QUARTERLY OF APPLIED MATHEMATICS

Vol. VII JULY, 1949 No. 2

THE SPECTRUM OF FREQUENCY-MODULATED WAVES AFTER RECEPTION IN RANDOM NOISE-I*

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Part I: Introduction and Discussion

The reception of radio, radar, and television signals is always accomplished in the presence of noise, which exclusive of outside sources—either man-made or natural—arises because of the random motion of electrons in the various circuit elements of the receiver, principally in the stages preceding the detecting devices. The noise is accordingly assumed to be of the fluctuation type; impulse noise is not considered in this paper. Knowledge of the spectral distribution and the power output following demodulation of a frequency-modulated carrier in such noise is essential to any critical theory of receiver behavior. Accordingly, the principal purpose of the present work is to investigate the spectral aspects of the problem. One of the chief reasons for our interest in the spectrum lies in the fact that the signal-to-noise ratio at the output as a function of the ratio at the input depends markedly on the spectral distribution, in the so-called case of *broad-band FM* (where the maximum change in carrier frequency may be several times the highest modulating components). The dependence in the instance of *narrow-band FM* (for which the maximum deviation in the carrier frequency is comparable with or less than the highest significant audio modulating frequency) is not nearly so critical.

Previous investigations, of which the work of Carson and Fry,[1] Crosby,[2] Blachman,[3] and Rice[4] are particularly to be noted here,** have dealt chiefly with the strong-carrier case, in which the noise is largely suppressed and appears linearly with the signal in

*Received May 27, 1948. The research reported in this document was made possible through support extended Cruft Laboratory, Harvard University, jointly by the Navy Department (Office of Naval Research) and the Signal Corps, U. S. Army, under contract N5ORI-76, T.O.I. The author wishes to thank Miss Marilyn Lang of the Electronics Research Laboratory, who performed most of the calculations in this paper.

[1]J. R. Carson and T. C. Fry, *Variable frequency electric circuit theory with application to the theory of frequency modulation*, Bell Syst. Tech. J. **16**, 513 (1937).

[2]M. C. Crosby, *Frequency modulation noise characteristics*, Proc. I. R. E. **25**, 472 (1937).

[3]N. M. Blachman, *A theoretical study of the demodulation of a frequency-modulated carrier and random noise by an FM receiver*, Thesis (Harvard, June, 1947). Technical Report No. 31, (1948), Cruft Laboratory, Harvard University. See also N. M. Blackman, J. Appl. Phys. **20**, 38 (1949).

[4]S. O. Rice, *Statistical properties of a sine-wave and random noise*, Bell Syst. Tech. J. **27**, 109 (1948).

**An interesting paper by F. L. H. M. Stumpers (Proc. I.R.E. 36, 1080 (1948) has since appeared, giving some of the same general conclusions reached here in the case of extreme limiting. The case of no limiting, however, is not discussed.

the output. Rice's work[4] is an exception, in that he determines the spectrum for the case of arbitrary input carrier-to-noise power ratio (p), but only for the idealized example of extreme limiting (as do likewise Carson and Fry[1] and Crosby[2]). Further, the carrier is not modulated. Blachman's interesting treatment,[3] on the other hand, is more comprehensive, since he includes modulation and a discussion of the case of no limiting. Similarly, the present paper is more general than Rice's work in that modulation and no limiting are also considered, and is an extension of Blachman's efforts[5] since spectra (and their associated correlation functions) are obtained for all values of the ratio p.

We consider the case of no limiting because it gives us one extreme of receiver performance as a function of limiting; superlimiting gives us the other. Since the passage from the former to the latter is monotonic, we may be able roughly to estimate the behavior for intermediate degrees of limiting. The ability to handle all values of input carrier-to-noise power ratios is particularly useful when we desire the spectrum for threshold signals ($p \backsim 1$ or less), an important region when contrasting the maximum sensitivities of *AM* and *FM* reception. Specific calculations of spectra are made, which utilize a Gaussian input noise distribution. This spectral shape is usually a somewhat better fit to actual responses than the rectangular spectrum. The results of the analysis are given in detail in Part II, while Part III contains a treatment of important special cases: (1) noise alone, (2) unmodulated carrier and noise, and (3) a sinusoidally modulated carrier in noise.

The actual nonlinear or demodulating elements in an *FM* receiver are too involved to yield satisfactorily to direct treatment, but a possible model of these essential circuit elements—the limiter, followed by the discriminator—may be constructed if we assume that the physical discriminator is replaced by an "ideal" one which responds everywhere linearly with frequency (the "quasi-stationary" hypothesis of Carson and Fry). Thus, the output current (or voltage) is directly proportional to the instantaneous difference frequency between the wave and the central or resonant frequency of the *IF*, limiter, and discriminator bands (all tuned to the same frequency and assumed to be symmetrical). The filter characteristic of the limiter is taken to be wide enough to pass the *IF* portion of the limited signal and noise without distortion due to frequency selection. In practice, of course, the real discriminator saturates for sufficiently large excursions about resonance, but such behavior is rare if only an insignificant portion of the wave's energy is redistributed into higher harmonics due to the "spreading" action of the limiter.[6] This is assumed to be true here even for the extremes of superlimiting, in which the incoming disturbance is heavily clipped at top and bottom.

[5]Blachman has shown (ref. 3, ch. IV) that when there is no limiting, the characteristic function method used by Rice and others[8] in the solution of *AM* receiver problems may be adapted for all carrier strengths to the problem in which the idealized discriminator is replaced by one closer to actual electronic practice, namely, two diode rectifiers π radians out of phase with one another. Unfortunately, the method breaks down when there is limiting, since it does not seem practicable to determine, in any useful form, the distribution functions W_1 and W_2 of the clipped, *filtered* noise and signal waves leaving the limiter to enter a second nonlinear device of the above type. Kac and Siegert (J. Appl. Phys. **18**, 383 (1947)) have calculated the first-order distribution W_1, but it is the second-order density W_2 that is needed for the spectrum. (The analysis of the present paper has been extended by the author to include arbitrary limiting, with the idealized discriminator and similar assumptions as to the width of the limiter's filter response. See Technical Report No. 62, Cruft Laboratory, Harvard University, Nov. 24, 1948).

[6]D. Middleton, *The response of biased, saturated linear and quadratic rectifiers to random noise*, J. Appl. Phys. **17**, 778 (1946).

The wave leaving the IF and entering the limiter is narrow-band, that is to say, it may be represented analytically by

$$V(t) = R(t) \cos [\omega_0 t + \theta(t)], \tag{1.1}$$

where $f(=\omega_0/2\pi)$ is the central frequency of the IF, limiter, and discriminator, R is the envelope and θ a phase angle; both R and θ are slowly varying functions of time compared with $\omega_0 t$. If we let $f(iz)$ be the Fourier transform of the limiter's dynamic characteristic $g(V)$, i.e.

$$f(iz) = \int_0^\infty e^{-izV} g(V)\, dV, \tag{1.2}$$

the output of the limiter may be written[7,8]

$$V_0(t) = \frac{1}{2\pi} \int_{\mathbf{C}} f(iz) \exp [iz R \cos (\omega_0 t + \theta)]\, dz. \tag{1.3}$$

The contour $\mathbf{C}$ extends along the real axis from $-\infty$ to $+\infty$ and is indented downward about a possible singularity at the origin. With the help of the expansion

$$\exp (ia \cos \phi) = \sum_{n=0}^{\infty} i^n \epsilon_n J_n(a) \cos n\phi, \qquad \epsilon_0 = 1, \qquad \epsilon_{n\geq 1} = 2, \tag{1.4}$$

we obtain from (1.3)

$$V_0(t) = \sum_{n=0}^{\infty} B_n(R) \cos n(\omega_0 t + \theta), \tag{1.5}$$

where

$$B_n(R) = \frac{i^n \epsilon_n}{2\pi} \int_{\mathbf{C}} f(iz) J_n(Rz)\, dz. \tag{1.6}$$

The quantities $B_n(R)$ are the *envelopes* of the n $(=0, 1, 2, \cdots)$ spectral zones that are produced in the limiter by its nonlinear action;[7,8] $n\theta$ is the phase relative to $n\omega_0 t$, the n-th harmonic of the original central (angular) frequency. Only the band centered about $f_0(n = 1)$ is passed to enter the discriminator. The input wave is therefore

$$V_{i-d} = B_1(R) \cos (\omega_0 t + \theta). \tag{1.7}$$

In vector language, this input is represented by $\mathbf{V}_{i-d} = V\mathbf{e}^{i\Phi}$, where real quantities correspond to the x-components, and imaginary quantities to the y-components. Here $V = B_1(R)$ and $\Phi = \omega_0 t + \theta$. Then we have

$$\dot{V}_{i-d} = \dot{V}(\cos \Phi + i \sin \Phi) + V\dot{\Phi}(-\sin \Phi + i \cos \Phi) = \mathbf{r}\dot{V} + \boldsymbol{\theta} V\dot{\Phi},$$

since the polar unit vectors $(\mathbf{r}, \boldsymbol{\theta})$ are related to the rectangular ones by the transformation

$$\mathbf{r} = \cos \Phi + i \sin \Phi, \qquad \boldsymbol{\theta} = -\sin \Phi + i \cos \Phi;$$

$\dot{V}$ is the time-rate-of-change of the modulus, and $V\dot{\Phi}$ is the time-derivative of the

[7] S. O. Rice, *Mathematical analysis of random noise*, Bell Syst. Tech. J. **24**, 46 (1945).

[8] D. Middleton, *Some general results in the theory of noise through nonlinear devices*, Quart. Appl. Math. **5**, 445 (1948). (D. M. I.)

phase, or, what is the same thing, the instantaneous frequency.[9] The output E_0' of our idealized discriminator is precisely this second term, namely

$$E_0'(t) = \kappa V\dot{\Phi} = \kappa B_1(R)(\omega_0 + \dot{\theta}),$$

where κ is a constant of proportionality with the dimensions (secs). In practice the high frequency term, represented above as the coefficient of ω_0, is not passed by the discriminator filter, so that the final discriminator output is

$$E_0(t) = \kappa B_1(R)\dot{\theta} = \frac{i\dot{\theta}\kappa}{\pi}\int_C f(iz)J_1(Rz)\,dz. \tag{1.9}$$

Notice that the effect of the limiter is contained in the term $B_1(R)$, the exact form of which depends on our choice of $g(V)$, cf. (1.2). We assume that the limiter amplifies linearly (for both positive and negative amplitudes) up to a certain level, denoted by

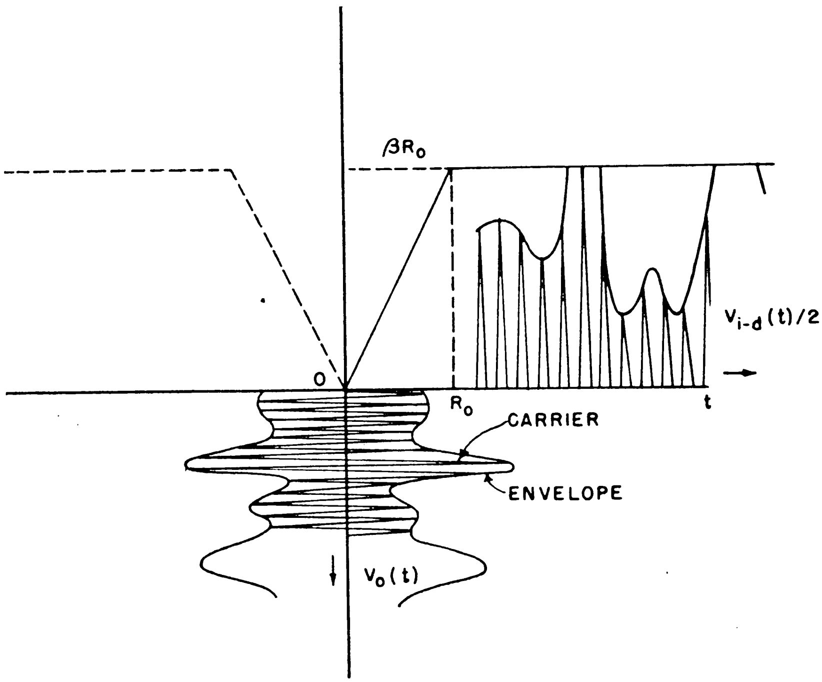

FIG. 1. A typical limiter characteristic.

R_0, beyond which point saturation occurs abruptly. The dynamic characteristic is shown schematically in Fig. 1. From (1.2) we find that here

$$f(iz) = 2\beta(1 - \exp[-iR_0z])/(iz)^2, \tag{1.10}$$

where β is the dynamic transfer constant of the limiter and the factor 2 is present be-

[9]B. van der Pol, *The fundamental principles of frequency modulation*, Jour. I. E. E. **93**, Part III, 153 (1946).

cause both the positive and negative portions contribute to the *envelope*. Accordingly, $B_1(R)$ in (1.9) is[10]

$$B_1(R) = -\frac{2\beta i}{\pi}\int_{\mathbf{C}}(1 - e^{-iR_0 z})J_1(Rz)\frac{dz}{z^2} = \beta R, \qquad 0 \le R \le R_0\,, \tag{1.11a}$$

$$\left.\begin{aligned} &= \frac{4}{\pi}\beta R_0\,{}_2F_1(-1/2,\,1/2;\,3/2;\,R_0^2/R^2), \\ &= \frac{2}{\pi}\beta R\left\{\frac{R_0}{R}\left(1 - \frac{R_0^2}{R^2}\right)^{1/2} + \sin^{-1}\frac{R_0}{R}\right\} \end{aligned}\right\} R_0 \le R. \tag{1.11b}$$

The two cases of chief interest are (a), *no limiting*, and (b), *superlimiting*. From (1.11a) and (1.11b) it is evident that $B_1(R) = \beta R$ (no limiting); and $B_1(R) \doteq (4/\pi)\beta R_0$ (superlimiting) in which the *rms* saturation level for the envelope is much less than the input noise level $(b_0)^{1/2}$, i.e. $R_0^2 \ll 2b_0$.

Although the spectral behavior of the discriminator output depends on a considerable number of parameters, chief among them (1) *carrier strength*, (2) *degree of limiting*, (3) *power and spectral distribution of the noise entering the demodulating elements*, (4) *deviation from resonance*, (5) *intensity and wave-form of the modulation*, etc., certain general observations can be made with regard to these factors and their effect on the power output and spectrum. We summarize the more significant features below, referring to the appropriate figures and equations, the analytical details of whose derivation are available in Parts II and III.

1. Signal Output (Power and Spectrum). Spectrally speaking, the modulation is reproduced without distortion (2.32), since our model of the discriminator responds linearly to all frequencies about resonance. In practice, this is only approximate, because of unavoidable nonlinearities which result in harmonics of the modulation in the low-frequency output. The effect is small, however, if the maximum deviation remains mostly in the linear portion of the discriminator characteristic.

Figure 2 illustrates the output signal power $R_0(0)_{(s\times s)}$ [Eq. (2.32)] as a function of the input carrier-to-noise power ratio p. A significant difference between the limiting and non-limiting conditions of receiver operation is immediately evident. For no limiting ($\lambda = 1$) and sufficiently large values of p ($p^2 \gg 1$) the output signal power is directly proportional to p, while for extreme limiting ($\lambda = 2$) saturation is observed: the output signal strength is independent ($\backsim p^0$) of the carrier. The amount of noise is assumed to be constant in both instances.

On the other hand, the threshold behavior for weak signals ($p \lesssim 1$) shows that the signal output of the discriminator is proportional to p^2, whether or not there is limiting. Thus, if the carrier is sufficiently weak, the limiter-discriminator combination behaves like any (half-wave) second detector acting on an AM wave,[11] as far as the dependence on carrier strength is concerned. Similar remarks apply for the *d-c* output (Fig. 2),

[10]The integration is readily accomplished by representing exp $(-iR_0z)$ as the sum of two Bessel functions and evaluating the resulting Weber-Schafheitlin integrals with the help of Eq. (2), p. 401, Watson, *Theory of Bessel functions* (Macmillan, 1945). See also D. M. I., Appendix III.

[11]D. Middleton, *Rectification of a sinusoidally modulated carrier in the presence of noise*, Proc. I. R. E., **36**, 1467 (1948).

provided that we replace the mean-square modulation by the square of the mean. Accordingly, when the modulation possesses no *d-c* component, there will be no *d-c* output, [cf. (2.36), (2.37)]. From these equations and from physical considerations it is also obvious that if the modulation is slow enough, $\dot{\eta} \doteq 0$, there will be no observable signal.

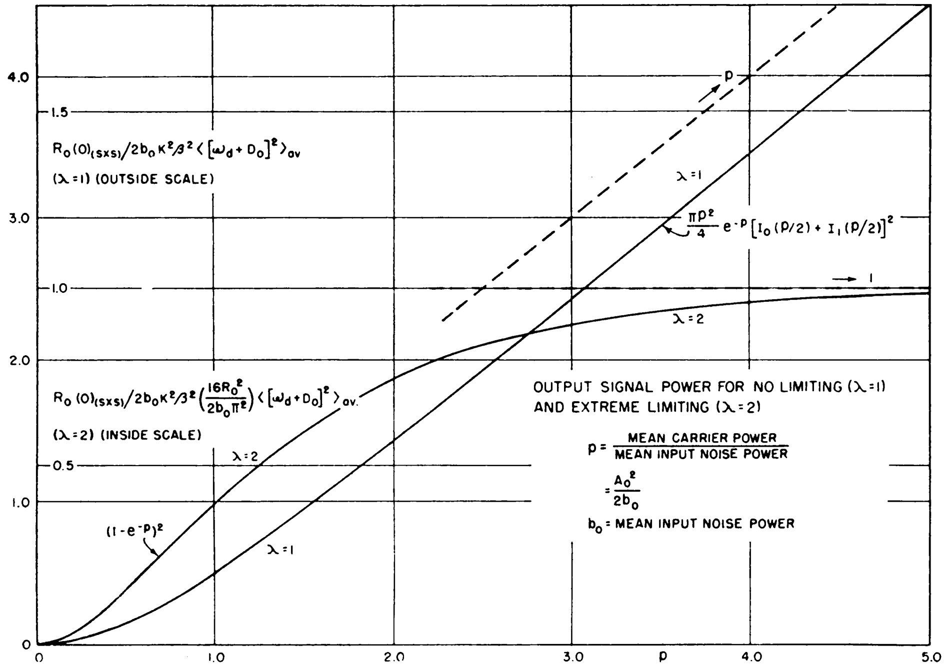

FIG. 2. Output signal power for no limiting ($\lambda = 1$) and extreme limiting ($\lambda = 2$).

2. Noise Output (Power). The low-frequency noise power from the discriminator is independent of the carrier when there is no limiting, provided the carrier is unmodulated [(Eq. (2.34)]. It is also true that the output does not depend on the modulating frequencies in either case, as can be verified from (2.34) and (2.35). However, unlike the analogous situation in *AM* reception[11] (mentioned above in connection with the signal) the magnitude of the noise power emanating from the discriminator *does* depend on the spectral distribution of the noise, i.e., on the *IF*-limiter-discriminator frequency response. The dependence is not very critical, covering a range of about 20-30 per cent between the extremes of rectangular and "optical," or single-tuned responses of the same energy (see Appendix V, and Table I of ref. 12).

Modulation suppression,[13] a phenomenon whereby the signal suppresses the noise when the signal is large enough, occurs in *FM* as well as in *AM* reception.[11] For no limiting ($\lambda = 1$) the low-frequency noise output is independent of carrier strength when $p^2 \gg 1$ and is still random noise, though mixed with components of the modulation [Eq. (2.40) et seq.]. A like behavior occurs in extreme limiting ($\lambda = 2$), except that

[12] D. Middleton, *Spurious signals caused by noise in triggered circuits*, J. Appl. Phys. **19**, 817 (1948).

[13] J. H. Van Vleck and D. Middleton, *A theoretical comparison of the visual, aural, and meter reception of pulsed signals in the presence of noise*, J. Appl. Phys. **17**, 940 (1946).

now the noise is absolutely, rather than relatively, suppressed, proportional to p^{-1}. The reverse situation arises when the carrier is weak relative to the noise. The noise is then dominant, and it is the signal which appears in the output as a perturbation of the noise. (See Sec. 1 above.)

