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ON REISSNER'S THEORY OF BENDING OF ELASTIC PLATES*

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1. Introduction. The classical theory of bending of elastic plates has recently been extended and improved by Reissner.[1,2,3] His theory takes into account the transverse-shear deformations of the plate and the equations of the theory are obtained by an application of Castigliano's theorem of minimum energy. The object of the present note is to show that Reissner's equations can be obtained directly from the stress equations of equilibrium and the stress-strain relations. Moreover, by consistent use of complex variable notation, the form of the results is simplified. The equations are first obtained for an isotropic material and are then extended to an aeolotropic material which is transversely isotropic in planes parallel to the faces of the plates.

2. Fundamental equations for isotropic plates. Consider Cartesian coordinates x, y, Z and let $z = x + iy$ denote the complex variable with $\bar{z} = x - iy$ the complex conjugate of z. Stresses connected with the coordinate Z are denoted by τ_{xz}, τ_{yz}, σ_z since there is no need to confuse the z in this notation with the complex variable. Attention is directed to stresses in plates bounded by the planes $Z = \pm h$.

When body forces are absent, Stevenson[4] has shown that the stress equations of equilibrium can be expressed in the form

$$\frac{\partial \Phi}{\partial z} + \frac{\partial \Theta}{\partial \bar{z}} + \frac{\partial \Psi}{\partial Z} = 0, \tag{1a}$$

$$\frac{\partial \Psi}{\partial z} + \frac{\partial \bar{\Psi}}{\partial \bar{z}} + \frac{\partial \sigma_z}{\partial Z} = 0, \tag{1b}$$

where a bar placed over a quantity denotes the complex conjugate of that quantity and where

$$\Theta = \sigma_x + \sigma_y, \qquad \Phi = \sigma_x - \sigma_y + 2i\tau_{xy}, \qquad \Psi = \tau_{xz} + i\tau_{yz}. \tag{2}$$

If u, v, w denote the Cartesian components of displacement and if $D = u + iv$, the complex form of the stress-strain relations is

$$(1 - 2\eta)\Theta = 2\mu\left\{\nabla_1^2(F + \bar{F}) + 2\eta \frac{\partial w}{\partial Z}\right\}, \tag{3a}$$

*Received Aug. 13, 1948.

[1]E. Reissner, J. Math. Phys. 23, 184-191 (1944).

[2]E. Reissner, J. Appl. Mech. 12, A68-A77 (1945).

[3]E. Reissner, Q. Appl. Math. 5, 55-68 (1947).

[4]A. C. Stevenson, Phil. Mag. (7) 33, 639-661 (1942).

$$(1 - 2\eta)\sigma_z = 2\mu\left\{\eta\nabla_1^2(F + \bar{F}) + (1 - \eta)\frac{\partial w}{\partial Z}\right\}, \tag{3b}$$

$$\Phi = 16\mu\frac{\partial^2 F}{\partial \bar{z}^2}, \tag{3c}$$

$$\Psi = 2\mu\frac{\partial}{\partial \bar{z}}\left(2\frac{\partial F}{\partial Z} + w\right), \tag{3d}$$

where

$$D = 4\frac{\partial F}{\partial \bar{z}}, \qquad \nabla_1^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{3e}$$

and where Poisson's ratio η and Young's modulus E are related to Lame's constants by

$$\eta = \frac{\lambda}{2(\lambda + \mu)}, \qquad \mu = \frac{E}{2(1 + \eta)}. \tag{4}$$

3. Change of axes. Stevenson has pointed out that the stress combinations Θ, Φ, Ψ are particularly suitable to the problem of transformation of stress and his main results are recorded here. Thus, if

$$\Theta' = \sigma_n + \sigma_s\,, \qquad \Phi' = \sigma_n - \sigma_s + 2i\tau_{ns}\,, \qquad \Psi' = \tau_{nz} + i\tau_{sz}\,, \tag{5a}$$

where

$$z' = n + is = ze^{-i\alpha}, \tag{5b}$$

then

$$\Theta' = \Theta, \qquad \Phi' = e^{-2i\alpha}\Phi, \qquad \Psi' = e^{-i\alpha}\Psi, \qquad D' = e^{-i\alpha}D. \tag{5c}$$

4. Formulation of the problem. It is assumed that the faces of the plate are free from applied shear stresses so that

$$\Psi = 0 \qquad (Z = \pm h), \tag{6a}$$

while the normal traction on the faces is such that

$$\sigma_z = \mp\tfrac{1}{2}p \qquad (Z = \pm h), \tag{6b}$$

where p is a given function of x and y.

