left. + \exp\left\{\frac{-(x + \xi)^2}{4\alpha^2[f(x) - f(\xi)]}\right\}\right] d\xi_{f(x)>f(\xi)}\ . \end{aligned} \tag{20}$$

Equation (20) is an integral equation for the determination of $t = f(x)$. To solve Eq. (20), we return to the designation $x = x(t)$. To do this let $t = f(x)$, $\tau = f(\xi)$, $x = x(t)$, and $\xi = x(\tau)$. This change is permissible if $t = f(x)$ is monotonically increasing. After making these substitutions in the right hand side of Eq. (20), replace $t - \tau$ by τ' and replace τ' by τ in the right hand side. Equation (20) becomes

$$\begin{aligned} 2\int_0^t \frac{g(t - \tau)}{\tau^{1/2}} \exp\left\{\frac{-x^2(t)}{4\alpha^2\tau}\right\} d\tau = A \int_0^t \frac{\dot{x}(t - \tau)}{\tau^{1/2}} &\left[\exp\left\{\frac{-[x(t) - x(t - \tau)]^2}{4\alpha^2\tau}\right\}\right. \\ \left. + \exp\left\{\frac{-[x(t) + x(t - \tau)]^2}{4\alpha^2\tau}\right\}\right] d\tau_{\tau>0}\ . \end{aligned} \tag{21}$$

A power series solution for $x = x(t)$ of equation (21) may be determined as follows: Let

$$x(t) = C_1 t + C_2 t^2 + \cdots , \tag{22}$$

and assume $g(t)$ a constant, g. The coefficients C_1 and C_2 are determined by using (22) to expand the integrands of equation (21) in powers of t (i.e., $x(t - \tau)$ is expanded about $\tau = 0$). We find that in the limit, as t tends to zero, equation (20) becomes

$$C_1 = \frac{g}{A} \cdot$$

Now, replacing $x(t)$ by $gt/A + C_2 t^2$ and $x(t - \tau)$ by

$$x(t - \tau) = x(t) - \tau \dot{x}(t) + \tau^2 \frac{\ddot{x}(t)}{2!}$$

in equation (21) and expanding about $t = 0$, Eq. (21) becomes

$$4gt^{1/2} - \frac{2g^2\pi^{1/2}}{\alpha A} t + \frac{g^3}{A^2\alpha^2} = 4gt^{1/2}$$

$$+ \frac{16}{3}\frac{A}{} C_2 t^{3/2} - \frac{2g^2\pi^{1/2}}{\alpha A} t + \frac{11}{3}\frac{g^3}{A^2\alpha^2} t^{3/2}$$

which yields

$$C_2 = -\frac{1}{2}\frac{g^3}{\alpha^2 A^3} \cdot$$

It is noticed that the coefficients determined in this manner are the same as the first two of equation (8). Similarly, when $g(t)$ is an analytic function of t, the first two coefficients, C_1 and C_2 , of equation (22) have been determined and agree with those of equation (12). The determination of the remaining coefficients by this method becomes extremely tedious.

3. Another problem which involves the determination of the curve of recrystallization would be to prescribe an initial temperature distribution in a metal slab of infinite thickness ($L = \infty$). The problem is then to determine the curve $x = x(t)$ where the temperature distribution is given by

$$u_{2_t}(x, t) = \alpha_2^2 u_{2_{xx}}(x, t) \qquad \text{for } 0 < x < x(t), \tag{23}$$

$$u[x(t), t] = 0, \tag{24}$$

$$u_{1_t}(x, t) = \alpha_1^2 u_{1_{xx}}(x, t) \qquad \text{for } x(t) < x \le \infty \tag{25}$$

with the boundary conditions

$$k_2 u_{2_x}[x(t), t] - k_1 u_{1_x}[x(t), t] = \rho H \dot{x}(t) = B \dot{x}(t), \tag{26}$$

$$u_1(x, 0) = \phi(x) \qquad \text{where} \qquad \phi(x) \le 0 \qquad \text{and} \qquad \phi(0) = 0, \tag{27}$$

$$u_{2_x}(0, t) = g(t). \tag{28}$$

In this problem $g(t)$ is assumed to be the same as in the previous problem and $\phi(x)$ is assumed to be expressible in a Taylor expansion about $x = 0$ with an infinite radius of convergence. The method of solution is again the same as that of Part I and the series representing $x(t)$ is given below through the second power of t.