3. Noise Output (Spectrum). It should be emphasized that discrimination, with or without limiting, is *not* a linear process, even though the noise output following it may be the linear sum of the effects of the transient pulses (and their derivatives) which make up the incoming noise wave. In general, from the spectral point of view the noise output after the nonlinear operations of limiting and discrimination will consist of three types of component, just as in conventional *AM* cases. These modulation products are produced by (a) beats between noise and noise (nxn), (b) beats between the signal and noise (sxn), and finally, (c) a signal contribution (sxs), due to products generated among the signal harmonics. For weak carriers the output spectrum is primarily determined by the distribution of the noise before discrimination (on the assumption of an ideal discriminator), while for strong carriers the spectrum of the discriminator output is not so heavily dependent on the spectral shape of the input, but is modified in a way that depends markedly on the degree of limiting. Limiting spreads the spectrum

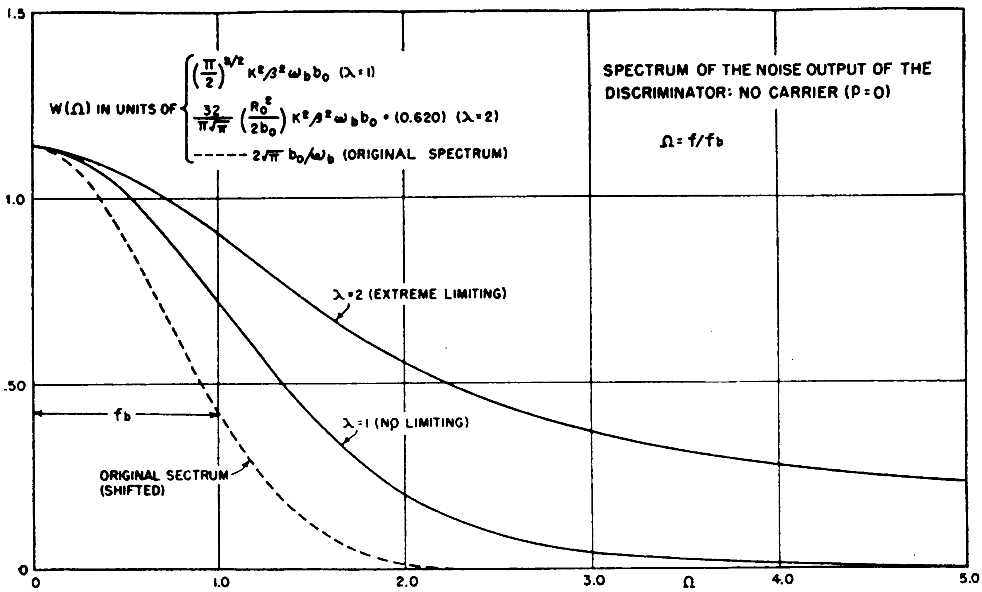

FIG. 3. Low-frequency noise spectrum following discrimination, no carrier.

by redistributing the energy into the higher harmonics of the distorted wave obtained in the clipping process.[14]

The significant feature of no limiting (strong carrier) is that the spectral intensity

[14]It has been implicitly assumed throughout that the limiter-discriminator filters are wide enough to pass the limited wave without appreciable distortion due to frequency selection. Thus, to consider spectra in the neighborhood of $\Omega = 5$, for example, is to assume these filters are at least five times wider than the *IF* band. The region of principal interest, however, is in most cases $\Omega < 2$, so that when there is limiting our model of a limiter-discriminator response several times as wide as the *IF* is not too far out of line with actual practice For no limiting this response need not be wider than the *IF*.

near the low-frequency end of the output does not as a rule vanish. For extreme limiting, however, the intensity is *always* very small near $f = 0$, unlike the analogous situation in AM reception. Moreover, the spectrum near $f = 0$ is proportional to f^2 and is roughy triangular, becoming more nearly rectangular as the carrier diminishes in power relative to the noise, in accordance with Crosby's results.[2] Consequently, in *broad-band FM* the amount of noise passed in the video or audio stages is noticeably reduced when there is heavy limiting as compared with the instance of no limiting (Part III). Compare the spectra in Figs. 4, 5 and 9, for example. This shows the necessity for limiting if there is to be an improvement (over AM, and FM without limiting) in the signal-to-noise ratio when the carrier is strong.[2,3] Limiting, however, is not mandatory for weak signals, to which FM receivers respond like AM receivers in their dependence on carrier strength (see Sec. 1 above). For narrow-band FM the dependence on spectral distribution is not critical, since it is the entire low-frequency spectrum, rather than a fraction of it, which is a measure of the interfering noise. Spectral shape is then unimportant. Figures 6-11 illustrate various other special cases.

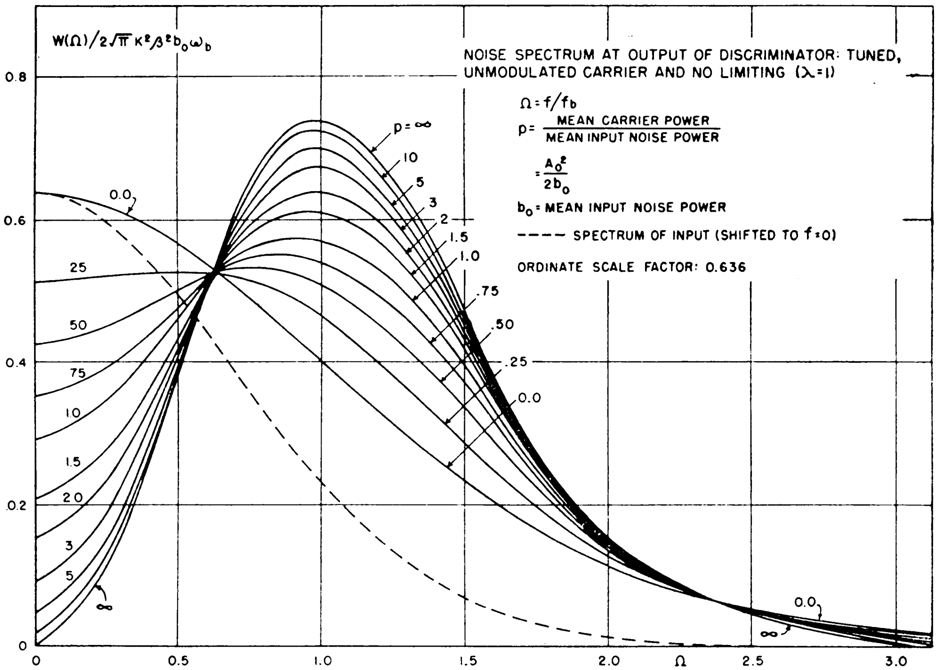

FIG. 4. Low-frequency noise spectrum following discrimination for a tuned, unmodulated carrier and no limiting.

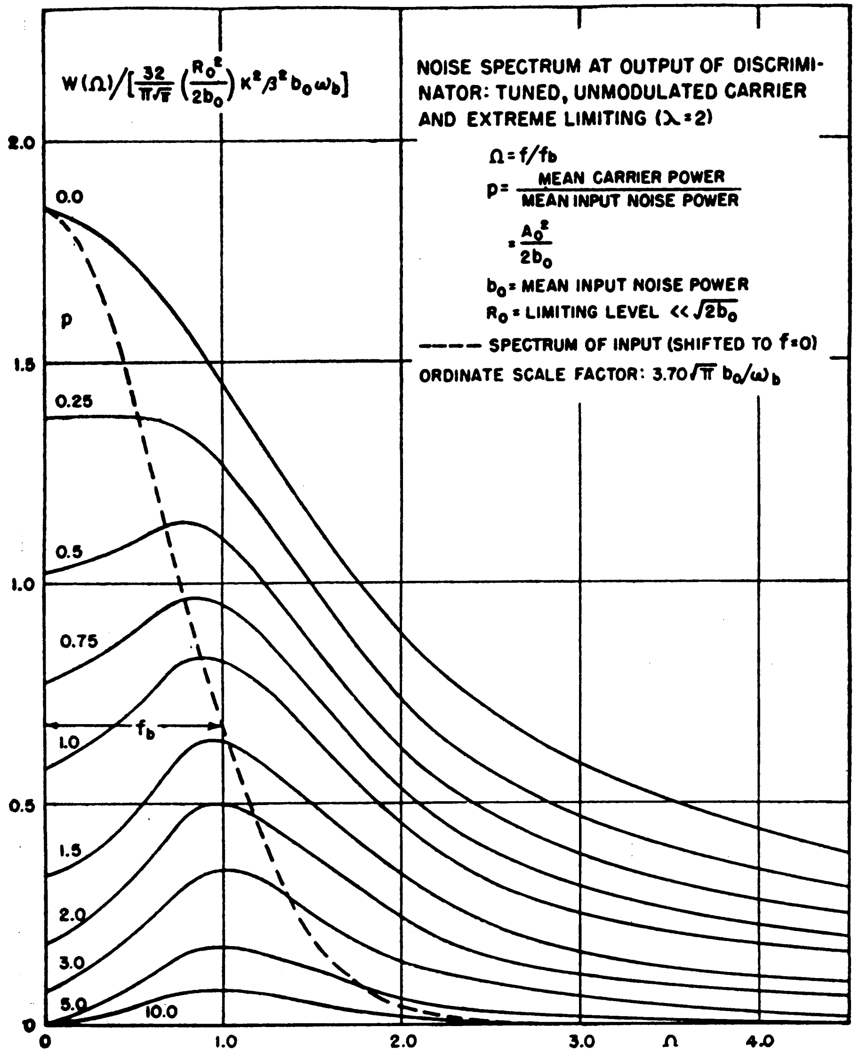

FIG. 5.* The same as Fig. 4, except now the limiting is extreme.

Correction added in proof: The ordinate scale factor in Fig. 5 is $2\pi^{1/2}b_0/1.85\omega_b = 1.08\,\pi^{1/2}b_0/\omega_b$.

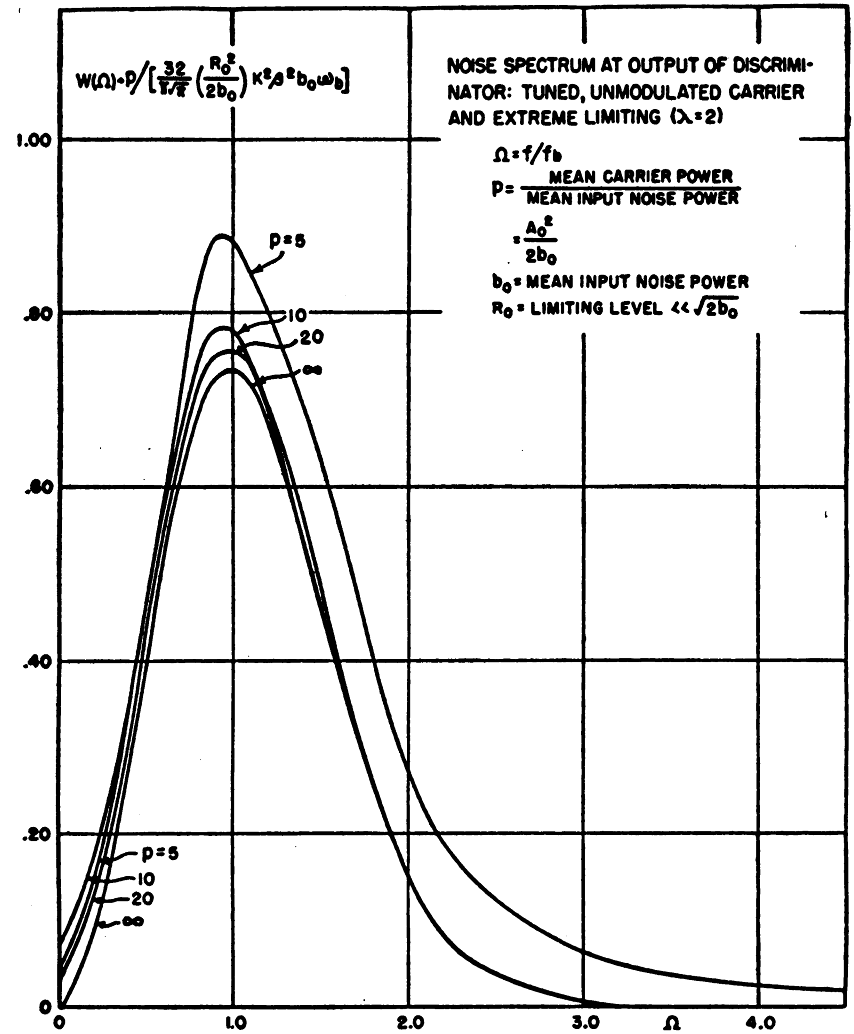

FIG. 6.* Figure 5, for large carrier amplitudes.

Correction added in proof: For $f < 0.3$, the curve for $p = 10$ in Fig. 6 should lie between the curves for $p = 5$ and $p = 20$.

Part II: Correlation Function of the Output (General Theory)

The correlation function of the low-frequency output of the discriminator is

$$R_0(t) = \overline{[E(t_0)E(t_0 + t)]_{\text{av.}}} = \overline{\kappa^2[B_1(R_1)B_1(R_2)\dot{\theta}_1\dot{\theta}_2]_{\text{av.}}} \,. \tag{2.1}$$

The bar indicates the statistical average over the random variables and $[\;]_{\text{av.}}$ denotes the average over the phases of the modulation, if any.[15] The mean power spectrum is given by the well-known Wiener[16]-Khintchine[17] relation

$$W(f) = 4\int_0^\infty R(t) \cos \omega t \, dt, \qquad \omega = 2\pi f, \tag{2.2}$$

with the transform

$$R(t) = \int_0^\infty W(f) \cos \omega t \, df. \tag{2.2a}$$

The input voltage to our limiter-discriminator combination may be written

$$V_0(t) = V_s + V_N = A_0 \cos (\omega_c t + \Psi) + \sum_{n=1}^{\infty} (a_n \cos \omega_n + b_n \sin \omega_n t), \tag{2.3}$$

where f_c is the carrier frequency (in the IF range) and

$$\Psi = \int^t D_0(t)\, dt, \qquad \text{with } D_0(t) \text{ the modulation.} \tag{2.3a}$$

The quantities a_n and b_n are random parameters with the properties

$$\overline{a_n} = \overline{b_m} = 0; \qquad \overline{a_n b_m} = 0; \qquad \overline{a_n a_m} = \overline{b_n b_m} = w(f_n)\Delta f \,\delta_m^n \,; \tag{2.4}$$

$w(f_n)\Delta f$ is the mean power dissipated in a unit resistance by the n-th component f_n ,[18] and δ_m^n is the familiar Kronecker delta, such that $\delta_m^n = 0$, $n \neq m$, $\delta_n^n = 1$. Here A_0 is a constant representing the peak amplitude of the frequency-modulated carrier. Since the disturbance is assumed to be narrow-band, let

$$\omega_n = \omega_0 + \omega_n' \qquad \text{and} \qquad \omega_c = \omega_0 + \omega_d \,, \tag{2.5}$$

where ω_0 is some constant (angular) frequency, in this case the resonant or central frequency of the IF-limiter-discriminator elements; ω_n' is then a frequency relative to ω_0 , and ω_d is simply the (fixed) deviation of the carrier from exact tuning ($\omega = \omega_0$). We have from (2.3) and (2.5)

$$V_N = \sum_{n=1}^{\infty} [a_n \cos (\omega_0 + \omega_n')t + b_n \sin (\omega_0 + \omega_n')t] = V_c \cos \omega_0 t - V_s \sin \omega_0 t, \tag{2.6}$$

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[15]The subscripts 1 and 2 on current or voltage amplitudes throughout refer to these amplitudes at the initial time $t_1 = t_0$ and a later time $t_2 = t_0 + t$($t \geq 0$), respectively.

[16]N. Wiener, Acta Math. **55**, 117 (1930).

[17]A. Khintchine, Math. Ann. **109**, 604 (1934).

[18]S. O. Rice, Bell Syst. Tech. J. **23**, 282 (1944) and M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945).

so that

$$V_c = \sum_{n=1}^{\infty} (a_n \cos \omega_n' t + b_n \sin \omega_n' t); \qquad V_s = \sum_{n=1}^{\infty} (a_n \sin \omega_n' t - b_n \cos \omega_n' t); \tag{2.7}$$

V_c and V_s are the portions of the slowly varying part of the wave that are in phase with $\cos \omega_0 t$ and $\sin \omega_0 t$, respectively. A similar treatment for the signal yields finally for the complete wave

$$\begin{aligned} V_0(t) &= [V_c + A_0 \cos (\omega_d t + \Psi)] \cos \omega_0 t - [V_s + A_0 \sin (\omega_d t + \Psi)] \sin \omega_0 t \\ &\equiv R \cos (\omega_0 t + \theta), \end{aligned} \tag{2.8}$$

so that

$$R = [(V_c + \alpha)^2 + (V_s + \beta)^2]^{1/2}, \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{V_s + \beta}{V_c + \alpha}\right) + n\pi \qquad n = 0, 1, 2, \cdots, \tag{2.9}$$

in which

$$\alpha = A_0 \cos (\omega_d t + \Psi) \qquad \text{and} \qquad \beta = A_0 \sin (\omega_d t + \Psi). \tag{2.9a}$$

From (2.9) we obtain

$$\dot\theta = [(V_c + \alpha)(\dot V_s + \dot\beta) - (V_s + \beta)(\dot V_c + \dot\alpha)]/R^2. \tag{2.10}$$

Consequently, the low-frequency output of our discriminator is [cf. (1.11)]

$$E_0(t) = \gamma_\lambda[(V_c + \alpha)(\dot V_s + \dot\beta) - (V_s + \beta)(\dot V_c + \dot\alpha)]/R^\lambda, \tag{2.11}$$

where $\gamma_\lambda = \kappa\beta$ for $\lambda = 1$ (no limiting), and $\gamma_\lambda \doteq 4\kappa\beta R_0/\pi$ for $\lambda = 2$ (superlimiting). The correlation function (2.1) is therefore given by the eightfold integral

$$\begin{aligned} R_0(t)_N = \gamma_\lambda^2 \int_{-\infty}^{\infty} dV_{c1} \int_{-\infty}^{\infty} dV_{c2} \cdots \int_{-\infty}^{\infty} d\dot V_{s2} W_2(V_{c1}, \cdots, \dot V_{s2}; t) \\ \times [(V_{c1} + \alpha_1)(\dot V_{c1} + \dot\beta) - (V_{s1} + \beta_1)(\dot V_{c1} + \dot\alpha_1)] \\ \times [(V_{c2} + \alpha_2)(\dot V_{s2} + \dot\beta_2) - (V_{s2} + \beta_2)(\dot V_{c2} + \dot\alpha_2)]/R_1^\lambda R_2^\lambda. \end{aligned} \tag{2.12}$$

The subscript N indicates that only the average over the random variables $V_{c1}, \cdots, \dot V_{s2}$ has been performed, and not that over the phases of the modulation. The complete correlation function requires the additional average. The quantity W_2 is the joint probability density of the eight random variables $V_{c1}, \cdots, \dot V_{s2}$.

The evaluation of (2.12) will require the Fourier transform of W_2, namely, its characteristic function $F_2(z_1, z_2, \cdots, z_8; t)$, which is specifically[19]

[19]The subscripts 1, 2, $\cdots$, 8 here clearly do not have the time significance of those on $V_{c1}, V_{c2}, \cdots$; cf. footnote 15.

$$F_2(z_1, z_2, \cdots, z_8; t) = [\exp\{iz_1V_{c1} + iz_2V_{c2} + iz_3V_{s1} + iz_4V_{s2} + iz_5\dot{V}_{c1}$$

$$+ iz_6\dot{V}_{c2} + iz_7\dot{V}_{s1} + iz_8\dot{V}_{s2}\}]\text{stat. av.}$$

$$= \exp\left\{-\frac{b_0}{2}(z_1^2 + z_2^2 + z_3^2 + z_4^2) - \frac{b_2}{2}(z_5^2 + z_6^2 + z_7^2 + z_8^2)\right. \tag{2.13}$$

$$- \phi_0(t)(z_1z_2 + z_3z_4) - \phi_1(t)(z_1z_6 - z_2z_5$$

$$\left. + z_3z_8 - z_4z_7) + \phi_2(t)(z_5z_6 + z_7z_8)\right\},$$

where

$$b_n = \int_0^\infty w(f)(\omega - \omega_0)^n\, df, \qquad \text{and} \qquad \phi_n(t) = \frac{\partial^n}{\partial t^n}\int_0^\infty w(f)\cos(\omega - \omega_0)t\, df. \tag{2.14}$$

Since $w(f)$ is the mean input power spectrum, determined by the shape of the IF filter response, b_0 is the mean input noise power and $\phi_0(t)$ is the correlation function associated with it, by virtue of (2.2a). Observe that $\phi_0(0) = b_0$; then, for later use we find it convenient to write $\phi_n(t)$ in semi-normalized form, viz.

$$\phi_n(t)/b_0 \equiv r_n(t), \qquad \text{and} \qquad r_n(t)_{\max} \geq r_{n-1}(t)_{\max} \geq \cdots \geq r_0(t)_{\max} = 1, \tag{2.15}$$

as is easily shown. The determination of W_2 and F_2 is given in Appendix I. Our result (2.13) is somewhat specialized in that the input spectrum $w(f)$ is assumed to be symmetrical about f_0.