Stress-resultants and stress-couples are defined by

$$M_x = \int_{-h}^{h} Z\sigma_x\,dZ, \qquad M_y = \int_{-h}^{h} Z\sigma_y\,dZ, \qquad H = \int_{-h}^{h} Z\tau_{xy}\,dZ,$$
$$V_x = \int_{-h}^{h} \tau_{xz}\,dZ, \qquad V_y = \int_{-h}^{h} \tau_{yz}\,dZ, \tag{7a}$$

or, in complex form,*

*These definitions are different from those given by Stevenson [4].

$$\Lambda = M_x + M_y = \int_{-h}^{h} Z\Theta \, dZ,$$

$$\Gamma = M_x - M_y + 2iH = \int_{-h}^{h} Z\Phi \, dZ, \tag{7b}$$

$$\Psi_0 = V_x + iV_y = \int_{-h}^{h} \Psi \, dZ.$$

By using the boundary conditions (6a) and (6b) and Eqs. (1a) and (1b) it is found that

$$\frac{\partial \Gamma}{\partial z} + \frac{\partial \Lambda}{\partial \bar{z}} - \Psi_0 = 0,$$

$$\frac{\partial \Psi_0}{\partial z} + \frac{\partial \bar{\Psi}_0}{\partial \bar{z}} - p = 0. \tag{8}$$

The boundary conditions (6b) may be satisfied by taking

$$\sigma_z = -\frac{p}{4}\left\{3\left(\frac{Z}{h}\right) - \left(\frac{Z}{h}\right)^3\right\}, \tag{9a}$$

and then, from (1b),

$$\Psi = \frac{3}{4h}\left\{1 - \left(\frac{Z}{h}\right)^2\right\}\Psi_0, \tag{9b}$$

which gives zero shear stresses at the faces of the plate $Z = \pm h$.

Weighted displacements are now defined by *

$$D^* = 4\frac{\partial F^*}{\partial \bar{z}}, \qquad F^* = \frac{3}{2h^3}\int_{-h}^{h} ZF \, dZ,$$

$$w^* = \frac{3}{4h}\int_{-h}^{h}\left\{1 - \left(\frac{Z}{h}\right)^2\right\} w \, dZ. \tag{10}$$

Relations between these weighted displacements and the stress-resultants and stress-couples can be found by multiplying Eqs. (3a) to (3d) by suitable functions of Z and integrating with respect to Z through the thickness of the plate. Before carrying out this process it is necessary to eliminate $\partial w/\partial Z$ from Eqs. (3a) and (3b) in order to avoid introducing further unknown quantities. Thus Eqs. (3a) and (3b) are replaced by

$$(1 - \eta)\Theta - 2\eta\sigma_z = 2\mu(1 + \eta)\nabla_1^2(F + \bar{F}), \tag{11}$$

and the remaining equation which involves $\partial w/\partial Z$ is not used. Hence

$$\Psi_0 = \frac{10\mu h}{3}\frac{\partial}{\partial \bar{z}}(w^* + 2F^*), \tag{12a}$$

*The choice of a weighting factor Z for the displacement D is natural. Equation (3d) then shows that w must be weighted with a quadratic factor in Z which vanishes at $Z = \pm h$, in order not to introduce further unknown quantities involving values of F at the faces of the plate.

$$\Gamma = \frac{32\mu h^3}{3} \frac{\partial^2 F^*}{\partial \bar{z}^2}, \tag{12b}$$

$$(1 - \eta)\Lambda = -\frac{4}{5} h^2 \eta p + \frac{4\mu h^3}{3} (1 + \eta)\nabla_1^2(F^* + \bar{F}^*). \tag{12c}$$