$$\begin{aligned} x(t) = \frac{1}{B}\,[k_2 g(0) - k_1\phi'(0)]t \\ + \frac{1}{2!B}\left\{k_2\left[-\frac{g(0)}{\alpha_2^2 B^2}\,(k_2 g(0) - k_1\phi'(0))^2\right.\right. \\ \left. + \dot{g}(0)\right] - k_1\left[\frac{\phi''(0)}{B}\,(k_2 g(0) - k_1\phi'(0))\right. \\ \left.\left. + \alpha_1^2\,\phi'''(0)\right]\right\} t^2 + \cdots . \end{aligned} \tag{29}$$

This problem is reducible to the second problem of Part I by taking $\phi(x) = 0$, and the coefficients in Eq. (29) then agree with those of Eq. (12).

Comments. The boundary condition (4), $u = 0$ for $x \geq x(t)$, used in the first two problems is not as specialized as may first appear. In most practical cases the metal is initially at a uniform temperature, say room temperature. If a metal slab, initially at a uniform temperature, is heated at one face and is insulated at the other, the temperature gradient remains zero for sufficiently small L. That is, the temperature rises uniformly until the critical temperature is reached providing the slab is thin. This result is seen if one lets $u = U > 0$ be the critical temperature and considers the slab initially to have the temperature distribution $u(x, 0) = 0$. One can then show that the temperature distribution is independent of the space coordinate as long as $u(x, t) < U$. The temperature distribution in the slab of thickness L is given by the differential equation

$$u_t = \alpha^2 u_{xx} \tag{30}$$

with the boundary conditions

$$u_x(0, t) = g(t), \tag{31}$$

$$u_x(L, t) = 0, \tag{32}$$

$$u(x, 0) = 0. \tag{33}$$

The Laplace transform of (30) is the equation

$$u_{xx}^{\square}(x, s) = \frac{s}{\alpha^2}\,u^{\square}(x, s),$$

the solution of which is

$$u^{\square}(x, s) = -\frac{\alpha g^{\square}(s)}{s^{1/2}}\,\frac{\cosh\,(L - x)s^{1/2}/\alpha}{\sinh\,Ls^{1/2}/\alpha} \tag{34}$$

for the transposed boundary conditions (31) and (32). Since the slab is considered thin, replace cosh $(L - x)\, s^{1/2}/\alpha$ and sinh $Ls^{1/2}/\alpha$ by the first terms of their expansions giving

$$u^{\square}(x, s) = -\frac{\alpha^2}{sL}\, g^{\square}(s). \tag{35}$$

In (35), $u^{\square}(x, s)$ is the transform of $u(x, t)$, $g^{\square}(s)$ of $g(t)$ and $-\alpha^2/(sL)$ is the transform of $-\alpha^2/L$; therefore, $u(x, t)$ may be written as the following convolution

$$u(x, t) = -\int_0^t \frac{\alpha^2}{L}\, g(y)\, dy.$$

This expression shows $u(x, t)$ to be a function only of the variable t, and is therefore a constant with respect to x.

Discussion of the curves of equations (8), (12), and (29). a) Figure 1 shows the degree of agreement of the first five sums of the series of equation (8) where $x_1(t)$ represents

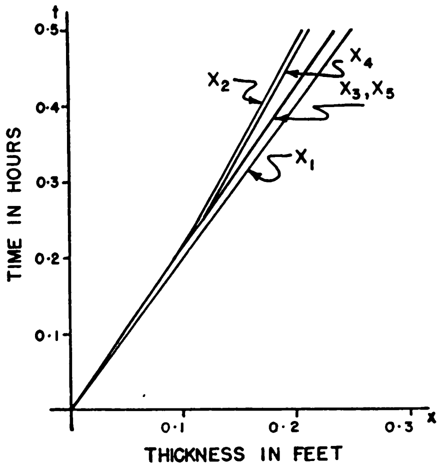

FIG. 1.