Now, from the definition of the characteristic function and its Fourier transform relation with W_2, the probability density may be expressed as

$$W_2(V_{c1}, V_{c2}, \cdots, \dot{V}_{s2}; t) = (2\pi)^{-8}\int_{-\infty}^\infty dz_1$$

$$\cdots \int_{-\infty}^\infty dz_8 \exp\{-i(z_1V_{c1} + z_2V_{c2} + \cdots + z_8\dot{V}_{s2})\}F_2(z_1, z_2, \cdots, z_8; t). \tag{2.16}$$

The correlation function (2.12) is then

$$R_0(t)_N = \gamma_\lambda^2\{I_1^{(\lambda)} - I_2^{(\lambda)} - I_3^{(\lambda)} + I_4^{(\lambda)}\}, \tag{2.17}$$

where

$$I_j^{(\lambda)} = \int_{-\infty}^\infty dV_{c1} \cdots \int_{-\infty}^\infty d\dot{V}_{s2}(2\pi)^{-8}\int_{-\infty}^\infty dz_1$$

$$\cdots \int_{-\infty}^\infty dz_8F_2(z_1, \cdots, z_8; t)G_j(V_{c1}, \cdots, \dot{V}_{s2}) \tag{2.18}$$

$$\times \exp\{-i(z_1V_{c1} + \cdots + z_8\dot{V}_{s2})\}R_1^{-\lambda}R_2^{-\lambda} \qquad (j = 1, 2, 3, 4)$$

with

$$
\begin{aligned}
G_1 &= (V_{c1} + \alpha_1)(\dot{V}_{s1} + \dot{\beta}_1)(V_{c2} + \alpha_2)(\dot{V}_{s2} + \dot{\beta}_2),\\
G_2 &= (V_{c1} + \alpha_1)(\dot{V}_{s1} + \dot{\beta}_1)(V_{s2} + \beta_2)(\dot{V}_{c2} + \dot{\alpha}_2),\\
G_3 &= (V_{s1} + \beta_1)(\dot{V}_{c1} + \dot{\alpha}_1)(V_{c2} + \alpha_2)(\dot{V}_{s2} + \dot{\beta}_2),\\
G_4 &= (V_{s1} + \beta_1)(\dot{V}_{c1} + \dot{\alpha}_1)(V_{s2} + \beta_2)(\dot{V}_{c2} + \dot{\alpha}_2).
\end{aligned} \tag{2.19}
$$

At this point we follow a procedure suggested by some recent work of Rice.[4] We integrate first over the random variables V_{c1} and V_{s1}, V_{c2} and V_{s2}, taken in pairs, following which we perform the integration over the $\dot{V}$. We need the value of

$$
K_1 \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{x}{(x^2 + y^2)^{\mu}} \exp\{-ixz - iy\xi\},
$$

and

$$
K_2 \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{y}{(x^2 + y^2)^{\mu}} \exp\{-ixz - iy\xi\}. \tag{2.20}
$$

Transformation to polar coordinates and integration give finally

$$
K_1 = \frac{-2\pi i z}{\Gamma(\mu) 2^{2\mu-2}(z^2 + \xi^2)^{2-\mu}},
$$

and

$$
K_2 = \frac{-2\pi i \xi}{\Gamma(\mu) 2^{2\mu-2}(z^2 + \xi^2)^{2-\mu}}, \qquad 2 > R(2\mu - 2) > -1. \tag{2.21}
$$

For *superlimiting* λ is 2 and μ is unity, while in the other extreme of *no limiting* λ is 1 and μ is 1/2. For this latter case the integrals (2.20) do not converge, but the results (2.21) still apply, as Appendix II demonstrates. Now, letting $x = \alpha_1 + V_{c1}$, $y = \beta_1 + V_{s1}$, $\cdots$ and using (2.20), we obtain the twelve-fold integral

$$
\begin{aligned}
I_1^{(\lambda)} = &\int_{-\infty}^{\infty} d\dot{V}_{c1} \int_{-\infty}^{\infty} d\dot{V}_{c2} \cdots \int_{-\infty}^{\infty} d\dot{V}_{s2} (2\pi)^{-8} \int_{-\infty}^{\infty} dz_1 \\
&\cdots \int_{-\infty}^{\infty} dz_8 \exp\{-i(z_5 \dot{V}_{c1} + \cdots + z_8 \dot{V}_{s2})\} F_2(z_1, \cdots, z_8; t) \\
&\times \frac{(-2\pi i)^2 z_1 z_2 \exp\{i(z_1\alpha_1 + z_2\alpha_2 + z_3\beta_1 + z_4\beta_2)\}}{(z_1^2 + z_3^2)^{2-\lambda/2}(z_2^2 + z_4^2)^{2-\lambda/2}} (\dot{V}_{s1} + \dot{\beta}_1)(\dot{V}_{s2} + \dot{\beta}_2);
\end{aligned} \tag{2.22}
$$

$I_2^{(\lambda)}$, $I_3^{(\lambda)}$, and $I_4^{(\lambda)}$ are found from (2.22) if we replace $z_1 z_2(\dot{V}_{s1} + \dot{\beta}_1)(\dot{V}_{s2} + \dot{\beta}_2)$ by $z_1 z_4(\dot{V}_{s1} + \dot{\beta}_1)(\dot{V}_{c2} + \dot{\alpha}_2)$, $z_2 z_3(\dot{V}_{c1} + \dot{\alpha}_1)(\dot{V}_{s2} + \dot{\beta}_2)$, $z_3 z_4(\dot{V}_{c1} + \dot{\alpha}_1)(\dot{V}_{c2} + \dot{\alpha}_2)$ respectively. The integrations with respect to $\dot{V}_{c1}, \cdots, \dot{V}_{s2}$ are effected with the help of

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} dY \int_{-\infty}^{\infty} \exp\{-izY\} g(z)\, dz = g(0),
$$

and

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} Y\, dY \int_{-\infty}^{\infty} \exp\{-izY\} g(z)\, dz = -i\left[\frac{\partial g}{\partial z}\right]_{z=0}. \tag{2.23}
$$

The first relation follows from Fourier's integral theorem and the second also, on integration by parts; in both instances it is assumed that $g(\pm\infty)$ vanishes properly. Application of (2.23) in (2.22) for $I_1^{(\lambda)}$ yields the fourfold integral

$$I_1^{(\lambda)} = -(2\pi)^{-2}\int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_4$$

$$\times \frac{z_1 z_2\{\phi_1^2 z_3 z_4 - \phi_2 + i\dot{\beta}_1\phi_1 z_3 - i\dot{\beta}_2\phi_1 z_4 + \dot{\beta}_1\dot{\beta}_2\}}{(z_1^2+z_3^2)^{2-\lambda/2}(z_2^2+z_4^2)^{2-\lambda/2}} \tag{2.24}$$

$$\times \exp\left\{-\frac{b_0}{2}(z_1^2+z_2^2+z_3^2+z_4^2) - \phi_0(z_1z_2+z_3z_4) + i(\alpha_1z_1+\alpha_2z_2+\beta_1z_3+\beta_2z_4)\right\}.$$

The other $I_j^{(\lambda)}$ are found from (2.24) if we replace its numerator, excluding the exponential, with

$$z_1z_4\{\phi_1^2 z_1 z_4 + i\dot{\beta}_1\phi_1 z_1 - i\dot{\alpha}_2\phi_1 z_4 + \dot{\alpha}_2\dot{\beta}_1\},$$

$$z_2z_3\{\phi_1^2 z_2 z_3 + i\dot{\alpha}_1\phi_1 z_3 - i\dot{\beta}_2\phi_1 z_2 + \dot{\alpha}_1\dot{\beta}_2\},$$

and

$$z_3z_4\{(\phi_1^2 z_1 z_2 - \phi_2) + i\dot{\alpha}_1\phi_1 z_1 - i\dot{\alpha}_2\phi_1 z_2 + \dot{\alpha}_1\dot{\alpha}_2\},$$

respectively for $j = 2, 3, 4$. Returning to (2.17) with these expressions for $I_j^{(\lambda)}$, adding and collecting terms give us finally

$$R_0(t)_N = \frac{\gamma_\lambda^2}{4\pi^2}\int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_4\{\phi_2(t)(z_1z_2+z_3z_4)$$

$$+ \phi_1(t)^2(z_1z_4 - z_2z_3)^2 - i\phi_1(t)(z_1z_4 - z_2z_3)(\dot{\alpha}_1z_3 + \dot{\alpha}_2z_4 - \dot{\beta}_1z_1 - \dot{\beta}_2z_2)$$

$$- (\dot{\beta}_1\dot{\beta}_2z_1z_2 + \dot{\alpha}_1\dot{\alpha}_2z_3z_4 - \dot{\alpha}_2\dot{\beta}_1z_1z_4 - \dot{\alpha}_1\dot{\beta}_2z_2z_3)\}(z_1^2+z_3^2)^{\lambda/2-2}(z_2^2+z_4^2)^{\lambda/2-2} \tag{2.25}$$

$$\times \exp\left\{-\frac{1}{2}b_0^2(z_1^2+z_2^2+z_3^2+z_4^2) - \phi_0(t)(z_1z_2+z_3z_4) + i(\alpha_1z_1+\alpha_2z_2+\beta_1z_3+\beta_2z_4)\right\}.$$

At this point we make the obvious transformation to polar coordinates[20]

$$z_1 = \rho_1\cos\theta_1\,, \qquad z_2 = \rho_2\cos\theta_2\,,$$

$$z_3 = \rho_1\sin\theta_1\,, \qquad z_4 = \rho_2\sin\theta_2\,,$$

[20] When $\lambda = 2$ (superlimiting), we can follow an alternative method suggested by Rice[4] for the case $(\beta = 0, \alpha^{\cdot} = \beta^{\cdot} = 0)$ in which the denominator $[(z_1^2+z_3^2)(z_2^2+z_4^2)]^{2-\lambda/2}$ is expressed as a pair of infinite integrals in (say) u and v, whose integrands have the form $\exp[-ub_0(z_1^2+z_3^2)/2]$ and $\exp[-vb_0(z_2^2+z_4^2)/2]$. The integration over $z_1 \cdots z_4$ may be achieved with the help of a principal-axis transformation, leaving a sum of double integrals, each of which may be further reduced to a single one after suitable changes of variable. The final integration, however, cannot be effected in closed form, except in special cases. When there is no limiting ($\lambda = 1$), this technique fails, as it is not possible to eliminate the radicals $[(z_1^2+z_3^2)(z_2^2+z_4^2)]^{3/2}$.

for which the Jacobian is simply $\rho_1\rho_2$. From (2.3) and (2.3a) we may define a phase η by

$$\eta = \omega_d t + \Psi,$$

whence

$$\dot{\eta} = \omega_d + \dot{\Psi},$$

and
$$\eta_1 = \omega_d t_0 + \Psi_1, \qquad \eta_2 = \omega_d(t_0 + t) + \Psi_2, \qquad t_2 \geq t_1 = t_0, \tag{2.26}$$

and since $(\alpha_{1,2}^2 + \beta_{1,2}^2)^{1/2} = A_0$ and $(\dot{\alpha}_{1,2}^2 + \dot{\beta}_{1,2}^2)^{1/2} = \dot{\eta}_{1,2}A_0$, we can write the correlation function in its general polar form

$$R_0(t)_N = \gamma_\lambda^2 \int_0^\infty \rho_1^{\lambda-2} \exp(-b_0\rho_1^2/2)\, d\rho_1 \int_0^\infty \rho_2^{\lambda-2} \exp(-b_0\rho_2^2/2)\, d\rho_2 \int_0^{2\pi} \frac{d\theta_1}{2\pi}$$

$$\times \int_0^{2\pi} \frac{d\theta_2}{2\pi} \exp\{-\phi_0\rho_1\rho_2 \cos(\theta_2 - \theta_1) + iA_0\rho_1 \cos(\theta_1 - \eta_1) + iA_0\rho_2 \cos(\theta_2 - \eta_2)\}$$

$$\times \{\phi_2 \cos(\theta_2 - \theta_1) + \phi_1^2\rho_1\rho_2 \sin^2(\theta_2 - \theta_1) + iA_0\phi_1 \sin(\theta_2 - \theta_1)[\dot{\eta}_1\rho_1 \cos(\theta_1 - \eta_1)$$

$$+ \dot{\eta}_2\rho_2 \cos(\theta_2 - \eta_2)] - A_0^2\dot{\eta}_1\dot{\eta}_2 \cos(\theta_1 - \eta_1) \cos(\theta_2 - \eta_2)\}. \tag{2.27}$$

The integration over θ_1 and θ_2 is accomplished in straight-forward fashion with the aid of the Bessel expansions of the various trigonometric exponentials [cf. (1.4)] and the orthonormality relations for the sine and cosine. The details are available in Appendix III. The correlation function becomes finally

$$\begin{aligned} R_0(t)_N = \gamma_\lambda^2 \sum_{k=0}^\infty \sum_{m=0}^\infty r_0(t)^{2m+k} \Bigg\{ \Bigg[&-\phi_2(t) \frac{\epsilon_k}{2} \Big(H_{k,2m,k+1}^{(0)^2} \cos(k+1)(\eta_2 - \eta_1) \\ &+ H_{k,2m,|k-1|}^{(0)^2} \cos(k-1)(\eta_2 - \eta_1) \Big) \\ &+ \phi_1(t)^2 \frac{\epsilon_k}{2} \Big(H_{k,2m+1,k}^{(1)^2} \cos k(\eta_2 - \eta_1) - \frac{1}{2} H_{k,2m+1,k+2}^{(1)^2} \cos(k+2)(\eta_2 - \eta_1) \\ &- \frac{1}{2} H_{k,2m+1,|k-2|}^{(1)^2} \cos(k-2)(\eta_2 - \eta_1) \Big) \Bigg] \\ &+ (2b_0 p)^{1/2} \phi_1(t) \frac{(\dot{\eta}_1 + \dot{\eta}_2)}{2} \Big[(1 - \delta_0^k) H_{k,2m,k+1}^{(0)} (H_{k,2m+1,k+2}^{(1)} - H_{k,2m+1,k}^{(1)}) \\ &\times \sin(k+1)(\eta_2 - \eta_1) \\ &- H_{k,2m,|k-1|}^{(0)} (H_{k,2m+1,k}^{(1)} - H_{k,2m+1,|k-2|}^{(1)}) \sin|k-1|(\eta_2 - \eta_1) \Big] \\ &+ 2b_0 p \dot{\eta}_1\dot{\eta}_2 \cos k(\eta_2 - \eta_1) \Big[\delta_0^k H_{k,2m,k+1}^{(0)^2} + \frac{1}{2} (H_{k,2m,k+1}^{(0)} - H_{k,2m,|k-1|}^{(0)})^2 \Big] \Bigg\}, \end{aligned} \tag{2.28}$$

where $p = A_0^2/2b_0$ is the ratio of the input mean-square carrier to mean-square noise voltage. The amplitude functions $H^{(n)}_{k,2m+n,q}$, $q = |k-1|, k+1; q(n=0)$; $q = k, k+2, |k-2|, (n=1)$ are obtained from

$$H^{(n)}_{k,2m+n,q}(p;\lambda) = \frac{b_0^{(2m+k)/2}}{2^{(2m+k)/2}[m!(m+k)!]^{1/2}} \times \int_0^\infty \rho^{2m+n+k+\lambda-2} \exp[-b_0\rho^2/2] J_q(A_0\rho)\, d\rho$$

$$= \frac{\Gamma[(k+2m+n+q+\lambda-1)/2]p^{q/2}}{q!b_0^{(n+\lambda-1)/2}2^{(3-n-\lambda)/2}[m!(m+k)!]^{1/2}} \times {}_1F_1([k+2m+n+q+\lambda-1]/2; q+1; -p), \tag{2.29}$$

where k, m, n, q are integral and positive. The integration is performed with the help of[21]

$$\int_0^\infty J_\nu(ax)\exp(-b^2x^2)x^{\mu-1}\,dx = \frac{\Gamma[(\nu+\mu)/2]}{2b^\mu\,\Gamma(\nu+1)}\left(\frac{a}{2b}\right)^\nu {}_1F_1((\nu+\mu)/2; \nu+1; -a^2/4b^2), \tag{2.30}$$

$$R(\nu+\mu) > 0$$

in which ${}_1F_1$ is a confluent hypergeometric function. Recurrence relations for the various H's are available in Appendix IV.[22] The complete correlation function $R(t)$ follows after the average over the phases of the modulation has been taken, viz.

$$R_0(t) = \frac{1}{T_0}\int_0^{T_0} R_0(t)_N\, dt_{(mod)}\ . \tag{2.31}$$

The mean power spectrum may be obtained in the usual way [cf. (2.2)], by determining the Fourier transform of $R_0(t)$. Specific cases are considered in Part III.

As in the analogous situation for amplitude modulation,[11] the output will consist of three sets of contributions: (1) (nxn) noise generated by the beating of the input noise components with each other, (2) (sxn) noise produced by the cross-modulation of the signal and the noise, and finally, (3), the discrete signal components (sxs). In our expression (2.28) for the correlation function, only the quantities in the first bracket [ ] for which $q = 0$ [cf. (2.29)] represent (nxn) terms; the remaining ones of the first and all those in the second and third brackets [ ] represent (sxn) noise, except in the latter when $m = k = 0$; these are signal components:

$$R_0(t)_{(sxs)} = \gamma_\lambda^2 2b_0p[\dot\eta_1\dot\eta_2]_{av}.H^2_{001}$$
$$= 2\gamma_\lambda^2 b_0pH^2_{001}\{[\omega_d + D_0(t_0)][\omega_d + D_0(t_0+t)]\}_{av}. \tag{2.32}$$

[21]G. N. Watson, *Theory of bessel functions*, Cambridge University Press, 1945.

[22]In its present form the series (2.28), or its Fourier transform, does not converge very rapidly when $p < 1$, so that it is more convenient to expand the amplitude functions $H(p;\lambda)$ and collect coefficients of p^n ($n = 0, 1, 2, \cdots$) before summing. Only a few such terms are then necessary when p is small, say 0.3 or less; interpolation between these values and those for $p \geq 1$ yields values of the correlation function and spectrum for the intermediate signal-to-noise ratios: $0.3 \leq p \leq 1$.

from (2.3a) and (2.28). Observe that the modulation is received without distortion since our idealized discriminator responds linearly to all frequencies.

The mean low-frequency power, W_0, may also be found from the correlation function on setting $t = 0$, according to a well-known theorem (D.M.I.). Now $\phi_1(0)$ vanishes, [cf. (2.14)], and $\eta_2 - \eta_1 = 0$, so that (2.28) reduces to

$$\begin{aligned} W_0 = R_0(0) = \overline{[E_0(t_0)^2]}_{\text{av.}} = \gamma_\lambda^2 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \Big\{ -\phi_2(0) \frac{\epsilon_k}{2} \\ \times (H^2_{k,2m,k+1} + H^2_{k,2m,|k-1|}) + 2b_0 p\{[\omega_d + D_0(t_0)]^2\}_{\text{av.}} \\ \times \left(\delta_0^k H^2_{k,2m,k+1} + \frac{1}{2}(H_{k,2m,k+1} - H_{k,2m,|k-1|})^2 \right) \Big\}, \end{aligned} \tag{2.33}$$

of which $2b_0 p \gamma_\lambda^2 H^2_{001} \{[\omega_d + D_0(t_0)]^2\}_{\text{av.}}$ is the signal power; the rest is attributable to the noise. A direct calculation of $\overline{[E_0(t_0)^2]}_{\text{av.}} = K^2[B_1(R)^2 \dot{\theta}^2]_{\text{av.}}$ using the first-order distribution density $W_1(R, \dot{\theta})$ gives us[3,23] for no limiting,

$$(W_0)_{\lambda=1} = \kappa^2\beta^2\left\{ -\phi_2(0) + 2b_0 p\left(1 - \frac{1 - e^{-p}}{2p}\right)\{[\omega_d + D_0(t_0)]^2\}_{\text{av.}}\right\}, \tag{2.34}$$

and for superlimiting

$$\begin{aligned} (W_0)_{\lambda=2} \doteq \frac{16}{\pi^2}\left(\frac{R_0^2}{2b_0}\right)\beta^2\kappa^2 e^{-p}\Big\{ (-\phi_2(0) + b_0 p\{[\omega_d + D_0(t_0)]^2\}_{\text{av.}}) \\ \times \left(\frac{\pi^2}{16} - 0.3444 - \mathrm{Ei}\left(-\frac{R_0^2}{2b_0}\right)\right) + \sum_{n=0}^{\infty} [(2n+3)p^{n+2}\{[\omega_d + D_0(t_0)]^2\}_{\text{av.}} \\ - (n+2)\phi_2(0)p^{n+1}]/(n+1)(n+2)! \Big\}, \qquad R_0^2 \ll 2b_0, \end{aligned} \tag{2.35}$$

where $\mathrm{Ei}(-x) = -\int_x^\infty e^{-y}\, dy/y$. The exponential integral is tabulated in Jahnke and Emde's *Tables of Functions*,[24] (pp. 6-9). Note from (2.34) that for no carrier or for an unmodulated one that is on tune, i.e., $\omega_d = 0$, the mean noise power output ($\backsim -\phi_2(0)$) is independent of the carrier when there is no limiting. Then the spectral distribution of the noise, whatever its shape (for different values of p, $\omega_d = 0$) will always enclose the same area. Equation (2.35) shows us that such is not true when extreme limiting exists (except, of course when $p = 0$). It is also evident that the total output power is independent of the modulation frequencies, since $[D_0(t_0)^2]_{\text{av.}}$ does not depend on frequency.