Equations (8) are now satisfied if

$$\Lambda + \frac{8h^3\mu}{3} \nabla_1^2 F^* = \frac{10\mu h}{3} (w^* + 2F^*), \tag{13a}$$

$$\nabla_1^2(w^* + F^* + \bar{F}^*) = \frac{3p}{5\mu h}. \tag{13b}$$

5. Solution of the differential equations. If

$$F^* = \phi + i\psi, \tag{14}$$

then, from (13b),

$$w^* + 2\phi = \frac{h^2}{5\mu} \left\{ \Omega'(z) + \bar{\Omega}'(\bar{z}) \right\} + \frac{3}{5\mu h} \nabla_1^2 P, \tag{15}$$

provided that the applied pressure p can be expressed in the form

$$p = \nabla_1^4 P. \tag{16}$$

Also, from (13a),

$$\psi - \frac{2h^2}{5} \nabla_1^2 \psi = 0, \tag{17a}$$

$$\Lambda + \frac{8h^3\mu}{3} \nabla_1^2 \phi = \frac{10\mu h}{3} (w^* + 2\phi), \tag{17b}$$

and, from (12c),

$$(1 - \eta)\Lambda + \frac{4}{5} h^2 \eta p = \frac{8\mu h^3}{3} (1 + \eta)\nabla_1^2 \phi. \tag{17c}$$

With the help of (15), (17b) and (17c), a simple calculation now gives

$$\phi = \frac{1 - \eta}{32\mu} \left\{ \bar{z}\Omega(z) + z\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z}) \right\} + \frac{3(1 - \eta)P}{8h^3\mu} + \frac{3\eta\nabla_1^2 P}{20h\mu}, \tag{18a}$$

$$\begin{aligned} w^* = &-\frac{1 - \eta}{16\mu} \left\{ \bar{z}\Omega(z) + z\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z}) \right\} + \frac{h^2}{5\mu} \left\{ \Omega'(z) + \bar{\Omega}'(\bar{z}) \right\} \\ &- \frac{3(1 - \eta)P}{4h^3\mu} + \frac{3(2 - \eta)\nabla_1^2 P}{10h\mu}, \end{aligned} \tag{18b}$$

$$\Lambda = \frac{(1 + \eta)h^3}{3} \left\{ \Omega'(z) + \bar{\Omega}'(\bar{z}) \right\} - \frac{2h^2\eta p}{5} + (1 + \eta)\nabla_1^2 P, \tag{18c}$$

$$\Gamma = \frac{32h^3\mu}{3}\frac{\partial^2}{\partial\bar{z}^2}(\phi + i\psi), \tag{18d}$$

$$\Psi_0 = \frac{10h\mu}{3}\frac{\partial}{\partial\bar{z}}(w^* + 2\phi + 2i\psi), \tag{18e}$$

$$D^* = 4\frac{\partial}{\partial\bar{z}}(\phi + i\psi). \tag{18f}$$

The functions $\Omega(z)$, $\omega(z)$ are arbitrary functions of z. These equations for the weighted displacements and for the stress-resultants and stress-couples are in a compact and convenient form for applications. With them are used formulas for the transformation of stress-resultants and stress-couples, which are easily found by integrating Eqs. (5c) with respect to Z, after multiplication by a suitable function of Z.

6. Results for certain aeolotropic materials. Only small changes are needed in the theory when the material of the plate is aeolotropic but is transversely isotropic in planes parallel to the faces of the plate. The stress-strain relations (3a) to (3d) are replaced by

$$\Theta = (c_{11} + c_{12})\nabla_1^2(F + \bar{F}) + 2c_{13}\frac{\partial w}{\partial Z}, \tag{19a}$$

$$\sigma_z = c_{13}\nabla_1^2(F + \bar{F}) + c_{33}\frac{\partial w}{\partial Z}, \tag{19b}$$

$$\Phi = 8(c_{11} - c_{12})\frac{\partial^2 F}{\partial\bar{z}^2}, \tag{19c}$$

$$\Psi = 2c_{44}\frac{\partial}{\partial\bar{z}}\left(2\frac{\partial F}{\partial Z} + w\right). \tag{19d}$$