the first term on the right-hand side of (8), $x_2(t)$ represents the first plus the second, $x_3(t)$ represents the first three terms, etc. The calculations for this figure are based on the following data: $k = 25$ BTU/(hr.)(sq. ft.)(°F/ft.), $c_p = 14$ BTU/(lb.)(°F), $\rho = 480$ lbs./cu. ft., $H = -6.76$ BTU/lb., and $g = -65$ BTU/(sq. ft.)(hr.). This data is approximately that for a cast iron.

b) Figure 2 shows a comparison of the results of Eqs. (8), (12), and (29). For this comparison the boundary and initial conditions are taken in the following way. The curve for Eq. (29) was drawn using the following data: $\rho = 480$ lbs./cu. ft., $H = -6.76$ BTU/lb., $k_2 = 25$ BTU/(hr.)(sq. ft.)(°F/ft.), $k_1 = 26$ BTU./(hr.)(sq. ft.)(°F/ft.), $c_{p_1} = .12$ BTU/(lb.)(°F), $c_{p_2} = .14$ BTU/(lb.)(°F). $g(t)$ was chosen to be $-117\, e^{-t}$ BTU/(sq. ft.)(hr.) and $\phi(x)$ to be $-50 \tan^{-1} x$ °F. This choice of $g(t)$ is such that the heat added per unit time decreases with increasing time, while $\phi(x)$ has the property that the initial temperature at $x = 0$ is the critical temperature and the temperature decreases with increasing x until it reaches 25π°F below critical at $x = \infty$. For the curve of Eq. (12) the data is acquired from that for Eq. (29) in the same manner that the solution (12) may be obtained from the solution (29), i.e., $k_2 = k$, $c_{p_2} = c_p$, $\phi(x) =$

$\phi(0) = 0$. The data for the curve of equation (8) is deduced from that for Eq. (12) similarly, i.e., $g(t) = g(0) = -117$ BTU/(sq. ft.)(hr.). Since only two terms of the solution (29) have been determined, all curves are drawn using only the first two terms of each series. The straight line shown in Figure 2 represents the time it would take a

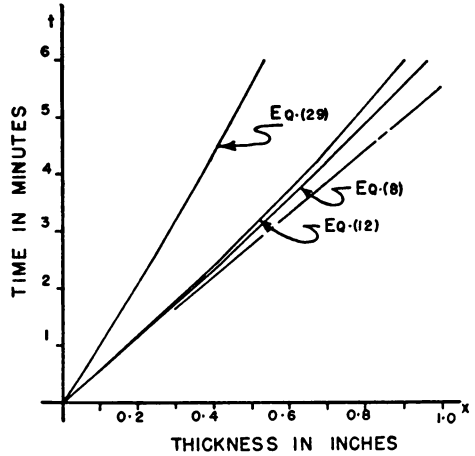

FIG. 2.

slab of thickness x at the critical temperature to recrystallize uniformly with a constant heat input of -117 BTU/(sq. ft.)(hr.).

According to (29), if t can be taken so small that all terms above the first are neglected, recrystallization cannot take place if $k_1\phi'(0) > k_2g(0)$. In fact, this condition would represent decrystallization and the curve would be in the negative x half plane.

BOOK REVIEWS

Electromagnetic theory. Proceedings of Symposia in Applied Mathematics, Volume II. American Mathematical Society, New York, 1950. iii + 91 pp. $3.00.

This volume contains the proceedings of the Second Symposium in Applied Mathematics, held at the Massachusetts Institute of Technology in July 1948. Of the seventeen papers included, ten are presented in the volume by abstract.

The subject matter covers a wide range under the general topic of Electromagnetic Theory including related quantum theory and statistics. The papers in full are: H. Feshbach, *The new quantum electrodynamics.* J. L. Synge, *Electromagnetism without metric.* W. H. Watson, *Discontinuity in electromagnetism.* L. Infeld, *The factorization method and its application to differential equations in theoretical physics.* R. J. Duffin, *Nonlinear electrical networks.* C. L. Pekeris, *Ray theory vs. normal mode theory in wave propagation problems.* A. E. Heins, *Systems of Wiener-Hopf integral equations and their application to some boundary value problems in electromagnetic theory.*

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