The d-c output is easily found from our result (2.28) if we allow t to become infinite. There is then no correlation between the wave at time $t_1 = t_0$ and at a later time t. Since $\phi_n(t)$, $n \geq 0$, vanishes in the limit $t \to \infty$, only the term $m = k = 0$ independent of ϕ_1 and ϕ_2 will contribute to give us the square of the *d-c* component. Taking the square-root, for no limiting we have ($\lambda = 1$)

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[23] D. Middleton, *On theoretical signal-to-noise ratios in FM receivers: a comparison with amplitude modulation*, to appear in J. Appl. Phys.

[24] E. Jahnke and F. Emde, *Tables of functions*, Dover Press, New York, 1943.

$$\left.\begin{aligned}\overline{[E_0(t_0)]}_{\text{av.}} &= \kappa\beta b_0^{1/2}[\omega_d + D_0(t_0)]_{\text{av.}}(\pi/2)^{1/2}p_1F_1(1/2;\,2;\,-p),\\ &= \kappa\beta b_0^{1/2}[\omega_d + D_0(t_0)]_{\text{av.}}(\pi/2)^{1/2}pe^{-p/2}[I_0(p/2) + I_1(p/2)].\end{aligned}\right\} \tag{2.36}$$

in terms of modified Bessel functions (see Appendix IV). The superlimiting case ($\lambda = 2$) yields

$$\left.\begin{aligned}\overline{[E_0(t_0)]}_{\text{av.}} &\doteq \frac{4}{\pi}\,\kappa\beta R_0[\omega_d + D_0(t_0)]_{\text{av.}}p\cdot{}_1F_1(1;\,2;\,-p),\\ &\doteq \frac{4}{\pi}\,\kappa\beta R_0[\omega_d + D_0(t_0)]_{\text{av.}}(1 - e^{-p}).\end{aligned}\right\} \tag{2.37}$$

Both (2.36) and (2.37) are in precise agreement with previous results.[3,23] If $[\omega_d + D_0(t_0)]_{\text{av.}}$ vanishes, there is no steady output component. (Figure 2 also illustrates the *square* of $\overline{[E_0(t_0)]}_{\text{av.}}$, provided we replace $\{[\omega_d + D_0(t_0)]^2\}_{\text{av.}}$ by $[\omega_d + D_0(t_0)]^2_{\text{av.}}$.)

We are now ready to examine the two important limiting cases which occur when the incoming carrier is strong, or weak, relative to the *rms* (input) noise level. Let us consider first the case when the carrier is strong, i.e., $p^2 \gg 1$. The asymptotic expansion of ${}_1F_1$ (see D.M.I.) is needed here, namely,

$$\begin{aligned}{}_1F_1(\alpha;\,\beta;\,-x) \backsimeq \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\,x^{-\alpha}\Big[1 &+ \frac{\alpha(\alpha-\beta+1)}{x}\\ &+ \frac{\alpha(\alpha+1)(\alpha-\beta+1)(\alpha-\beta+2)}{2!x^2} + \cdots\Big].\end{aligned} \tag{2.38}$$

1. Strong Carrier. For large ratios (p) we find accordingly that

$$\begin{aligned}H^{(n)}_{k,2m+n,q}(p;\,\lambda) &\backsimeq \frac{\Gamma[(k+2m+n+q+\lambda-1)/2]p^{-(k+2m+n+\lambda-1)/2}}{b_0^{(n+\lambda-1)/2}2^{(3-n-\lambda)/2}[m!(m+k)!]^{1/2}}\\ &\times \frac{[1 + (k+2m+n+q+\lambda-1)(k+2m+n-q+\lambda-1)/4p + \cdots]}{\Gamma[(q+3-k-2m-n-\lambda)/2]}.\end{aligned} \tag{2.39}$$

The correlation function (2.28) becomes

$$\begin{aligned}R_0(t) \backsimeq \frac{\gamma_\lambda^2\Gamma(\lambda/2)^2p^{1-\lambda}}{b_0^{\lambda-1}2^{3-\lambda}\Gamma[(4-\lambda)/2]^2}\,&\{-\phi_2(t)[\cos(\eta_2-\eta_1)]_{\text{av.}}\\ &+ (\lambda-1)\phi_1(t)[(\dot\eta_1+\dot\eta_2)\sin(\eta_2-\eta_1)]_{\text{av.}}\\ &+ (\lambda-1)^2\phi_0(t)[\dot\eta_1\dot\eta_2\cos(\eta_2-\eta_1)]_{\text{av.}} + 2b_0p[\dot\eta_1\dot\eta_2]_{\text{av.}}\},\qquad p^2 \gg 1.\end{aligned} \tag{2.40}$$

The first three terms represent (sxn) noise, and the last term is the signal contribution (sxs); (nxn) noise is suppressed. Note that the contribution of the noise enters linearly in the correlation function: only ϕ_0 and its derivatives appear. This means that the output noise is still random in the "pure" sense in which it entered the limiter-discriminator elements. In other words, the effects of the transients which gave rise to the original noise in the *IF* still add linearly in the output. In spite of this, discrimination remains a nonlinear process, as the mixing of noise and signal indicates (Eq. 2.40). That the noise can still remain random in the present case is due to the fact that discrimination is (for our ideal discriminator) frequency selective and not amplitude selective: the noise loses its "randomness" (normal properties) only when subject to amplitude limitation, i.e., distortion and/or clipping.

From (2.2) and with the help of (2.14) we may write the output noise spectrum when there is no limiting as

$$\begin{aligned} W_0(f)_{\lambda=1} &\backsimeq -4\kappa^2\beta^2 \int_0^\infty \phi_2(t)[\cos(\eta_2 - \eta_1)]_{\text{av.}} \cos \omega t \, dt, \qquad p^2 \gg 1, \\ &\backsimeq 4\kappa^2\beta^2 \sum_{n=0}^\infty A_n \int_0^\infty \cos(n\omega_a + \omega_d)t \cos \omega t \, dt \int_0^\infty (\omega' - \omega_0)^2 \\ &\quad \times w(f') \cos(\omega' - \omega_0)t \, df', \qquad (f' = \omega'/2\pi), \end{aligned} \tag{2.41}$$

after expanding $[\cos(\eta_2 - \eta_1)]_{\text{av.}}$ in a Fourier series; here $f_a(=\omega_a/2\pi)$ is the fundamental frequency of the modulation. Integration using the Dirac delta-function[6-8] gives us

$$\begin{aligned} W_0(f)_{\lambda=1} &\backsimeq \kappa^2\beta^2 \sum_{n=0}^\infty A_n \int_0^\infty (\omega' - \omega_0)^2 w(f')[\delta(\omega' - \omega_0 - n\omega_a - \omega_d - \omega) \\ &\quad + \delta(\omega' - \omega_0 - n\omega_a - \omega_d + \omega)] \, df' \\ &\backsimeq \kappa^2\beta^2 \sum_{n=0}^\infty A_n[(n\omega_a + \omega_d + \omega)^2 w(\omega_0 + n\omega_a + \omega_d + \omega) \\ &\quad + (n\omega_a + \omega_d - \omega)^2 w(\omega_0 + n\omega_a + \omega_d - \omega)]. \end{aligned} \tag{2.42}$$

Observe that the spectral density at and near $f = 0$ is finite and nonvanishing, no matter what distribution the input spectrum w has.

The case of superlimiting ($\lambda = 2$) when the carrier is strong is particularly interesting. If we let $x = t_0$ and $y = t_0 + t$, the three noise terms in (2.40) may be written

$$\begin{aligned} R_0(t)_{\text{noise}(\lambda=2)} &\backsimeq \frac{16}{\pi^2} \kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\Bigg\{-\phi_2(t)\left[\cos\left(\omega_d t + \int_x^y D_0(t_0')\,dt'\right)\right]_{\text{av.}} \\ &\quad + \phi_1(t)\left\{[2\omega_d + D_0(x) + D_0(y)] \sin\left(\omega_d t + \int_x^y D_0(t')\,dt'\right)\right\}_{\text{av.}} \\ &\quad + \phi_0(t)\left\{[\omega_d + D_0(x)][\omega_d + D_0(y)] \cos\left(\omega_d t + \int_x^y D_0(t_0')\,dt'\right)\right\}_{\text{av.}}\Bigg\} \\ &\backsimeq \frac{16}{\pi^2} \kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\left\{\left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)\phi_0(t)\right. \\ &\quad \left. \times \int_0^{x_0} \cos\left[\omega_d t + \int_x^y D_0(t')\,dt'\right] dx/x_0\right\}, \end{aligned} \tag{2.43}$$

where $x_0(=2\pi/\omega_a)$ is the period of the modulation and $\phi_n(t) \equiv (\partial^n/\partial t^n)\phi_0(t)$. Expansion of the integrand in a Fourier series, followed by the average over x, gives

$$\frac{1}{x_0}\int_0^{x_0} \cos\left[\omega_d t + \int_x^{y=x+t} D_0(t')\,dt'\right] dx = \sum_{n=0}^{\infty} A_n \cos(n\omega_a + \omega_d)(y - x), \qquad y - x = t, \tag{2.44}$$

since the result must be even in t. The coefficients A_n are the same as those for the development in (2.41). Next, we substitute (2.44) into (2.43), which shows that

$$R_0(t)_{\mathrm{noise}(\lambda=2)} \backsimeq \frac{16}{\pi^2}\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\sum_{n=0}^{\infty} A_n[-\phi_2(t)\cos(n\omega_a + \omega_d)t + 2(n\omega_a + \omega_d)\phi_1(t)\sin(n\omega_a + \omega_d)t + (n\omega_a + \omega_d)^2\phi_0(t)\cos(n\omega_a + \omega_d)t]. \tag{2.45}$$

Use of the integral form of $\phi_n(t)$ gives us, finally, a relation similar to (2.41) for the noise spectrum:

$$W_0(f)_{\lambda=2} \backsimeq \frac{64}{\pi^2}\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\sum_{n=0}^{\infty} A_n\left\{\int_0^{\infty}\cos(n\omega_a + \omega_d)t\cos\omega t\,dt \times \int_0^{\infty}[(\omega' - \omega_0)^2 + (n\omega_a + \omega_d)^2]w(f')\cos(\omega' - \omega_0)t\,df' \right. \tag{2.46}$$
$$\left. -2\int_0^{\infty}(n\omega_a + \omega_d)\sin(n\omega_a + \omega_d)t\cos\omega t\,dt\int_0^{\infty}(\omega' - \omega_0)w(f')\sin(\omega' - \omega_0)t\,df'\right\},$$

$$W_0(f)_{\lambda=2} \backsimeq \frac{16}{\pi^2}\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\sum_{n=0}^{\infty} A_n\left\{\int_0^{\infty}[(\omega' - \omega_0)^2 + (n\omega_a + \omega_d)^2 - 2(n\omega_a + \omega_d) \times (\omega' - \omega_0)]w(f')[\delta(\omega' - \omega_0 - n\omega_a - \omega_d + \omega) + \delta(\omega' - \omega_0 - n\omega_a - \omega_d - \omega)]\,df'\right\}$$

which is

$$W_0(f)_{\lambda=2} \backsimeq \frac{16}{\pi^2}\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\sum_{n=0}^{\infty} A_n\omega^2[w(\omega_0 + n\omega_a + \omega_d + \omega) + w(\omega_0 + n\omega_a + \omega_d - \omega)], \qquad p^2 \gg 1. \tag{2.47}$$

The significant fact about (2.47) is that the spectral intensity for superlimiting always vanishes at $f = 0$ and is small in the vicinity of $f = 0$. Compared with our result for the unlimited case [cf. (2.42)], this shows why *for a sufficiently strong carrier, limiting gives a noticeable improvement over no limiting, provided we observe the signal component(s) in a band, near zero frequency, which is narrow relative to the width* ($\backsim f_b$) *of the (combined) IF-limiter-discriminator filter response*, i.e., $f \ll f_b$. This result is identical with Blachman's,[3] which was derived by a different method, and it agrees with the earlier work of Carson and Fry.[1]

From (2.40) we observe that for $p^2 \gg 1$ the noise is always suppressed, its contribution being proportional to $p^{1-\lambda}$ and that of the signal to $p^{2-\lambda}$. Accordingly, when there is no limiting, the noise power is constant and the output signal strength increases as p^1, whereas for extreme limiting, the noise actually vanishes as $p \to \infty$, and the signal reaches a limiting value independent of p. These comments are illustrated in Fig. 2. We notice also from (2.40) that when $\lambda = 1$, the noise power output is independent of modulation; this is not true for extreme limiting ($\lambda = 2$).

2. Weak Signals. When the signal is weak or, at most, of the same order of magnitude as the noise, the correlation function (2.28) is most conveniently described[22] by a power series in p; if we use (2.29) and omit terms in $p^n (n \geq 3)$, we obtain finally after considerable reduction the following result:

$$R_0(t) = \gamma_\lambda^2 \Gamma(\lambda/2)^2 2^{\lambda-3} b_0^{2-\lambda} \{G_0(t) + pG_1(t) + p^2[G_2(t) + 2\{\dot{\eta}_1\dot{\eta}_2\}_{av.}] + \cdots\} \tag{2.48}$$
$$p^3 \ll 1,$$

where

$$G_0(t) = (r_1^2 - r_0 r_2)\, {}_2F_1(\lambda/2, \lambda/2; 2; r_0^2), \tag{2.49a}$$

$$\begin{aligned} G_1(t) = {} & (r_1^2 - r_0 r_2)\Big\{-\lambda\, {}_2F_1(\lambda/2, \lambda/2 + 1; 2; r_0^2) \\ & + r_0 \frac{\lambda^2}{4}\, {}_2F_1(\lambda/2 + 1, \lambda/2 + 1; 3; r_0^2)[\cos(\eta_2 - \eta_1)]_{av.}\Big\} + \Big\{r_0[\dot{\eta}_1\dot{\eta}_2 \cos(\eta_2 - \eta_1)]_{av.} \\ & - r_1[(\dot{\eta}_1 + \dot{\eta}_2)\sin(\eta_2 - \eta_1)]_{av.} - r_2[\cos(\eta_2 - \eta_1)]_{av.}\Big\}\, {}_2F_1(\lambda/2, \lambda/2; 2; r_0^2), \end{aligned} \tag{2.49b}$$

and

$$\begin{aligned} G_2(t) = {} & (r_1^2 - r_0 r_2)\Big\{\frac{\lambda^2}{4}\, {}_2F_1(\lambda/2 + 1, \lambda/2 + 1; 2; r_0^2) + \lambda/4(\lambda/2 + 1)\cdot {}_2F_1(\lambda/2, \lambda/2 + 2; 2; r_0^2) \\ & - \frac{r_0\lambda^2}{4}(\lambda/2 + 1)\, {}_2F_1(\lambda/2 + 1, \lambda/2 + 2; 3; r_0^2)[\cos(\eta_2 - \eta_1)]_{av.} \\ & + \frac{\lambda^2}{16}[\cos 2(\eta_2 - \eta_1)]_{av.}[2\, {}_2F_1(\lambda/2 + 1, \lambda/2 + 1; 2; r_0^2) - {}_2F_1(\lambda/2 + 1, \\ & \lambda/2 + 1; 3; r_0^2)]\Big\} + {}_2F_1(\lambda/2, \lambda/2 + 1; 2; r_0^2)\Big\{-\frac{3\lambda}{2} r_0[\dot{\eta}_1\dot{\eta}_2 \cos(\eta_2 - \eta_1)]_{av.} \\ & + \lambda r_1[(\dot{\eta}_1 + \dot{\eta}_2)\sin(\eta_2 - \eta_1)]_{av.} + r_2\, \lambda/2[\cos(\eta_2 - \eta_1)]_{av.}\Big\} \\ & + {}_2F_1(\lambda/2 + 1, \lambda/2 + 1; 3; r_0^2)\Big\{\frac{-r_0 r_1 \lambda^2}{8}[(\dot{\eta}_1 + \dot{\eta}_2)\sin 2(\eta_2 - \eta_1)]_{av.} \\ & + \frac{r_0^2\lambda^2}{8}[\dot{\eta}_1\dot{\eta}_2 \cos 2(\eta_2 - \eta_1)]_{av.} - \frac{r_1^2\lambda^2}{8}[\cos 2(\eta_2 - \eta_1)]_{av.}\Big\} \\ & + 2[\dot{\eta}_1\dot{\eta}_2]_{av.}[{}_2F_1(\lambda/2, \lambda/2; 1; r_0^2) - 1]. \end{aligned} \tag{2.49c}$$

The reduction is accomplished with the aid of the recurrence relation

$$ {}_2F_1(a, b; c; x) - {}_2F_1(a, b; c+1; x) = \frac{abx}{c(c+1)} \, {}_2F_1(a+1, b+1; c+2; x). \tag{2.50} $$

We see at once from (2.48) and (2.49) that the output noise is no longer "pure," i.e., the transients producing the noise no longer add linearly in the output, unlike the case of a strong carrier and weak noise.

The first term in (2.48) represents the correlation function for the noise output that would exist in the absence of a carrier, and $2p^2[\dot{\eta}_1\dot{\eta}_2]_{av.}$ is the expression for the output signal or modulation. *In threshold perception* ($p \lesssim 1$) *the observed signal is suppressed by the noise and the output signal-to-noise (power) ratio is therefore directly proportional to* p^2 *(for the leading term), just as in the amplitude modulated cases.*[11,13*] This is true whether or not there is limiting.[25] Figure 2 shows the output signal power as a function of the input carrier-to-noise power.

Part III: Special Cases

1. Noise alone. The first important example occurs when the demodulated wave is narrow-band noise. Here $\eta_1 = \eta_2 = \dot{\eta}_1 = \dot{\eta}_2 = 0$ and $p = 0$ also, so that the correlation function (2.48) is simply.

$$ \begin{aligned} R_0(t) &= \gamma_\lambda^2(r_1^2 - r_0 r_2) b_0^{2-\lambda} 2^{\lambda-3} \sum_{m=0}^{\infty} \frac{r_0^{2m}\Gamma(m+\lambda/2)^2}{m!(m+1)!}, \\ &= \gamma_\lambda^2(r_1^2 - r_0 r_2) b_0^{2-\lambda} 2^{\lambda-3}\Gamma(\lambda/2)^2 \, {}_2F_1(\lambda/2, \lambda/2; 2; r_0^2). \end{aligned} \tag{3.1} $$

In the instance of no limiting we obtain

$$ \begin{aligned} R_0(t)_{\lambda=1} &= \frac{\pi}{4} \kappa^2\beta^2 b_0(r_1^2 - r_0 r_2) \, {}_2F_1(1/2, 1/2; 2; r_0^2), \\ &= \kappa^2\beta^2 \frac{b_0}{2} (r_1^2 - r_0 r_2) K(r_0^2), \end{aligned} \tag{3.2} $$

where $K(r_0^2)$ is an elliptic integral of the first kind, with modulus r_0. With the aid of

$$ {}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, $$

$$ R(\gamma) \neq 0, -1, -2, \qquad R(\gamma-\alpha-\beta) \neq 0, -1, -2, \cdots, $$

we find the mean low-frequency power to be

$$ (W_0)_{\lambda=1} = R_0(0)_{\lambda=1} = -\kappa^2\beta^2\phi_2(0) = \kappa^2\beta^2 \int_0^\infty (\omega - \omega_0)^2 w(f)\, df, \tag{3.3} $$

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*See, in particular, Sec. VII, Part I and Sec. II, Part II, ref. 13.

[25] See references 23 and 3 for a discussion of signal-to-noise ratios.

a special case of (2.33) or (2.34) when $p = 0$. The spectrum follows from (2.2) and (3.1); it is

$$W_0(f)_{\lambda=1} = \pi b_0 \kappa^2 \beta^2 \sum_{m=0}^{\infty} \frac{(1/2)_m^2}{m!(2)_m} \int_0^{\infty} [r_1(t)^2 - r_0(t) r_2(t)] r_0^{2m}(t) \cos \omega t \, dt. \tag{3.4}$$

At this point, and for all subsequent calculations, we assume a Gaussian spectral distribution for the incoming noise, representing the composite IF-limiter-discriminator frequency response, viz.,

$$w(f) = W_0 \exp \{-(\omega - \omega_0)^2/\omega_b^2\}, \qquad \text{and} \qquad b_0 = W_0 \omega_b / 2\pi^{1/2}, \tag{3.5}$$

where W_0 is the maximum spectral intensity of the noise entering the limiter. From (2.14) and (2.15) we find that

$$r_0(t) = \exp\{-\omega_b^2 t^2/4\}, \qquad r_1(t) = -\omega_b^2 t r_0(t)/2, \qquad r_2(t) = \omega_b^2(\omega_b^2 t^2 - 2) r_0(t)/4, \tag{3.6}$$

so that $(r_1^2 - r_0 r_2) = \omega_b^2 r_0(t)^2/2$. Similar relations for the "optical" or single-tuned characteristic and for rectangular spectra are discussed briefly in Appendix V. Substitution of (3.6) into (3.4) gives

$$W_0(f)_{\lambda=1} = \left(\frac{\pi}{2}\right)^{3/2} \kappa^2 \beta^2 \omega_b b_0 \sum_{m=0}^{\infty} \frac{(1/2)_m^2}{(m!)^2 (m+1)^{3/2}} \exp\{-\Omega^2/(2m+2)\}, \tag{3.7}$$
$$\text{where } \Omega \equiv \omega/\omega_b = f/f_b .$$

The distribution is shown in Fig. 3.