Since the details of the analysis are similar to those for an isotropic material only the final results are given here. Thus

$$\phi = \frac{c_{33}}{16(c_{11}c_{33} - c_{13}^2)}\left\{\bar{z}\Omega(z) + z\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z})\right\} + \frac{3c_{13}\nabla_1^2 P}{10h(c_{11}c_{33} - c_{13}^2)} + \frac{3c_{33}P}{4h^3(c_{11}c_{33} - c_{13}^2)}, \tag{20a}$$

$$w^* = -\frac{c_{33}}{8(c_{11}c_{33} - c_{13}^2)}\left\{\bar{z}\Omega(z) + z\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z})\right\} + \frac{h^2}{5c_{44}}\left\{\Omega'(z) + \bar{\Omega}'(\bar{z})\right\} - \frac{3c_{33}P}{2h^3(c_{11}c_{33} - c_{13}^2)} + \frac{3}{5h}\left\{\frac{1}{c_{44}} - \frac{c_{13}}{c_{11}c_{33} - c_{13}^2}\right\}\nabla_1^2 P, \tag{20b}$$

$$\Lambda = \frac{h^3}{3}\left\{\frac{c_{33}(c_{11} + c_{12}) - 2c_{13}^2}{c_{11}c_{33} - c_{13}^2}\right\}\left\{\Omega'(z) + \bar{\Omega}'(\bar{z})\right\} - \frac{2h^2c_{13}(c_{11} - c_{12})p}{5(c_{11}c_{33} - c_{13}^2)} + \frac{c_{33}(c_{11} + c_{12}) - 2c_{13}^2}{c_{11}c_{33} - c_{13}^2}\nabla_1^2 P, \tag{20c}$$

$$\Gamma = \frac{16h^3(c_{11} - c_{12})}{3} \frac{\partial^2}{\partial \bar{z}^2} (\phi + i\psi), \tag{20d}$$

$$\Psi_0 = \frac{10hc_{44}}{3} \frac{\partial}{\partial \bar{z}} (w^* + 2\phi + 2i\psi), \tag{20e}$$

$$D^* = 4 \frac{\partial}{\partial \bar{z}} (\phi + i\psi). \tag{20f}$$

The function ψ satisfies the equation

$$\psi - \frac{c_{11} - c_{12}}{c_{44}} \cdot \frac{h^2}{5} \nabla_1^2 \psi = 0. \tag{20g}$$

SOME REMARKS ON THE FLAT PLATE BOUNDARY LAYER*

By J. A. LEWIS and G. F. CARRIER (*Brown University*)

1. **Introduction.** In a recent paper[1], the Blasius solution for the boundary layer flow past a flat plate was supplemented by an investigation of the character of the flow field near the leading edge of the plate. In that paper, the leading edge solution (of the Stokes type) was matched numerically to the Blasius solution. At first glance, therefore, it would seem desirable to find the solution of the Oseen type and either verify or improve this match. In the present paper we shall compute the exact solution of the Oseen equations associated with the flow past a semi-infinite plate and shall present arguments which verify our belief that this solution is physically not acceptable. However, we shall introduce a modification of the Oseen linearization such that the exact solution of the modified equations completes the flow pattern with a reasonable degree of accuracy.

We shall also indicate an iteration procedure from which the exact flow pattern (i.e., the exact solution of the non-linear equations) can be obtained as the limit of a rapidly converging sequence of functions. Since the actual calculation associated with each step of this iteration would be very tedious, and since the solution given by Blasius[1] and the present result give the most interesting information, we shall not complete the integrations.

Finally, we shall indicate the "modified Oseen solution" for the flow past a flat plate of finite length. Again, the interest does not seem to justify the necessary algebra so that no numerical results are obtained.

2. **The Oseen linearization.** The equations which govern the flow of a viscous, incompressible fluid

$$u_i \frac{\partial u_j}{\partial x_i} + \rho^{-1} \frac{\partial p}{\partial x_j} = \nu \frac{\partial^2 u_j}{\partial x_i \, \partial x_i}, \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

*Received Nov. 5, 1948.

[1]G. F. Carrier and C. C. Lin, *On the nature of the boundary layer near the leading edge of a flat plate,* Q. Appl. Math. **6**, 63-68 (1948).