The correlation function for extreme limiting assumes the form

$$\begin{aligned} R_0(t)_{\lambda=2} &\doteq \frac{16}{\pi^2} \beta^2 \kappa^2 \left(\frac{R_0^2}{2b_0}\right) b_0 (r_1^2 - r_0 r_2) \, {}_2F_1(1, 1; 2; r_0^2) \\ &= \frac{-16}{\pi^2} \beta^2 \kappa^2 \left(\frac{R_0^2}{2b_0}\right) b_0 \left(\frac{r_1^2 - r_0 r_2}{r_0^2}\right) \log_e (1 - r_0^2). \end{aligned} \tag{3.8}$$

Although $R_0(0)_{(\lambda=2)}$ is logarithmically divergent, the mean (low-frequency) output power is actually not infinite, but zero, since (3.8) applies rigorously when $t \to 0$ only if $R_0^2/2b_0 \to 0$ also; in other words, the limiting is so extreme that essentially no low-frequency components are produced. All the energy goes into higher harmonics.[6] For computation of spectra, however, it is sufficient that $R_0^2/2b_0 \ll 1$.

We obtain the spectrum ($\lambda = 2$) at once from (3.1) or (3.8) and (2.2) for the Gaussian distribution (3.5). We have

$$W_0(f)_{\lambda=2} \doteq \frac{32}{\pi(2\pi)^{1/2}} \left(\frac{R_0^2}{2b_0}\right) \kappa^2 \beta^2 \omega_b b_0 \sum_{m=0}^{\infty} \frac{\exp\{-\Omega^2/(2m+2)\}}{(m+1)^{3/2}}. \tag{3.9}$$

Figure 3 also illustrates $W_0(f)_{\lambda=2}$ as a function of Ω. The spectral intensity at $\Omega = 0$ is

$$W_0(0)_{\lambda=2} \doteq \frac{32}{\pi(2\pi)^{1/2}} \left(\frac{R_0^2}{2b_0}\right) \beta^2 \kappa^2 \omega_b b_0 \zeta(3/2), \tag{3.10}$$

where ζ is Riemann's zeta-function[26] and $\zeta(3/2)$ has the value 2.612. Some details of the evaluation of (3.7) and (3.9) are available in Appendix V. From (A5-2) it also

[26]Jahnke and Emde, *loc. cit.*, pp. 269-273.

follows that when Ω becomes infinite, the spectral intensity (3.9) falls to zero as $W_0(f)_{\lambda=2} \simeq 16\beta^2\kappa^2\omega_b R_0^2/\pi\,\Omega(\Omega\to\infty)$.

2. Tuned, unmodulated carrier. The unmodulated carrier in noise is the next more complicated case. Now $\eta_1 = \eta_2 = 0$ and $\dot{\eta}_1 = \dot{\eta}_2 = 0$ also, provided the carrier is tuned to the center of the band, i.e., $\omega_d = 0$. The correlation function (2.28) then becomes

$$R_0(t) = \gamma_\lambda^2 \sum_{k=0}^{\infty} \frac{\epsilon_k}{2} \sum_{m=0}^{\infty} r_0(t)^{2m+k} \left\{ -\phi_2(t)(H^2_{k,2m,k+1} + H^2_{k,2m,|k-1|}) + \phi_1(t)^2 \left(H^2_{k,2m+1,k} - \frac{1}{2} H^2_{k,2m+1,k+2} - \frac{1}{2} H^2_{k,2m+1,|k-2|} \right) \right\} \tag{3.11}$$

with $p = A_0^2/2b_0$ as before. The power output associated with the low-frequency terms is obtained at once from (2.34) and (2.35), viz:

$$(W_0)_{\lambda=1} = -\kappa^2\beta^2\phi_2(0),$$
$$(W_0)_{\lambda=2} \doteq -\frac{16}{\pi^2}\left(\frac{R_0^2}{2b_0}\right)\beta^2\kappa^2\phi_2(0)e^{-p}\left\{\frac{\pi^2}{16} - Ei\left(-\frac{R_0^2}{2b_0}\right) - 0.3444 + \sum_{n=0}^{\infty} \frac{p^{n+1}}{(n+1)!(n+1)}\right\}. \tag{3.12}$$

Observe that the mean-square low-frequency output of the ideal discriminator is independent of carrier strength when there is no limiting. This is what one would expect, since the carrier is tuned to the central frequency of the symmetrical band, and its *d-c* therefore vanishes.

The mean power spectrum follows as before from (2.2) and (3.11). For the Gaussian input noise distribution it is

$$W_0(f) = \gamma_\lambda^2 2\pi^{1/2}\omega_b b_0 \left\{ \sum_{k=0}^{\infty} \frac{\epsilon_k}{2} \sum_{m=0}^{\infty} \left[\left(2m + k + \frac{2\Omega^2}{2m+k+2}\right)(H^2_{k,2m,k+1} + H^2_{k,2m,|k-1|}) \frac{\exp\{-\Omega^2/(2m+k+1)\}}{(2m+k+1)^{3/2}} + b_0\left(1 - \frac{2\Omega^2}{2m+k+2}\right) \times \left(H^2_{k,2m+1,k} - \frac{1}{2}H^2_{k,2m+1,k+2} - \frac{1}{2}H^2_{k,2m+1,|k-2|}\right) \frac{\exp\{-\Omega^2/(2m+k+2)\}}{(2m+k+2)^{3/2}} \right] \right\}. \tag{3.13}$$

The spectra for no limiting are shown in Fig. 4 for various values of p, and for superlimiting in Figs. 5 and 6. The significance of the results is discussed briefly in Part I.

When the carrier is large, the spectrum is found at once from (2.42) or (2.47). These results give the leading term, while a further use of the asymptotic development (2.39) gives the succeeding one. Here

$$A_n = \delta_n^0$$

and $$w(f) = 2b_0\omega_b^{-1}\pi^{1/2}\exp\{-(\omega-\omega_0)^2/\omega_b^2\},$$ so that

$$W_0(f)_{\lambda=1} \simeq 4\pi^{1/2}\kappa^2\beta^2 b_0\omega_b \left\{ \Omega^2\exp(-\Omega^2) - \frac{1}{2p}\left[\Omega^2\exp(-\Omega^2) - \frac{1}{4\cdot 2^{1/2}}(1+\Omega^2)\exp\{-\Omega^2/2\}\right] + \cdots \right\}, \quad p^2 \gg 1. \tag{3.14}$$

A similar procedure for superlimiting [cf. (2.47)] yields

$$W_0(f)_{\lambda=2} \simeq \frac{64}{\pi^{3/2}}\,\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)b_0\omega_b\,\frac{1}{p}\left\{\Omega^2\exp(-\Omega^2)+\frac{\exp\{-\Omega^2/2\}}{2^{3/2}p}+\cdots\right\}. \tag{3.15}$$

The mean output power corresponding to (3.12) when $p^2 \gg 1$ is simply $-\kappa^2\beta^2\phi_2(0)$ when $\lambda = 1$ and is

$$(W_0)_{\lambda=2} = \int_0^\infty W_0(f)_{\lambda=2}\,df \simeq \frac{8}{\pi^2}(\kappa\beta\omega_b)^2\left(\frac{R_0^2}{2b_0}\right)\frac{b_0}{p}\left\{1+\frac{2^{-7/4}}{p}+\cdots\right\}. \tag{3.16}$$

We notice that when there is extreme limiting ($R_0^2 \ll 2b_0$) and the carrier is strong ($p^2 \gg 1$), suppression of the noise occurs, analogous to the similar situation in the reception of amplitude-modulated waves.[11,13] The spectra for the strong carrier case are illustrated in Fig. 6.

The small-signal problem, on the other hand, gives us a different output spectrum. From (2.2), (2.48), and (2.49) the spectrum of the noise may be written

$$W_0(f) \doteq \gamma_\lambda^2\Gamma(\lambda/2)^2 2^{\lambda-1}b_0^{2-\lambda}\int_0^\infty [G_0(t)+pG_1(t)+p^2G_2(t)+\cdots]\cos\omega t\,dt, \tag{3.17}$$

which in the instance of the Gaussian distribution (3.5) becomes finally

$$\begin{aligned}
W_0(f) \doteq{}& W_0(f)_{\text{noise}}p^0 + \gamma_\lambda^2\Gamma(\lambda/2)^2\omega_b\pi^{1/2}2^{\lambda-2}b_0^{2-\lambda}\\
&\times\left\{p\left[\sum_{m=0}^\infty \frac{-\lambda(\lambda/2)_m(\lambda/2+1)_m\exp\{-\Omega^2/(2m+2)\}}{m!(m+1)!(2m+2)^{1/2}}\right.\right.\\
&+\frac{2(\lambda/2)^2(\lambda/2+1)_m^2\exp\{-\Omega^2/(2m+3)\}}{m!(m+2)!(2m+3)^{1/2}}\\
&-\frac{(\lambda/2)_m^2}{m!(m+1)!(2m+1)^{1/2}}\left[\left(1-\frac{2\Omega^2}{2m+1}\right)\frac{1}{(2m+1)}\right.\\
&\left.\left.-1\right]\exp\{-\Omega^2/(2m+1)\}\right]\\
&+p^2\left[\sum_{m=0}^\infty\left(\left[\frac{\lambda^2}{4}\left(\frac{\lambda}{2}+1\right)_m^2+\frac{\lambda}{4}\left(\frac{\lambda}{2}+1\right)\left(\frac{\lambda}{2}\right)_m\left(\frac{\lambda}{2}+2\right)_m\right.\right.\right.\\
&\left.\left.+\frac{1}{2}\left(\frac{\lambda}{2}\right)^2\left(\frac{\lambda}{2}+1\right)_m^2\left(\frac{m+1}{m+2}\right)\right]\\
&\times\frac{\exp\{-\Omega^2/(2m+2)\}}{m!(m+1)!(2m+2)^{1/2}}\\
&-2\frac{(\lambda/2)^2(\lambda/2+1)(\lambda/2+1)_m(\lambda/2+2)_m\exp\{-\Omega^2/(2m+3)\}}{m!(m+2)!(2m+3)^{1/2}}\\
&-\frac{(\lambda/2)(\lambda/2)_m(\lambda/2+1)_m}{m!(m+1)!(2m+1)^{1/2}}\left[\left(1-\frac{2\Omega^2}{2m+1}\right)\frac{1}{(2m+1)}-1\right]\exp\{-\Omega^2/(2m+1)\}\\
&\left.\left.-\frac{(\lambda/2)^2(\lambda/2+1)_m^2}{m!(m+2)!(2m+2)^{3/2}}\left(1-\frac{\Omega^2}{m+1}\right)\exp\{-\Omega^2/(2m+2)\}\right]+\cdots\right\}.
\end{aligned} \tag{3.18}$$

The curves of Figs. 4 and 5 for $p < 0.3$ have been calculated with the help of (3.18).[22]

3. Off-tune, unmodulated carrier. This is a generalization of the preceding case to include the effects of a constant deviation of the carrier from the IF-limiter-discriminator resonant frequency f_0. It is now evident that $\eta_1 = \omega_d t_0$, $\eta_2 = \omega_d(t_0 + t)$, and so $\dot{\eta}_1 = \dot{\eta}_2 = \omega_d$. The correlation function (2.28) reduces to

$$R_0(t) = \gamma_\lambda^2 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} r_0^{2m+k} \Big\{ -\phi_2 \frac{\epsilon_k}{2} \Big(H_{k,2m,k+1}^2 \cos (k + 1)\omega_d t$$

$$+ H_{k,2m,|k-1|}^2 \cos (k - 1)\omega_d t \Big)$$

$$+ \phi_1^2 \frac{\epsilon_k}{2} \Big(H_{k,2m+1,k}^2 \cos k\omega_d t - \frac{1}{2} H_{k,2m+1,k+2}^2 \cos (k + 2)\omega_d t$$

$$- \frac{1}{2} H_{k,2m+1,|k-2|}^2 \cos (k - 2)\omega_d t \Big) \tag{3.19}$$

$$+ (2b_0 p)^{1/2} \omega_d \phi_1 \Big[(1 - \delta_0^k) H_{k,2m,k+1} (H_{k,2m+1,k+2} - H_{k,m+1,k}) \sin (k + 1)\omega_d t$$

$$- H_{k,2m,|k-1|} (H_{k,2m+1,k} - H_{k,2m+1,|k-1|}) \sin |k - 1| \omega_d t \Big]$$

$$+ 2b_0 \omega_d^2 p \cos k\omega_d t \Big[\delta_0^k H_{k,2m,k+1}^2 + \frac{1}{2} (H_{k,2m,k+1} - H_{k,2m,|k-1|})^2 \Big] \Big\}.$$

The effects of the carrier with constant deviation are discussed in Part I. The contribution of the signal component is illustrated in Fig. 2, where it is seen (from the ordinate scale) that the signal power varies as the square of the deviation f_d. The general determination of the noise spectrum may be made in the usual way with the aid of (2.2) and (3.5).

The example of a strong carrier is instructive. Then, with the Gaussian distribution as a particular example, we see from (2.42) and (2.47) that when there is no limiting the output spectrum is

$$W_0(f)_{\lambda=1} \backsimeq 2\pi^{1/2} \kappa^2 \beta^2 \omega_b b_0 [(\Omega_d + \Omega)^2 \exp \{-(\Omega_d + \Omega)^2\}$$
$$+ (\Omega_d - \Omega)^2 \exp \{-(\Omega_d - \Omega)^2\}], \qquad \Omega_d \equiv \frac{\omega_d}{\omega_b}, \tag{3.20}$$

since here the Fourier development of $[\cos (\eta_2 - \eta_1)]_{\mathrm{av.}}$ is simply $A_n \cos \omega_d t$, $A_n = \delta_n^0$. The mean output power is $(\omega_b \kappa \beta)^2 b_0/2$, which is proportional to $-\phi_2(0)$, and is independent of modulation, which in this case is the deviation of the carrier from resonance. This can be observed at once from (2.40) or directly by integration of (3.20). Figure 7 shows the spectrum (3.20) for various ratios Ω_d.

The spectrum after extreme limiting is found in a similar manner to be

$$W_0(f)_{\lambda=2} \backsimeq \frac{32}{\pi^{3/2}} \kappa^2 \beta^2 \Big(\frac{R_0^2}{2b_0}\Big) \omega_b b_0 p^{-1} [\Omega^2 \exp \{-(\Omega_d + \Omega)^2\} + \Omega^2 \exp \{-(\Omega_d - \Omega)^2\}],$$
$$p^2 \gg 1. \tag{3.21}$$

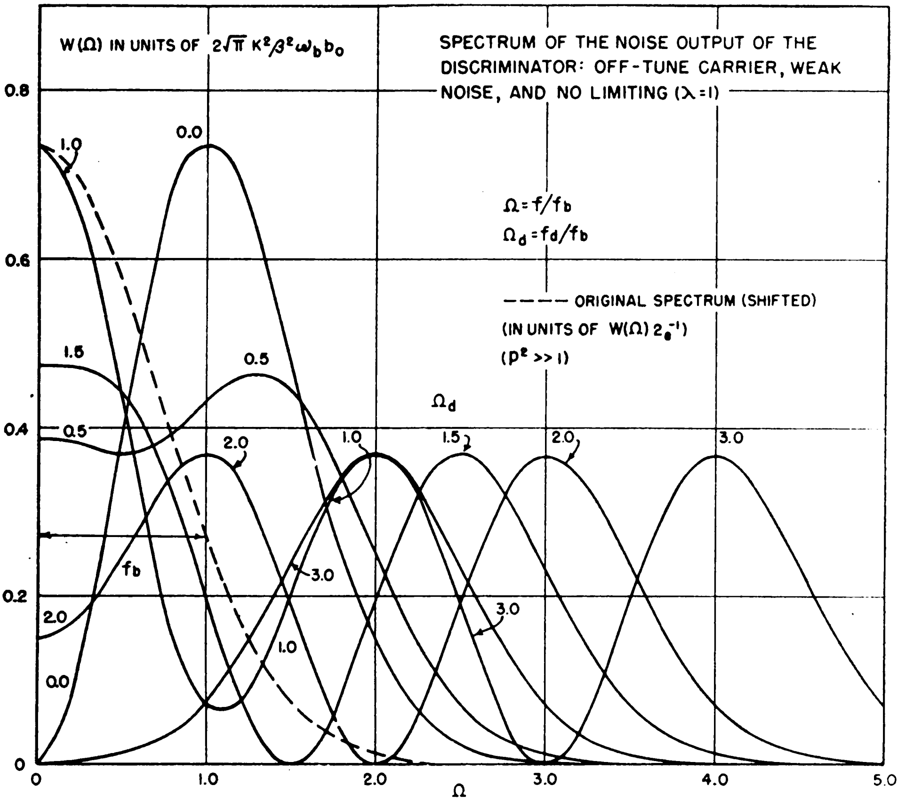

FIG. 7. Low-frequency noise spectrum following discrimination, for a weak noise and an off-tune carrier and no limiting.

Figure 8 illustrates the spectrum in this instance. The mean power output of the discriminator may be obtained from

$$(W_0)_{\lambda=2} \backsimeq \frac{32}{\pi^{3/2}}\,\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)\omega_b b_0 p^{-1}\int_0^\infty \Omega^2[\exp\{-(\Omega_d+\Omega)^2\}+\exp\{-(\Omega_d-\Omega)^2\}]\,d\Omega\frac{\omega_b}{2\pi}$$

$$\backsimeq \frac{16}{\pi^2}(\kappa\beta\omega_b)^2\left(\frac{R_0^2}{2b_0}\right)\frac{b_0}{p}\left(\Omega_d^2+\frac{1}{2}\right), \tag{3.22}$$

showing that the noise depends on Ω_d^2, as can be seen from Fig. 8.

The distribution in the neighborhood of $f = 0$ is easily determined with the help of Taylor's series

$$\exp\{-(\Omega_d \pm \Omega)^2\} = (2\pi)^{1/2}\sum_{l=0}^{\infty}(\pm1)^l\frac{(2\Omega^2)^{l/2}}{l!}\phi^{(l)}(2^{1/2}\Omega_d);$$

$$\phi^{(l)} \equiv \frac{1}{(2\pi)^{1/2}}\frac{d^l}{d(2^{1/2}\Omega_d)^l}(\exp\{-\Omega_d^2\}), \tag{3.23}$$

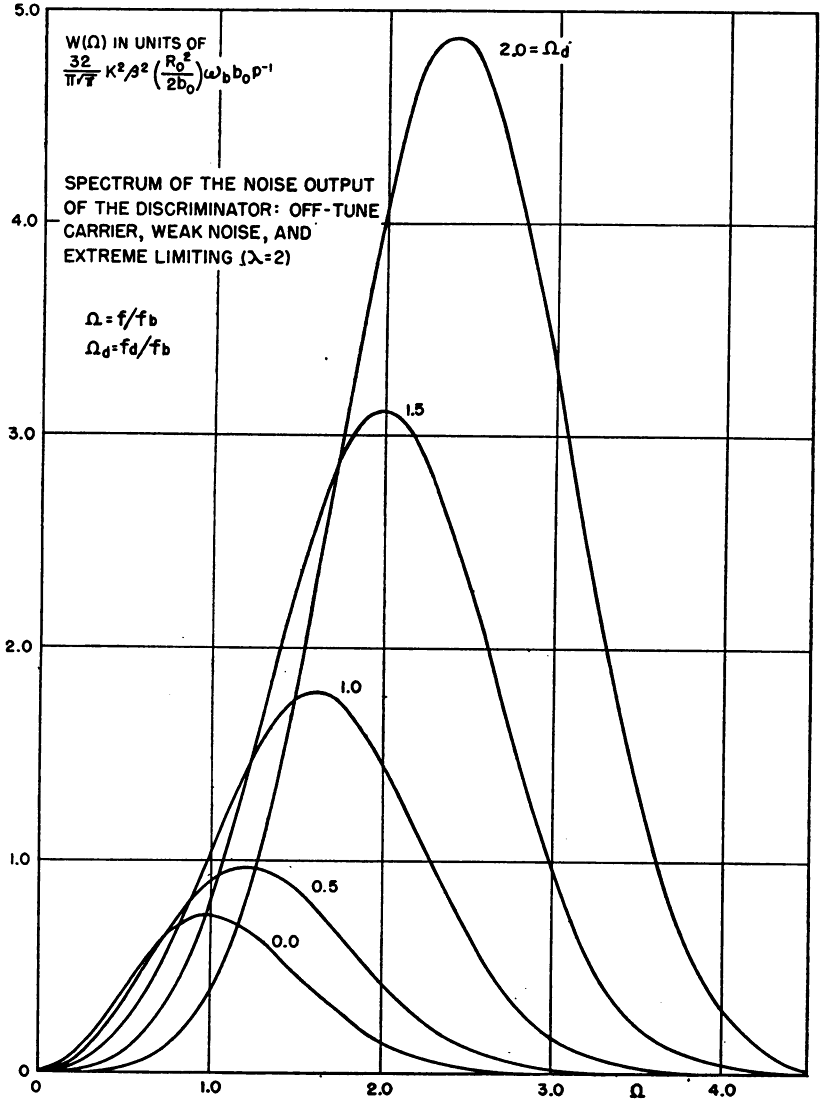

FIG. 8. The same as Fig. 7, but for extreme limiting.

a development which is valid for *all* Ω and Ω_d. Equations (3.20) and (3.21) are accordingly modified to

$$W_0(f)_{\lambda=1} \simeq 4\pi^{1/2}\kappa^2\beta^2\omega_b b_0\{\Omega_d^2 - 2\Omega\Omega_d^3 + \Omega^2(1 - 5\Omega_d^2 + 2\Omega_d^4) + \cdots\} \exp\{-\Omega_d^2\}, \tag{3.24}$$

$$\Omega^3 \ll 1$$

and

$$W_0(f)_{\lambda=2} \simeq \frac{64}{\pi^{3/2}} \kappa^2\beta^2\omega_b b_0 p^{-1}\left(\frac{R_0^2}{2b_0}\right)\{\Omega^2 + \cdots\} \exp\{-\Omega_d^2\}. \tag{3.25}$$

A comparison of (3.24) and (3.25) bears out the general observation made following Eq. (2.47).

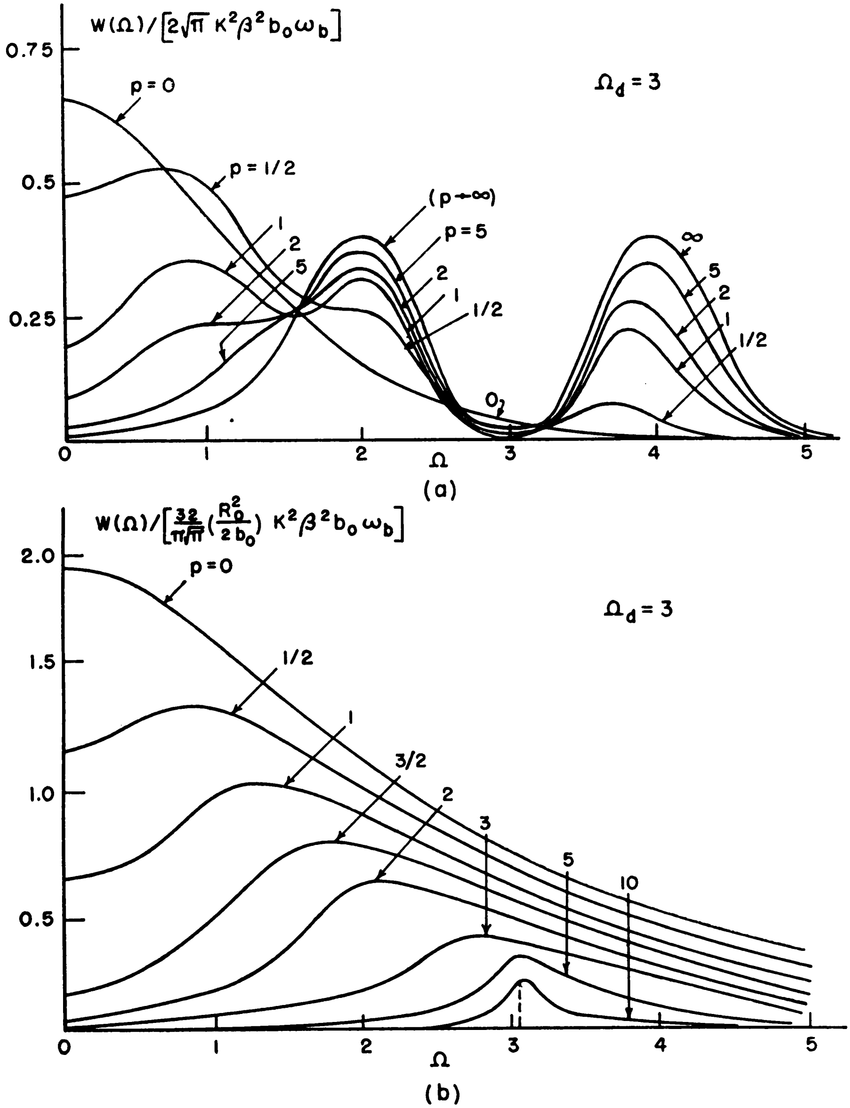

FIG. 9. Typical low-frequency spectra for an off-tune carrier: (a) no limiting, (b) extreme limiting.

4. On- and off-tune carriers modulated by a sine wave. In this case we have

$$\dot{\eta} = \omega_d + D_0 \cos \omega_a t, \quad \text{and} \quad \therefore \ \eta = \omega_d t + \mu \sin \omega_a t; \quad \mu \equiv \frac{D_0}{\omega_a} = \left(\frac{D_0}{\omega_b}\right) \cdot \frac{1}{\Omega_a}. \tag{3.26}$$

Here μ is the modulation index, and D_0 is the maximum deviation of the modulation, while ω_a is its angular frequency. (For such modulations the inequalities $D_0 \ll \omega_0$,

$\omega_a \ll \omega_0$ must hold.)[9] Referring to the general correlation function (2.28) we see that the modulation enters by virtue of the coefficients

$$[\cos l(\eta_2 - \eta_1)]_{\text{av.}}\,, \qquad [(\dot{\eta}_1 + \dot{\eta}_2) \sin l(\eta_2 - \eta_1)]_{\text{av.}}\,,$$

and

$$[\dot{\eta}_1\dot{\eta}_2 \cos l(\eta_2 - \eta_1)]_{\text{av.}}\,.$$

A straightforward application of the techniques of Appendix III allows us to write these averages as

$$[\cos l(\eta_2 - \eta_1)]_{\text{av.}} = \sum_{n=0}^{\infty} \epsilon_n J_n(l\mu)^2 \cos (n\omega_a + l\omega_d)t, \tag{3.27}$$

$$\begin{aligned}[(\dot{\eta}_1 + \dot{\eta}_2) \sin l(\eta_2 - \eta_1)]_{\text{av.}} = \sum_{n=0}^{\infty} \epsilon_n J_n(l\mu)\Big\{\frac{D_0}{2} J_{n-1}(l\mu) \sin [(n-1)\omega_a + l\omega_d]t \\ + \Big[\omega_d J_n(l\mu) + \frac{D_0}{2}(J_{n+1}(l\mu) + J_{n-1}(l\mu))\Big] \sin (n\omega_a + l\omega_d)t \\ + \frac{D_0}{2} J_{n+1}(l\mu) \sin [(n+1)\omega_a + l\omega_d]t\Big\},\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}[\dot{\eta}_1\dot{\eta}_2 \cos l(\eta_2 - \eta_1)]_{\text{av.}} = \sum_{n=0}^{\infty} \epsilon_n J_n(l\mu)\Big\{\Big[\frac{D_0\omega_d}{2} J_{n-1}(l\mu) + \frac{D_0^2}{4} J_n(l\mu)\Big] \\ \times \cos [(n-1)\omega_a + l\omega_d]t \\ + \Big[\omega_d^2 J_n(l\mu) + \frac{D_0\omega_d}{2}(J_{n+1}(l\mu) + J_{n-1}(l\mu)) + \frac{D_0^2}{4}(J_{n+2}(l\mu) \\ + J_{n-2}(l\mu))\Big] \cos (n\omega_a + l\omega_d)t \\ + \Big[\frac{D_0\omega_d}{2} J_{n+1}(l\mu) + \frac{D_0^2}{4} J_n(l\mu)\Big] \cos [(n+1)\omega_a + l\omega_d]t\Big\}.\end{aligned} \tag{3.29}$$

When the carrier is tuned to resonance, ω_d vanishes and (3.27) to (3.29) simplify considerably. However, a precise determination of the spectrum in either instance ($\omega_d = 0$, or $\omega_d \neq 0$) is a very tedious process. Fortunately, for many purposes the qualitative and semiquantitative conclusions obtained from the limiting behavior of the general case are sufficient; (see Part I).

We note that for strong carriers ($p^2 \gg 1$) the output spectrum may be found as before from (2.42) or (2.47). Equation (3.26) shows that the Fourier expansion of $[\cos (\eta_2 - \eta_1)]_{\text{av.}}$ is $A_n \cos (n\omega_a + \omega_d)t$, where $A_n = \epsilon_n J_n(\mu)^2$, so that for no limiting and our usual Gaussian noise distribution we find

$$\begin{aligned}W_0(f)_{\lambda=1} \simeq 2\pi^{1/2}\kappa^2\beta^2\omega_b b_0 \sum_{n=0}^{\infty} \epsilon_n J_n^2(\mu)\Big\{(n\Omega_a + \Omega_d + \Omega)^2 \exp\{-(n\Omega_a + \Omega_d + \Omega)^2\} \\ + (n\Omega_a + \Omega_d - \Omega)^2 \exp\{-(n\Omega_a + \Omega_d - \Omega)^2\}\Big\}, \qquad \Omega_a \equiv \frac{\omega_a}{\omega_b},\end{aligned} \tag{3.30}$$

and the mean output power is simply $(\kappa^2\beta^2\omega_b^2)b_0/2$, [cf. (2.40)]. For superlimiting we obtain

$$W_0(f)_{\lambda=2} \simeq \frac{32}{\pi^{3/2}}\,\kappa^2\beta^2\omega_b b_0\left(\frac{R_0^2}{2b_0}\right)p^{-1}\Omega^2 \sum_{n=0}^{\infty} \epsilon_n J_n^2(\mu)\Big\{\exp\{-(n\Omega_a + \Omega_d + \Omega)^2\} + \exp\{-(n\Omega_a + \Omega_d - \Omega)^2\}\Big\}. \tag{3.31}$$

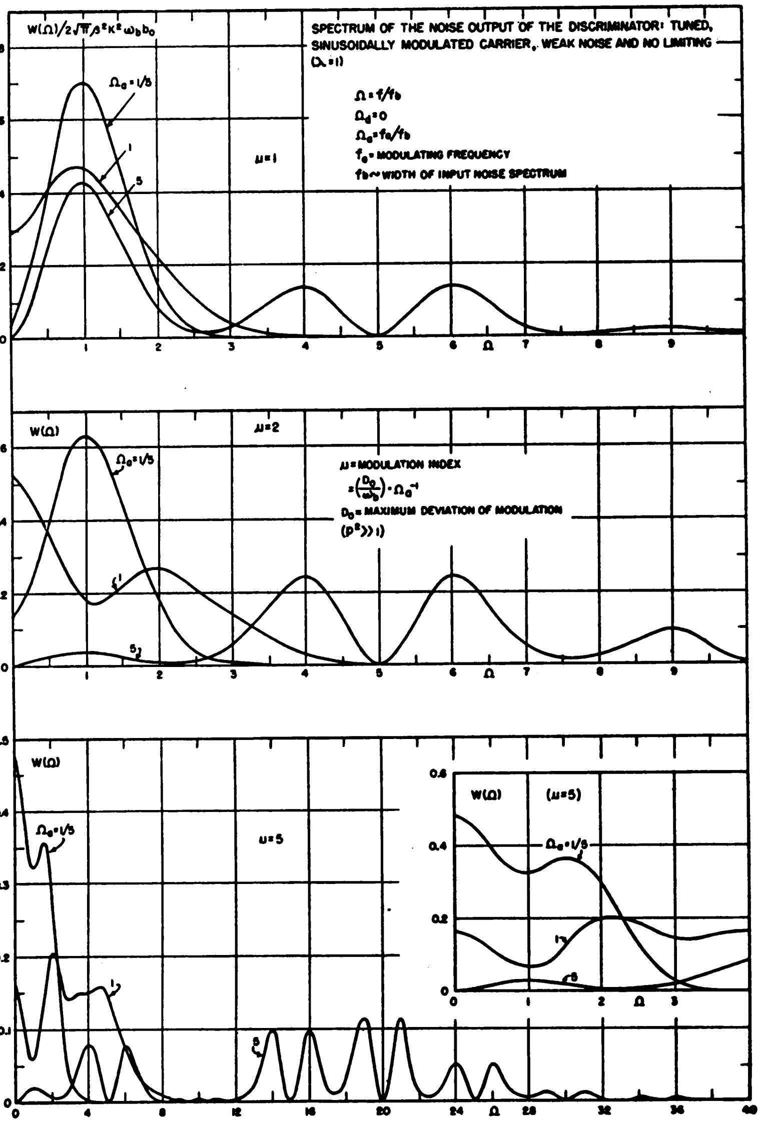

FIG. 10. Low-frequency noise spectra following discrimination, for a weak noise and a sinusoidally-modulated carrier on tune.

The mean output power associated with the noise in this instance is

$$(W_0)_{\lambda=2} = \int_0^\infty W_0(f)_{\lambda=2}\,df \simeq \frac{16}{\pi^2}(\kappa\beta\omega_b)^2 b_0\left(\frac{R_0^2}{2b_0}\right)p^{-1}\left\{\frac{1}{2} + \Omega_d^2 + \sum_{n=0}^{\infty} \epsilon_n(n^2\Omega_a^2 + 2n\Omega_a\Omega_d)J_n^2(\mu)\right\}. \tag{3.32}$$

Figures 10 and 11 illustrate the variation in spectra for different modulation indices and modulation frequencies in the somewhat less general case of zero displacement from resonance, $\Omega_d = 0$.

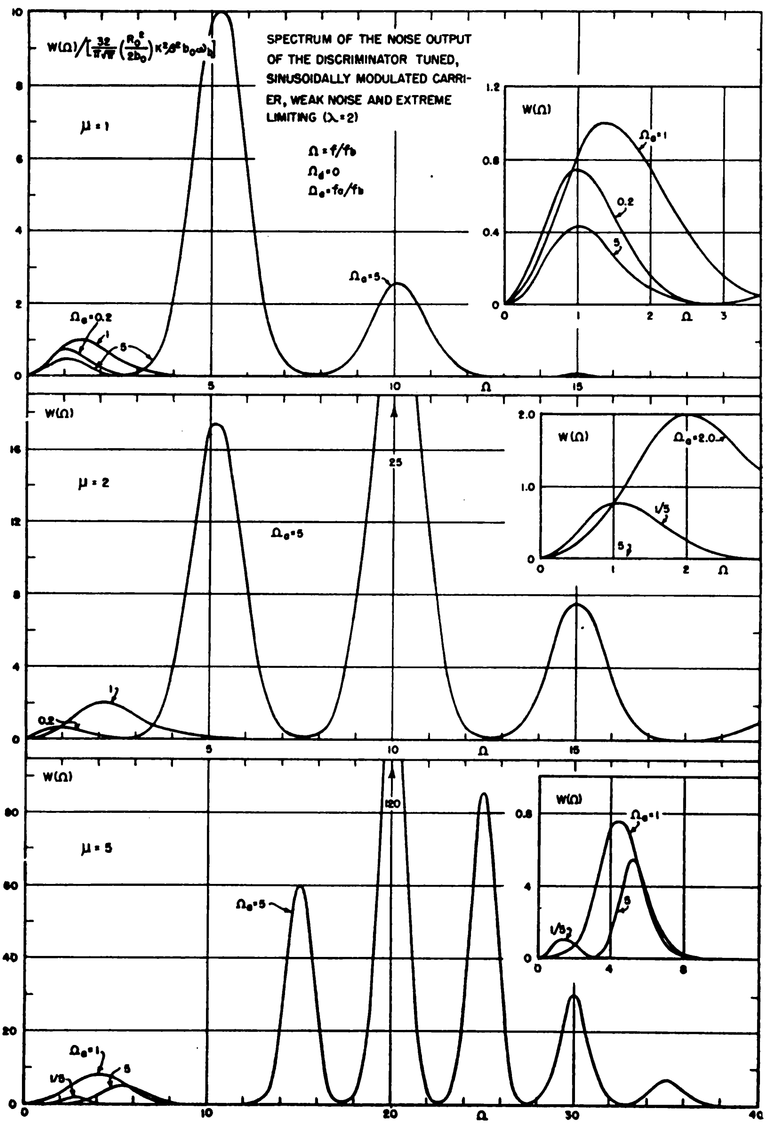

FIG. 11. The same as Fig. 10, but for extreme limiting.

APPENDIX I: Second-order probability density, its characteristic function and associated moments

Let X_k $(k = 1, 2, 3, \cdots, s)$ be s random variables, each of which in turn is the sum of a large number (N) of independent random variables x_{ik} $(i = 1, 2, 3, \cdots, N)$, distributed according to some law. Then the central limit theorem[27] states that the distribution of the X's is normal (i.e., multivariate Gaussian) in s dimensions, subject to certain restrictions [see e.g. S. O. Rice,[27] Eqs. (2.10-3)] on the distribution of the x_{ik}'s comprising X_k, restrictions which are assumed to be satisfied here. The probability density of the X_k's is

$$W_2(X_1, \cdots, X_s; t) = \frac{\exp\{-\tilde{\mathbf{X}}\mathbf{M}\,\mathbf{X}/2\}}{[(2\pi)^s \mid \mu \mid]^{1/2}}$$
$$= \frac{1}{(2\pi)^{s/2} \mid \mu \mid^{1/2}} \exp\left[-\frac{1}{2}\sum_{k=1}^{s}\sum_{l=1}^{s}\frac{\mu^{kl}}{\mid \mu \mid} X_k X_l\right], \tag{A1.1}$$

where $\mathbf{M}$ is the reciprocal matrix to μ, with elements $\boldsymbol{\mu}^{kl}/\mid \mu \mid$; μ^{kl} is the cofactor of μ_{kl}; $\mid \mu \mid$ is the determinant of μ, and $\tilde{\mathbf{X}}$ is the transpose of $\mathbf{X}$. The matrix $\boldsymbol{\mu}$ is symmetrical and has elements

$$\mu_{kl} = \sum_{i}^{N} \overline{x_{ik}x_{il}}, \tag{A1.2}$$

in which x_{ik} is the i-th element in the sum comprising X_k and x_{il} is a similar element in X_l. We assume here, without loss of generality, that $\sum_i \overline{x_{ik}} = 0$, i.e., that $\overline{X_k} = 0$. All odd moments are therefore zero. The Fourier transform of $W_2(X_1, \cdots, X_s; t)$ is known as the *characteristic function* and is[8]

$$F_2(z_1, \cdots, z_s; t) = \exp\left[-\frac{1}{2}\sum_{k=1}^{s}\sum_{l=1}^{s}\mu_{kl}z_k z_l\right]. \tag{A1.3}$$

Now, in particular, we denote here the various "in-phase" and "out-of-phase" components $V_{c1}, \cdots, V_{s2}$ and their derivatives $\dot{V}_{c1}, \cdots, \dot{V}_{s2}$ [which by (2.6), (2.7) and the properties (2.4) are random variables] by $X_1, \cdots, X_8$, $(s = 8)$. The subscript 1 on V_c, $\dot{V}_c$, or V_s, $\dot{V}_s$ refers to an initial time and the subscript 2 to a time t later. The elements of our matrix $\boldsymbol{\mu}$ may be computed as follows, from (2.6) and (2.7) in (A1.2). For example, in the instance of μ_{11} we obtain

$$\mu_{11} = \overline{V_{c1}V_{c1}} = \left\{\sum_{n=1}^{N}(a_n \cos\omega_n' t_0 + b_n \sin\omega_n' t_0)\right.$$
$$\left.\times \sum_{n=1}^{N}(a_m \cos\omega_m' t_0 + b_m \sin\omega_m' t_0)\right\}_{\text{stat. av.}} = \sum_{n=1}^{N}(\overline{a_n^2} + \overline{b_n^2})/2 \rightarrow \int_0^\infty w(f)\,df, \tag{A1.4}$$

with the aid of the characteristic properties (2.4) on passing to the limit $N \rightarrow \infty$. It is

[27] See S. O. Rice, Bell Syst. Tech. J. **23**, 282 (1944), sec. 2.10. D. M. I. also contains a discussion of some of these questions—Appendix I.

assumed throughout that $w(f)$ is such that the various integrals involving $w(f)$ converge. All 64 moments ($\mu^{kl} = \mu^{lk}$, $\mu_{kl} = \mu_{lk}$) are similarly found to be

$$\mu_{11} = \overline{V_{c1}^2} = \mu_{22} = \overline{V_{c2}^2} = \mu_{33} = \overline{V_{s1}^2} = \mu_{44} = \overline{V_{s2}^2} = b_0 ;$$

$$\mu_{55} = \overline{\dot{V}_{c1}^2} = \mu_{66} = \overline{\dot{V}_{c2}^2} = \mu_{77} = \overline{\dot{V}_{s1}^2} = \mu_{88} = \overline{\dot{V}_{s2}^2} = b_2 ;$$

$$\mu_{17} = \overline{V_{c1}\dot{V}_{s1}} = \mu_{28} = \overline{V_{c2}\dot{V}_{s2}} = -\mu_{35} = -\overline{V_{s1}\dot{V}_{c1}} = -\mu_{46} = -\overline{V_{s2}\dot{V}_{c2}} = b_1 ;$$

$$\mu_{13} = \overline{V_{c1}V_{s1}} = \mu_{24} = \overline{V_{c2}V_{s2}} = \mu_{15} = \overline{V_{c1}\dot{V}_{c1}}$$

$$= \mu_{37} = \overline{V_{s1}\dot{V}_{s1}} = \mu_{26} = \overline{V_{c2}\dot{V}_{c2}} = \mu_{48} = \overline{V_{s2}\dot{V}_{s2}} = 0;$$

$$\mu_{12} = \overline{V_{c1}V_{c2}} = \mu_{34} = \overline{V_{s1}V_{s2}} = \phi_0(t);$$

$$\mu_{14} = \overline{V_{c1}V_{s2}} = -\mu_{23} = \overline{-V_{c2}V_{s1}} = \lambda_0(t); \tag{A1.5}$$

$$\mu_{15} = \overline{V_{c1}\dot{V}_{c2}} = \mu_{38} = \overline{V_{s1}\dot{V}_{s2}} = -\mu_{52} = -\overline{\dot{V}_{c1}V_{c2}} = -\mu_{74} = -\overline{\dot{V}_{s1}V_{s2}} = \phi_1(t);$$

$$\mu_{18} = \overline{V_{c1}\dot{V}_{s2}} = \mu_{27} = \overline{V_{c2}\dot{V}_{s1}} = -\mu_{54} = -\overline{\dot{V}_{c1}V_{s2}} = -\mu_{63} = -\overline{\dot{V}_{c2}V_{s1}} = \lambda_1(t);$$

$$\mu_{56} = \overline{\dot{V}_{c1}\dot{V}_{c2}} = \mu_{78} = \overline{\dot{V}_{s1}\dot{V}_{s2}} = -\phi_2(t);$$

$$\mu_{58} = \overline{\dot{V}_{c1}\dot{V}_{s2}} = -\mu_{67} = -\overline{\dot{V}_{c2}\dot{V}_{s1}} = -\lambda_2(t).$$

Here we have

$$b_n = \int_0^\infty (\omega - \omega_0)^n w(f)\, df, \qquad \phi_n(t) = \frac{\partial^n}{\partial t^n} \int_0^\infty w(f) \cos(\omega - \omega_0)t\, df,$$

$$\lambda_n(t) = \frac{\partial^n}{\partial t^n} \int_0^\infty w(f) \sin(\omega - \omega_0)t\, df. \tag{A1.6}$$

When the input spectrum $w(f)$ is symmetrical, $\lambda_n(t)$ vanishes, as does also b_{2j+1} ($j = 0, 1, 2, \cdots$). The probability density (A1.1) in such cases is given by

$$W_2(V_{c1}, \cdots, \dot{V}_{s2}; t) = (2\pi)^{-4} \mid \mu \mid^{-1/2} \exp\left[-\frac{1}{2\mid \mu \mid}\left\{\mu^{11}(V_{c1}^2 + V_{c2}^2 + V_{s1}^2 + V_{s2}^2)\right.\right.$$

$$+ \mu^{33}(\dot{V}_{c1}^2 + \dot{V}_{c2}^2 + \dot{V}_{s1}^2 + \dot{V}_{s2}^2) + 2\mu^{12}(V_{c1}V_{c2} + V_{s1}V_{s2})$$

$$+ 2\mu^{34}(\dot{V}_{c1}\dot{V}_{c2} + \dot{V}_{s1}\dot{V}_{s2}) + 2\mu^{13}(V_{c1}\dot{V}_{c1} + V_{s1}\dot{V}_{s1} - V_{c2}\dot{V}_{c2} - V_{s2}\dot{V}_{s2}) \tag{A1.7}$$

$$\left.\left. + 2\mu^{14}(V_{c1}\dot{V}_{c2} + V_{s1}\dot{V}_{s2} - V_{c2}\dot{V}_{c1} - V_{s2}\dot{V}_{s1})\right\}\right].$$

The determinant $|\mu|$ and cofactors μ^{kl} are

$$|\mu| = [(b_0b_2 - \phi_1^2)^2 + (\phi_1^2 - \phi_0\phi_2)^2 - (\phi_1^4 + b_0^2\phi_2^2 + b_2^2\phi_0^2)]^2$$

$$\left.\begin{aligned}
\mu^{11} &= |\mu|^{1/2}[b_0(b_2^2 - \phi_2^2) - b_2\phi_1^2], & \mu^{12} &= |\mu|^{1/2}[-\phi_1^2\phi_2 - \phi_0(b_2^2 - \phi_2^2)],\\
\mu^{13} &= -|\mu|^{1/2}\phi_1[\phi_0 b_2 + b_0\phi_2], & \mu^{14} &= -|\mu|^{1/2}\phi_1[\phi_0 b_2 + \phi_0\phi_2 - \phi_1^2],\\
\mu^{33} &= |\mu|^{1/2}[b_2(b_0^2 - \phi_0^2) - b_0\phi_1^2], & \mu^{34} &= |\mu|^{1/2}[\phi_2(b_0^2 - \phi_0^2) + \phi_1^2 b_0],
\end{aligned}\right\}. \tag{A1.8}$$

The characteristic function (A1.3) is represented by (2.13). When the spectrum $w(f)$ is not symmetrical, the resulting expressions for W_2 and F_2 are correspondingly more involved, containing $\lambda_n(t)$, $(n = 0, 1, 2)$. Specifically, the characteristic function is now

$$\begin{aligned}
F_2(z_1, \cdots, z_8; t) = \exp\Bigg(-\frac{1}{2}\Big\{& b_0(z_1^2 + z_2^2 + z_3^2 + z_4^2) + b_2(z_5^2 + z_6^2 + z_7^2 + z_8^2)\\
&+ 2\phi_0(t)(z_1z_2 + z_3z_4) + 2\phi_1(t)(z_1z_6 - z_2z_5 + z_8z_3 - z_4z_7)\\
&- 2\phi_2(t)(z_5z_6 + z_7z_8) + 2\lambda_0(t)(z_1z_4 - z_2z_3) + 2\lambda_1(t)(z_1z_8 + z_2z_7 - z_3z_6 - z_4z_5)\\
&- 2\lambda_2(t)(z_5z_8 - z_6z_7)\Big\}\Bigg).
\end{aligned} \tag{A1.9}$$

APPENDIX II: Evaluation of K_1 and K_2 when $R_0 \to \infty$

The integrals K_1 and K_2 (Eq. 2.20) apparently are divergent when there is no limiting ($\mu = 1/2$ or $\lambda = 1$, and $R_0 \to \infty$), and it is not immediately evident that the results of integration (2.21) still apply in this case. The difficulty may be removed if we return to the general form (1.11) for arbitrary clipping. Our expressions for K_1 and K_2 accordingly become

$$\begin{aligned}
K_1 = -\frac{2i}{\pi} \lim_{(R_0 \to \infty)} \int_C \left(\frac{1 - \exp\{-iR_0\lambda\}}{\lambda^2}\right) d\lambda \iint_{-\infty}^{\infty} dx\, dy\\
\times \frac{xJ_1(\lambda(x^2 + y^2)^{1/2})}{x^2 + y^2} \exp\{-ixz - iy\xi\}
\end{aligned} \tag{A2.1}$$

and

$$\begin{aligned}
K_2 = -\frac{2i}{\pi} \lim_{(R_0 \to \infty)} \int_C \left(\frac{1 - \exp\{-iR_0\lambda\}}{\lambda^2}\right) d\lambda \iint_{-\infty}^{\infty} dx\, dy\\
\times \frac{yJ_1(\lambda(x^2 + y^2)^{1/2})}{x^2 + y^2} \exp\{-ixz - iy\xi\}.
\end{aligned} \tag{A2.2}$$

Transforming to polar coordinates gives us for K_1

$$K_1 = -\frac{2i}{\pi} \lim_{(R_0\to\infty)} \int_{\mathbf{C}} \left(\frac{1 - \exp\{-iR_0\lambda\}}{\lambda^2}\right) d\lambda \int_0^\infty dr\, J_1(\lambda r) \times \int_0^{2\pi} \cos\theta \exp\{iZr\cos(\theta - \psi)\}\, d\theta, \tag{A2.3}$$

where

$$\cos\psi = \frac{z}{(z^2 + \xi^2)^{1/2}}, \qquad \sin\psi = \frac{\xi}{(z^2 + \xi^2)^{1/2}}, \qquad \text{and} \qquad Z = (z^2 + \xi^2)^{1/2} \geq 0.$$

We employ next Eq. (1.4), so that (A2.3) is found to be

$$K_1 = -\frac{4z}{Z} \lim_{(R_0\to\infty)} \int_{\mathbf{C}} \left(\frac{1 - \exp\{-iR_0\lambda\}}{\lambda^2}\right)\left[\int_0^\infty J_1(r\lambda) J_1(Zr)\, dr\right] d\lambda. \tag{A2.4}$$

The integral in r is a special case of Weber and Schafheitlin's integral[10] and has the values

$$\left.\begin{aligned} \int_0^\infty J_1(r\lambda) J_1(rZ)\, dr &= \frac{\lambda}{2Z^2}\cdot {}_2F_1(1/2, 3/2; 2; \lambda^2/Z^2), \qquad Z \geq \lambda \geq 0 \\ &= \frac{Z}{2\lambda^2}\cdot {}_2F_1(1/2, 3/2; 2; Z^2/\lambda^2), \qquad \lambda \geq Z \geq 0 \end{aligned}\right\}. \tag{A2.5}$$

At $Z = \lambda$ the functions are identical, viz., $2\Gamma(0)/\pi$, but logarithmically divergent. Since this divergence is only logarithmic, the integral in λ is still convergent, including the point $Z = \lambda$, and this independently of R_0. We observe now that (A2.5) may be extended to negative values of λ if we replace λ by $|\lambda|$ and multiply the expressions (A2.5) by (-1). The integrands containing the real part of $(1 - \exp[-iR_0\lambda])$ then cancel, and we are left with

$$K_1 = -\frac{4zi}{Z} \lim_{(R_0\to\infty)} \left[\frac{1}{Z^2}\int_0^Z \frac{\sin R_0\lambda}{\lambda}\, {}_2F_1(1/2, 3/2; 2; \lambda^2/Z^2)\, d\lambda + Z\int_Z^\infty \frac{\sin R_0\lambda}{\lambda}\, {}_2F_1(1/2, 3/2; 2; Z^2/\lambda^2)\, d\lambda\right]. \tag{A2.6}$$

Utilizing the fact that $\lim_{(R_0\to\infty)} (\sin R_0\lambda)/\lambda$ is a delta-function,[6,8] with the value $\pi\delta(\lambda - 0)$, we have finally

$$K_1 = -\frac{2\pi iz}{Z^3} = -\frac{2\pi iz}{(z^2 + \xi^2)^{3/2}}; \tag{A2.7}$$

the second integral in (A2.6) vanishes because of the property $\int_a^b f(x)\delta(x - 0)\, dx = 0$, $b > a > 0$. A corresponding expression for K_2 is obtained on replacing z by ξ in the numerator of (A2.7). If we set $\mu = 1/2$ in (2.21) we see that the result is precisely that given above, cf. (A2.7).

APPENDIX III: Integration for $\mathbf{R}(t)_N$

The Bessel expansions for the trigonometric exponentials in (2.27) are

$$\exp\{-\phi_0\rho_1\rho_2\cos(\theta_2-\theta_1)+iA_0\rho_1\cos(\theta_1-\eta_1)+iA_0\rho_2\cos(\theta_2-\eta_2)\}$$

$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}(-1)^k\epsilon_k I_k(\phi_0\rho_1\rho_2)i^{l+n}\epsilon_l\epsilon_n J_l(A_0\rho_1)J_n(A_0\rho_2) \tag{A3.1}$$

$$\cos k(\theta_2-\theta_1)\cos l(\theta_1-\eta_1)\cos n(\theta_2-\eta_2).$$

The integration over θ_1 and θ_2 indicated in (2.27) is accomplished with the help of

$$\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}\sin\psi k\cdot\sin\psi m\\ \cos\psi k\cdot\cos\psi m\end{pmatrix}d\psi=\delta_m^k/\epsilon_m \quad\text{and}\quad \frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}\sin\psi k\cdot\cos\psi m\\ \cos\psi k\cdot\sin\psi m\end{pmatrix}d\psi=0. \tag{A3.2}$$

We at length obtain the following five expressions for the average over θ_1 and θ_2 :

$$C_{1,ln}^{(k)}=\frac{\cos(n\eta_2-l\eta_1)}{2\epsilon_l\epsilon_n}[\delta_l^{k+1}\delta_n^{k+1}+\delta_l^{|k-1|}\delta_n^{|k-1|}], \tag{A3.3}$$

$$C_{2,ln}^{(k)}=\frac{\cos(n\eta_2-l\eta_1)}{2\epsilon_l\epsilon_n}\left[\delta_l^k\delta_n^k-\frac{1}{2}\delta_l^{k+2}\delta_n^{k+2}-\frac{1}{2}\delta_l^{|k-2|}\delta_n^{|k-2|}\right], \tag{A3.4}$$

$$\begin{aligned}C_{3,ln}^{(k)}=\frac{1}{4\epsilon_l\epsilon_n}&[\delta_n^{k+1}(\delta_l^{k+2}+\delta_l^k)\cos\eta_1\sin(n\eta_2-l\eta_1)\\&+\delta_n^{k+1}(\delta_l^{k+2}-\delta_l^k)\sin\eta_1\cos(n\eta_2-l\eta_1)-\delta_n^{k-1}\delta_l^{|k-1|+1}\sin n\eta_2\cos(l-1)\eta_1\\&+\delta_n^{|k-1|}\delta_l^k\cos n\eta_2\sin(l-1)\eta_1-\delta_n^{k-1}\sin n\eta_2(\delta_l^{|k-1|-1}\cos\eta_1\cos l\eta_1\\&-\delta_l^{|k-1|-1}\sin\eta_1\sin l\eta_1)+\delta_n^{|k-1|}\cos n\eta_2(\delta_l^{k-2}\cos\eta_1\sin l\eta_1+\delta_l^{|k-2|}\sin\eta_1\cos l\eta_1)],\end{aligned} \tag{A3.5}$$

$$\begin{aligned}C_{4,ln}^{(k)}=\frac{1}{4\epsilon_l\epsilon_n}&[\delta_l^{k+1}(\delta_n^{k+2}+\delta_n^k)\cos\eta_2\sin(n\eta_2-l\eta_1)\\&-\delta_l^{k+1}(\delta_n^{k+2}-\delta_n^k)\sin\eta_2\cos(n\eta_2-l\eta_1)\\&+\delta_l^{k-1}\delta_n^{|k-1|+1}\sin l\eta_1\cos(n-1)\eta_2-\delta_l^{|k-1|}\delta_n^k\cos l\eta_1\sin(n-1)\eta_2\\&+\delta_l^{k-1}\sin l\eta_1(\delta_n^{|k-1|-1}\cos\eta_2\cos n\eta_2-\delta_n^{|k-1|-1}\sin\eta_2\sin n\eta_2)\\&-\delta_l^{|k-1|}\cos l\eta_1(\delta_n^{k-2}\cos\eta_2\sin n\eta_2+\delta_n^{|k-2|}\sin\eta_2\cos n\eta_2)],\end{aligned} \tag{A3.6}$$

and

$$
\begin{aligned}
C_{5,ln}^{(k)} = \frac{1}{4\epsilon_n\epsilon_l} [&\cos \eta_2 \cos n\eta_2 \cos \eta_1 \cos l\eta_1(\delta_n^{k+1} + \delta_n^{|k-1|})(\delta_l^{k+1} + \delta_l^{|k-1|}) \\
&+ \cos \eta_2 \cos n\eta_2 \sin \eta_1 \sin l\eta_1(\delta_n^{k+1} + \delta_n^{|k-1|})(\delta_l^{k+1} - \delta_l^{k-1}) \\
&+ \sin \eta_2 \sin n\eta_2 \cos \eta_1 \cos l\eta_1(\delta_n^{k+1} - \delta_n^{k-1})(\delta_l^{k+1} + \delta_l^{|k-1|}) \\
&+ \sin \eta_2 \sin n\eta_2 \sin \eta_1 \sin l\eta_1(\delta_n^{k+1} - \delta_n^{k-1})(\delta_l^{k+1} - \delta_l^{k-1}) \\
&+ \cos \eta_2 \sin n\eta_2 \cos \eta_1 \sin l\eta_1(\delta_n^{k+1} + \delta_n^{k-1})(\delta_l^{k+1} + \delta_l^{k-1}) \\
&+ \cos \eta_2 \sin n\eta_2 \sin \eta_1 \cos l\eta_1(\delta_n^{k+1} + \delta_n^{k-1})(\delta_l^{|k-1|} - \delta_l^{k+1}) \\
&+ \sin \eta_2 \cos n\eta_2 \cos \eta_1 \sin l\eta_1(\delta_n^{|k-1|} - \delta_n^{k+1})(\delta_l^{k+1} + \delta_l^{k-1}) \\
&+ \sin \eta_2 \cos n\eta_2 \sin \eta_1 \cos l\eta_1(\delta_n^{|k-1|} - \delta_n^{k+1})(\delta_l^{|k-1|} - \delta_l^{k+1})]
\end{aligned}
\tag{A3.7}
$$

with the convention that $\delta_b^{-a} = -\delta_b^a$. The correlation function may now be written

$$
\begin{aligned}
R_0(t)_N = \gamma_\lambda^2 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_k\epsilon_l\epsilon_n(-1)^{k+(l+n)/2} \int_0^\infty \rho_1^{\lambda-2} J_l(\rho_1 A_0) \exp(-b_0\rho_1^2/2)\, d\rho_1 \\
\times \int_0^\infty \rho_2^{\lambda-2} J_n(\rho_2 A_0) \exp(-b_0\rho_2^2/2) I_k(\phi_0\rho_1\rho_2)\, d\rho_2 \\
\times \{\phi_2 C_{1,ln}^{(k)} + \phi_1^2\rho_1\rho_2 C_{2,ln}^{(k)} + iA_0\phi_1\dot{\eta}_1\rho_1 C_{3,ln}^{(k)} + iA_0\phi_1\dot{\eta}_2\rho_2 C_{4,ln}^{(k)} - A_0^2\dot{\eta}_1\dot{\eta}_2 C_{5,ln}^{(k)}\}.
\end{aligned}
\tag{A3.8}
$$

The series in n and l are eliminated because of the Kronecker deltas in (l, k) and (n, k), which appear everywhere in pairs in the various C's. If we separate the variables ρ_1 and ρ_2 by expanding $I_k(\phi_0\rho_1\rho_2)$, integrate, and reduce the C's, we finally obtain (2.28).

APPENDIX IV: Some properties of $H_{k,2m+n,q}(p; \lambda)$

The determination of spectra requires a wide range of values of the order $(k, 2m + n, q)$ of the amplitude functions $H_{k,2m+n,q}(p; \lambda)$. These values are best found with the help of recurrence relations, which greatly facilitate the work of computation. The amplitude function H is described by (Eq. 2.29)

$$
\begin{aligned}
H_{k,2m+n,q}(p; \lambda) = \frac{\Gamma[(k + 2m + n + q + \lambda - 1)/2]p^{q/2}}{q! b_0^{(n+\lambda-1)/2} 2^{(3- \ -\lambda)/2}[m!(m + k)!]^{1/2}} \\
\times {}_1F_1([k + 2m + n + q + \lambda - 1]/2; q + 1; -p),
\end{aligned}
\tag{A4.1}
$$

$k, m, n, q \geq 0$ and integral. Here q is equal to k plus an integer, and n is either zero or unity. Using this fact we distinguish, for convenience, two classes of functions, denoted respectively by superscript zero and superscript unity. The amplitude functions are thus only of double order (in k and m).

Recurrence relations between the H's are readily found from those of the confluent hypergeometric function ${}_1F_1$, a table of which we list below,[28] after multiplication by the appropriate factors [cf. (A4.1)]. We have

	$F_{\alpha+}$	$F_{\alpha-}$	$F_{\beta+}$	$F_{\beta-}$	F
1.	α	$\alpha - \beta$			$\beta - 2\alpha \mp p$
2.	$\alpha\beta$		$\pm p(\beta - \alpha)$		$-\beta(\alpha \pm p)$
3.	α			$1 - \beta$	$(\beta - \alpha - 1)$
4.		$-\beta$	$\mp p$		β
5.		$\alpha - \beta$		$\beta - 1$	$1 - \alpha \mp p$
6.			$\pm p(\beta - \alpha)$	$\beta(\beta - 1)$	$\beta(1 - \beta \mp p)$

(A4.2)

The subscripts on F heading the columns refer to an addition or subtraction of unity in α or β of $F = {}_1F_1(\alpha; \beta; \pm p)$, the rows list the factors which multiply the heading F's, and the upper sign wherever $(\pm)$ occurs refers to ${}_1F_1(\alpha; \beta; +p)$, while the lower refers to ${}_1F_1(\alpha; \beta; -p)$. The sum of each row is zero.

There are five distinct H's, corresponding to $q = k + 1, |k - 1|$, with $n = 0$, and $q = k, k + 2, |k - 2|$, with $n = 1$. For each amplitude function, two recurrence formulas are needed to cover the complete manifold (k, m): one for k fixed, $m = 0, 1, \cdots$, the other for m constant, $k = 0, 1, 2, \cdots$. Since m appears only in α of ${}_1F_1(\alpha; \beta; -p)$ [see (A4.1) above] the first row of (A4.2) may be used directly in the former case, viz:

$$[(\beta - \alpha){}_1F_1(\alpha - 1; \beta; -p) + (2\alpha - \beta - p){}_1F_1(\alpha; \beta; -p)]/\alpha = {}_1F_1(\alpha + 1; \beta; -p). \tag{A4.3}$$

In the latter, where k is varied, both α and β contain k additively, and so we seek a relation between F, $F_{\alpha-\beta-}$, and $F_{\alpha+\beta+}$. Now if we change β to $\beta - 1$ in 4. of (A4.2), and again alter α to $\alpha + 1$, use 5. and eliminate $F_{\beta-}$ and $F_{\alpha+}$ from these three relations, we obtain

$$\frac{\beta}{p\alpha}\{-(\beta - 1){}_1F_1(\alpha - 1; \beta - 1; -p) + (\beta - 1 + p){}_1F_1(\alpha; \beta; -p)\} = {}_1F_1(\alpha + 1; \beta + 1; -p). \tag{A4.4}$$

[28] See Appendix 4 in ref. 7.

The general recurrence formulas follow from (A4.3) and (A4.4), and are

$$\frac{1}{[(m+2)(m+2+k)]^{1/2}}\left\{-\left(\frac{k+2m+n+\lambda-1-q}{2}\right)\right.$$

$$\times\left(\frac{k+2m+n+q+\lambda-1}{2}\right)\frac{H^{(n)}_{k,2m+n,q}}{[(m+1)(m+k+1)]^{1/2}} \tag{A4.5}$$

$$\left.+(2m+n+k+\lambda-p)H^{(n)}_{k,2m+n+2,q}\right\} = H^{(n)}_{k,2m+n+4,q}$$

and

$$\frac{1}{(m+k+2)^{1/2}}\left\{-\left(\frac{k+2m+n+q+\lambda-1}{2}\right)\frac{H^{(n)}_{k,2m+n,q}}{(m+k+1)^{1/2}}\right.$$

$$\left.+\frac{(q+p+1)}{p^{1/2}}H^{(n)}_{k+1,2m+n,q+1}\right\} \tag{A4.6}$$

$$= H^{(n)}_{k+2,2m+n,q+2}\,.$$

We observe from (A4.1) that $\alpha = (k+2m+n+q+\lambda-1)/2$ is either a positive integer or half-integer. When α is a half-integer we use [29]

$$ {}_1F_1(\mu+1/2;2\mu+1;\pm p) = \frac{2^{2\mu}\Gamma(\mu+1)}{(\pm p)^{\mu}}\exp\{\pm p/2\}I_\mu(\pm p/2). \tag{A4.7}$$

This is a convenient form for computation because of the available tables of $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$.[21] When α is integral, ${}_1F_1$ may be expressed in closed form as a polynomial in p and e^{-p}. In general, four functions must be known initially for $n=0$ or $n=1$, in order that (A4.5) and (A4.6) can be used to give us a complete set of H's.

The initial functions [30] to be used in the recurrence relations when $n=0$ $(q=k+1)$ to give us $H^{(0)}_{k,2m,k+1}$ are

$$H^{(0)}_{001}(p;\lambda) = a_\lambda p^{1/2}{}_1F_1(\lambda/2;2;-p) = a_1p^{1/2}e^{-p/2}[I_0(p/2)+I_1(p/2)], \qquad (\lambda=1) \tag{A4.8a}$$

$$= a_2p^{-1/2}(1-e^{-p}) \qquad , \qquad (\lambda=2) \tag{A4.8b}$$

$$H^{(0)}_{021} = \frac{\lambda a_\lambda}{2}p^{1/2}{}_1F_1(\lambda/2+1;2;-p) = a_1p^{1/2}e^{-p/2}[I_0(p/2)-I_1(p/2)]/2 \tag{A4.9a}$$

$$= a_2p^{1/2}e^{-p}, \tag{A4.9b}$$

$$H^{(0)}_{102} = \frac{\lambda a_\lambda}{4}p\,{}_1F_1(\lambda/2+1;3;-p) = a_1e^{-p/2}I_1(p/2) \tag{A4.10a}$$

$$= a_2(1-e^{-p}-pe^{-p})/p \tag{A4.10b}$$

[29] Ref. 21, pp. 191.
[30] W. R. Bennett, J. Acoust. Soc. Amer. 15, 164 (1944).

and

$$H^{(0)}_{122} = (\lambda/2 + 1)\frac{\lambda a_\lambda}{4\cdot 2^{1/2}}\, p\ {}_1F_1(\lambda/2 + 2; 3; -p)$$

$$\left.\begin{aligned} &= a_1 e^{-p/2}[pI_0(p/2) - (p+1)I_1(p/2)]/2^{3/2} \\ &= a_2 p e^{-p}/2^{1/2} \end{aligned}\right\} = \frac{a_\lambda}{c_\lambda 2^{1/2}} H^{(1)}_{032} \qquad \begin{aligned}&(\text{A4.11a})\\&(\text{A4.11b})\end{aligned}$$

where $a_\lambda = \Gamma(\lambda/2)/b_0^{(\lambda-1)}2^{(3-\lambda)/2}$;

$$\text{in particular, } a_1 = \pi^{1/2}/2 \text{ and } a_2 = (2b_0)^{-1/2}. \qquad (\text{A4.12})$$

A similar technique is applied to obtain $H^{(0)}_{k,2m,|k-1|}$ ($n = 0$, $q = |k - 1|$). When $k = 0$, $H^{(0)}_{0,2m,1}$ may be obtained from (A4.8-A4.11) above. The initial functions here are

$$H^{(0)}_{100} = a_\lambda\ {}_1F_1(\lambda/2; 1; -p) \qquad = a_1 e^{-p/2} I_0(p/2) \qquad (\text{A4.13a})$$

$$= a_2 e^{-p}, \qquad (\text{A4.13b})$$

$$H^{(0)}_{120} = \frac{\lambda a_\lambda}{2^{3/2}}\ {}_1F_1(\lambda/2 + 1; 1; -p) \qquad = a_1 e^{-p/2}[(1 - p)I_0(p/2) + pI_1(p/2)]/2^{3/2} \qquad (\text{A4.14a})$$

$$= a_2(1 - p)e^{-p}/2^{1/2}, \qquad (\text{A4.14b})$$

$$H^{(0)}_{201} = \frac{\lambda a_\lambda}{2^{3/2}}\, p^{1/2}\ {}_1F_1(\lambda/2 + 1; 2; -p) = a_1 p^{1/2} e^{-p/2}[I_0(p/2) - I_1(p/2)]/2^{3/2} \qquad (\text{A4.15a})$$

$$= a_2 p^{1/2} e^{-p}/2^{1/2}, \qquad (\text{A4.15b})$$

and

$$H^{(0)}_{221} = (\lambda/2 + 1)\frac{\lambda a_\lambda}{2\cdot 6^{1/2}}\, p^{1/2}\ {}_1F_1(\lambda/2 + 2; 2; -p)$$

$$\left.\begin{aligned} &= a_1 \frac{p^{1/2}e^{-p/2}}{2\cdot 6^{1/2}}[(3/2 - p)I_0(p/2) + (p - 1/2)I_1(p/2)] \\ &= 2a_2 p^{1/2} e^{-p}(1 - p/2)/6^{1/2} \end{aligned}\right\} = \frac{a_\lambda}{c_\lambda} H^{(1)}_{311} \qquad \begin{aligned}&(\text{A4.16a})\\&(\text{A4.16b})\end{aligned}$$

We continue in the same way to obtain $H^{(1)}_{k,2m+1,k}$ ($n = 1$, $q = k$). The initial functions are

$$H^{(1)}_{010} = c_\lambda\ {}_1F_1(\lambda/2; 1; -p) \qquad = c_1 e^{-p/2} I_0(p/2) \qquad (\text{A4.17a})$$

$$= c_2 e^{-p}, \qquad (\text{A4.17b})$$

$$H^{(1)}_{030} = \frac{\lambda c_\lambda}{2}\ {}_1F_1(\lambda/2 + 1; 1; -p) \qquad = \frac{c_1 e^{-p/2}}{2}[(1 - p)I_0(p/2) + pI_1(p/2)] \qquad (\text{A4.18a})$$

$$= c_2(1 - p)e^{-p}, \qquad (\text{A4.18b})$$

$$H_{111}^{(1)} = \lambda/2\, c_\lambda p^{1/2}{}_1F_1(\lambda/2 + 1; 2; -p) = c_1 \frac{p^{1/2}e^{-p/2}}{2} [I_0(p/2) - I_1(p/2)] \tag{A4.19a}$$

$$= c_2 p^{1/2} e^{-p}, \tag{A4.19b}$$

and

$$H_{131}^{(1)} = (\lambda/2 + 1)\frac{\lambda c_\lambda}{2^{3/2}} p^{1/2}{}_1F_1(\lambda/2 + 2; 2; -p)$$

$$= \frac{p^{1/2}c_1e^{-p/2}}{2^{3/2}}[(3/2 - p)I_0(p/2) + (p - 1/2)I_1(p/2)] = 3^{1/2}H_{311}^{(1)} \tag{A4.20a}$$

$$= 2c_2p^{1/2}e^{-p}(1 - p/2)/2^{1/2}, \tag{A4.20b}$$

and here

$$c_\lambda = \Gamma(\lambda/2)/b_0^{\lambda/2}2^{(2-\lambda)/2}; \text{ specifically, } c_1 = (\pi/2b_0)^{1/2} \text{ and } c_2 = b_0^{-1}. \tag{A4.21}$$

There remain $H_{k,2m+1,k+2}^{(1)}$ $(n = 1, q = k + 2)$ and $H_{k,2m+1,|k-2|}^{(1)}$, $(n = 1, q = |k - 2|)$. For the former we find the initial functions to be

$$H_{012}^{(1)} = (\lambda/4)\, c_\lambda p\cdot{}_1F_1(\lambda/2 + 1; 3; -p) = c_1e^{-p/2}I_1(p/2) \tag{A4.22a}$$

$$= c_2(1 - e^{-p} - pe^{-p})/p, \tag{A4.22b}$$

$$H_{032}^{(1)} = (\lambda/2 + 1)\frac{\lambda c_\lambda}{4} p\cdot{}_1F_1(\lambda/2 + 2; 3; -p) = c_1\frac{e^{-p/2}}{2}[pI_0(p/2) - (p + 1)I_1(p/2)] \tag{A4.23a}$$

$$= c_2pe^{-p}, \tag{A4.23b}$$

$$H_{113}^{(1)} = (\lambda/2 + 1)\frac{\lambda c_\lambda}{12} p^{3/2}{}_1F_1(\lambda/2 + 2; 4; -p) = \frac{c_1e^{-p/2}}{2p^{1/2}}[(4 + p)I_1(p/2) - pI_0(p/2)] \tag{A4.24a}$$

$$= c_2p^{-3/2}(2 - 2e^{-p} - 2pe^{-p} - p^2e^{-p}), \tag{A4.24b}$$

and

$$H_{133}^{(1)} = \frac{(2 + \lambda/2)(1 + \lambda/2)\lambda p^{3/2}c_\lambda}{3\cdot 2^{5/2}}\, {}_1F_1(\lambda/2 + 3; 4; -p)$$

$$= \frac{c_1}{2^{5/2}} p^{-1/2}e^{-p/2}[(2p^2 + p)I_0(p/2) - (2p^2 + 3p + 4)I_1(p/2)] \tag{A4.25a}$$

$$= c_2p^{3/2}e^{-p}/2^{1/2}. \tag{A4.25b}$$

We have finally, for $H_{k,2m+1,|k-2|}^{(1)}$,

$$H_{210}^{(1)} = \frac{\lambda c_\lambda}{2^{3/2}}\, {}_1F_1(\lambda/2 + 1; 1; -p) = \frac{c_1e^{-p/2}}{2^{3/2}}[(1 - p)I_0(p/2) + pI_1(p/2)] \tag{A4.26a}$$

$$= \frac{c_2}{2^{1/2}}(1 - p)e^{-p}, \tag{A4.26b}$$

Spectrum	*Analytic Form* ($x = \omega_b t$)	$r_0(t)$	$r_1(t)$	$r_2(t)$	$\Delta(t) \equiv r_1^2 - r_0 r_2$
(1.) *Gaussian*	$w(f) = W_0 \exp\{-(\omega - \omega_0)^2/\omega_b^2\}$	$\exp(-x^2/4)$	$-\omega_b \dfrac{x}{2} \exp(-x^2/4)$	$\dfrac{\omega_b^2}{2}\left(\dfrac{x^2}{2} - 1\right) \cdot \exp(-x^2/4)$	$\dfrac{\omega_b^2}{2} \exp(-x^2/2)$
(2.) *Rectangular*	$w(f) = W_0, \; f_0 - f_b < f < f_0 + f_b$ $= 0, \; f_0 - f_b > f$ $f_0 + f_b < f$	$\dfrac{\sin x}{x}$	$-\omega_b\left(\dfrac{\sin x}{x^2} - \dfrac{\cos x}{x}\right)$	$\omega_b^2\left\{2\dfrac{\sin x}{x^3} - 2\dfrac{\cos x}{x^2} - \dfrac{\sin x}{x}\right\}$	$\omega_b^2\left(\dfrac{x^2 - \sin^2 x}{x^4}\right)$
(3.) "*Optical*"	$w(f) = \dfrac{W_0}{1 + [(\omega - \omega_0)/\omega_b]^2}$	$\exp(-\lvert x \rvert)$	$\left.\begin{array}{l} -\omega_b \exp(-\lvert x \rvert), \; x > 0 \\ 0, \; x = 0 \end{array}\right\}$	$\left.\begin{array}{l} \omega_b^2 \exp(-\lvert x \rvert), \; x > 0 \\ -\omega_b^2 \delta(x - 0), \; x = 0 \end{array}\right\}$	$\left.\begin{array}{l} 0, \; x > 0 \\ \omega_b^2 \delta(x - 0), \; x = 0 \end{array}\right\}$

$$H_{230}^{(1)} = (\lambda/2 + 1) \frac{\lambda c_\lambda}{2 \cdot 6^{1/2}} {}_1F_1(\lambda/2 + 2; 1; -p) = \frac{c_1 e^{-p/2}}{2 \cdot 6^{1/2}} [(p^2 - 3p + 3/2) I_0(p/2) + (2p - p^2) I_1(p/2)] \tag{A4.27a}$$

$$= \frac{c_2 e^{-p}}{6^{1/2}} (2 - 4p + p^2), \tag{A4.27b}$$

$$H_{311}^{(1)} = (\lambda/2 + 1) \frac{\lambda c_\lambda p^{1/2}}{2 \cdot 6^{1/2}} {}_1F_1(\lambda/2 + 2; 2; -p)$$

$$= \frac{c_1 p^{1/2}}{2 \cdot 6^{1/2}} e^{-p/2} [(3/2 - p) I_0(p/2) + (p - 1/2) I_1(p/2)] \tag{A4.28a}$$

$$= 2 c_2 p^{1/2} e^{-p} (1 - p/2), \tag{A4.28b}$$

and

$$H_{331}^{(1)} = (\lambda/2 + 2)(\lambda/2 + 1) \frac{\lambda c_\lambda}{4 \cdot 6^{1/2}} p^{1/2} {}_1F_1(\lambda/2 + 3; 2; -p)$$

$$= \frac{c_1 p^{1/2} e^{-p/2}}{4 \cdot 6^{1/2}} [(p^2 - 9p/2 + 15/4) I_0(p/2) - (p^2 - 7p/2 + 3/4) I_1(p/2)] \tag{A4.29a}$$

$$= \frac{c_2 p^{1/2} e^{-p}}{2 \cdot 6^{1/2}} (p^2 - 6p + 6). \tag{A4.29b}$$

APPENDIX V: Evaluation of Series (Noise alone); Remarks on Optical and Rectangular Spectra

We note here some possible methods of summing the series (3.7) and (3.9) in the instance of noise alone. For no limiting ($\lambda = 1$) Kummer's transformation[31] puts the series (3.7) in a form that is more rapidly convergent. Here we write

$$A_m = 2(m + 1) \exp \{\beta/(m + 1)\}; \quad a_m = (1/2)_m^2 \exp \{-\beta/(m + 1)\}/(m!)^2 (m + 1)^{3/2};$$

$$2\beta = \omega^2/\omega_b^2 ,$$

and

$$\lambda_m = A_m - A_{m+1} \frac{a_{m+1}}{a_m}$$

$$= 2(m + 1) \exp \{\beta/(m + 1)\} \left[1 - \left(\frac{m + 1}{m + 2} \right)^{1/2} \frac{(m + 1/2)^2}{(m + 1)^2} \right].$$

The series becomes

$$\sum_{m=0}^{\infty} a_m = A_0 a_0 + \sum_{m=0}^{\infty} (1 - \lambda_m) a_m = 2 + \sum_{m=0}^{\infty} (1 - \lambda_m) a_m . \tag{A5.1}$$

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[31] E. P. Adams, *Smithsonian mathematical formulas and tables of elliptic functions*, Publication No. 2672, Smithsonian Institution, 1939.

Because $\lim_{m\to\infty} \lambda_m \to 1$, the series $\sum_m (1 - \lambda_m)a_m$ converges more rapidly than $\sum_m a_m$ alone.

On the other hand, for superlimiting ($\lambda = 2$), we may use Euler's summation formula[32] to sum the remainder after $m + 1 = M$ terms. Our series (3.9) becomes

$$\sum_{m=0}^{\infty} \frac{\exp\{-\beta/(m+1)\}}{(m+1)^{3/2}} = \sum_{m=0}^{M-1} \frac{\exp\{-\beta/(m+1)\}}{(m+1)^{3/2}} + (\pi/\beta)^{1/2}\,\Theta\left([\beta/M]^{1/2}\right)$$
$$+ M^{-3/2}\exp\{-\beta/M\}\left\{\left[\frac{1}{2} + \frac{1}{8M} - \frac{\beta}{12M^2}\right]\right. \tag{A5.2}$$
$$\left. + \frac{1}{720M^3}\left[-\frac{105}{8} + \frac{105\beta}{4M} - \frac{21\beta^2}{2M^2} + \frac{\beta^3}{M^3}\right] + \cdots\right\}$$

The function $\Theta(x) = 2\pi^{-1/2}\int_0^x \exp(-y^2)\,dy$ is the error-function.[24] Observe that when $\beta \to \infty$ the series approaches the value $(\pi/\beta)^{1/2}$.

* * * * * *

When there is only random noise entering the limiter-discriminator elements, the correlation function of the output [cf. (3.2) or (3.8)] in the two extremes of no limiting and superlimiting is proportional to $\Delta(t) \equiv [r_1^2(t) - r_0(t)r_2(t)]$, where r_0, r_1, r_2 are defined by (2.14) and (2.15). It is instructive to determine $\Delta(t)$ for the case of "optical" and rectangular spectra, as well as for the Gaussian distribution used extensively in the present paper. The results appear in the table. Here $\delta(t - 0)$ is the familiar delta-function of Dirac, and the spectral "width-factors" ω_b for the different distributions are related by

$$(\omega_b)_{\text{Gauss}} = \frac{2}{\pi^{1/2}}(\omega_b)_{\text{rect.}} = \pi^{1/2}(\omega_b)_{\text{opt.}}, \tag{A5.3}$$

if the input noise power (b_0) and the maximum spectral intensity W_0 are the same in these cases.

When the input spectrum is "optical," the output noise spectrum is found from (2.2) and (3.1) to be

$$W(f)_{\text{opt.}} = \gamma_\lambda^2 b_0^{2-\lambda}2^{\lambda-1}\Gamma(\lambda/2)^2\frac{\omega_b}{\pi}\int_0^\infty \cos\omega t\;{}_2F_1(\lambda/2, \lambda/2; 2; r_0^2)\,\delta(t-0)\,dt$$
$$= \gamma_\lambda^2 b_0^{2-\lambda}2^{\lambda-1}\frac{\omega_b^2}{2\pi}\frac{\Gamma(\lambda/2)^2\Gamma(2-\lambda)}{\Gamma(2-\lambda/2)^2}, \qquad (f > 0). \tag{A5.4}$$

There is no *d-c* component, since $\Delta(\infty) = 0$. The spectrum has a uniform intensity for *all* frequencies, and hence the power output is infinite. This catastrophe is never realized physically, because it is not possible to produce an exact optical distribution. Owing to parasitic effects in the circuit elements the falling-off of $w(f)$ at the higher and lower frequencies, away from resonance, is always greater than the ideal optical behavior, and so $\Delta(t)$ is finite for all values of t. The output spectrum, widely spread when parasitic effects are minimal, nevertheless encloses a finite area and hence finite power. The saturation of the discriminator in practical cases imposes a further deviation from the optical shape.

[32]Reference 31, Section 1.86. We follow a suggestion of Rice[4] here.