—NOTES—

A NOTE ON A SOLUTION TO POSSIO'S INTEGRAL EQUATION FOR AN OSCILLATING AIRFOIL IN SUBSONIC FLOW*

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Summary. Following a procedure analogous to that used by Schwarz for solving the integral equation for an oscillating airfoil in incompressible flow, a method of attack on the same problem for subsonic, compressible flow is outlined. The circulation and kernel to the circulation integral equations are expanded in series with the coefficients M^{2n} and $M^{2n}\ln M$, and a corresponding expansion for the Green's function is thereby implicitly determined. The solution up to (but not including) terms of $O(M^4, M^4 \ln M)$ is stated. A method of iteration for improving a given solution is also indicated.

The circulation integral equation. The problem to be considered is that of a thin airfoil located near the x axis between $x = \pm 1$ in an airstream having a subsonic velocity U in the positive x direction. The theory is linearized in the sense that the downwash $w(x, t)$ (positive down) may be specified along the x axis between $x = \pm 1$ and is small compared to U. The function which it is desired to obtain from $w(x, t)$ is the circulation, $\gamma(x, t)$ which is so defined that the pressure jump, $\Pi(x, t)$ at any point on the airfoil is given by the Kutta-Joukowsky law

$$\Pi(x, t) = \rho U \gamma(x, t). \tag{1}$$

For a harmonic time variation of the form $\exp(i\omega t)$, Küssner[1] has shown that $\gamma(x, t)$ and $w(x, t)$ are connected by the integral equation

$$w(x, t) = \frac{1}{2\pi\mu} \int_{-1}^{+1} \gamma(\xi, t) \exp [ik(\xi - x)] R[M, \mu^2 k(x - \xi)](x - \xi)^{-1}\, d\xi, \tag{2}$$

where M is Mach's modulus, μ is the Lorentz contraction factor $(1 - M^2)^{-1/2}$, k is the frequency parameter $\omega b/U$ (b, the semi-chord, is the unit of length used throughout the anslysis), and $R(M, y)$ is given by

$$R(M, y) = \frac{i\pi}{2} My \int_{-\infty}^{y} \exp (iu) H_1^{(2)}(M \mid u \mid) \mid u \mid^{-1}\, du, \tag{3}$$

$H_1^{(2)}(z)$ being Hankel's cylindrical function of the second kind.[2] The notation has been converted to that generally used in this country; Küssner uses $\beta = M$, $\nu = \omega$, $\omega = ik$.

The incompressible problem. Soehngen[3] has shown that the only solution to

$$g(x) = \frac{1}{2\pi} \int_{-1}^{1} f(\xi)(x - \xi)^{-1}\, d\xi \tag{4}$$

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[1]H. G. Küssner, *Algemeine Tragflächentheorie*, Luftfahrtforschung **17**, 337-354 (1940).

[2]G. N. Watson, *Bessel functions*, Macmillan, New York, 1945.

[3]H. Soehngen, *Die Lösungen der Integralgleichung*

$$g(x) = (2\pi)^{-1} \int_{-a}^{a} (x - \xi)^{-1} f(\xi)\, d\xi$$

und deren Anwendung in der Tragflügeltheorie, Math. Z. **45** (1939).

for which $f(1)$ is finite (in the present problem, the Kutta condition), is given by

$$f(x) = -\frac{2}{\pi}\left(\frac{1-x}{1+x}\right)^{1/2}\int_{-1}^{1} g(\xi)\left(\frac{1+\xi}{1-\xi}\right)^{1/2}(x-\xi)^{-1}\,d\xi. \tag{5}$$

Using this result, Schwarz[4] has given the incompressible ($M = 0$) solution to (1) and the Kutta condition

$$\gamma(1, t) = 0. \tag{6}$$

His solution is

$$\gamma(\cos\theta, t) = \frac{1}{\pi}\int_0^{\pi} G_0(k, \theta, \varphi)w(k, \varphi, t)\,d\varphi, \tag{7}$$

$$x = \cos\theta, \qquad \xi = \cos\varphi, \tag{8}$$

$$\begin{aligned} G_0(k, \theta, \varphi) = ik \ln\left[\frac{1-\cos(\theta+\varphi)}{1-\cos(\theta-\varphi)}\right] + 2(\cos\varphi - \cos\theta)^{-1}\sin\theta \\ + [(1-\cos\varphi) + T(k)(1+\cos\varphi)]\tan\left(\frac{\theta}{2}\right), \end{aligned} \tag{9}$$

$$T(k) = \frac{-iH_0^{(2)}(k) + H_1^{(2)}(k)}{iH_0^{(2)}(k) + H_1^{(2)}(k)}, \tag{10}$$

where $H_0^{(2)}$ and $H_1^{(2)}$ are Hankel functions in Watson's notation.[2]

Compressible case. In seeking a solution to (2) for $M \neq 0$, it is expedient to introduce the modified functions

$$w^*(x) = \mu \exp(-i\kappa x)w(x), \tag{11}$$

$$\gamma^*(x) = \exp(-i\kappa x)\gamma(x), \tag{12}$$

$$\kappa = \left(\frac{M^2}{1-M^2}\right)k = M^2k^*, \tag{13}$$

$$k^* = \mu^2 k \tag{14}$$

and rewrite the integral Eq. (2) as

$$w^*(x, t) = \int_{-1}^{1} K[M, k(x-\xi)]\gamma^*(\xi, t)\,d\xi, \tag{15}$$

$$K(M, y) = \frac{1}{2\pi}y^{-1}\exp(-iy)R(M, y). \tag{16}$$

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[4] L. Schwarz, *Berechnung der Druckverteilung einer harmonisch sich verformenden Tragfläche in ebener Strömung,* Luftfahrtforschung **17**, 379-386 (1940).

Expanding the Hankel function in (3), the kernel in (16) may be written

$$K(M, y) = \frac{1}{2\pi} \exp(-iy) \int_{-\infty}^{y} \exp(iu)\Big\{-u^{-2} + \sum_{m=0}^{\infty} \frac{(-)^m M^{2m+2} u^{2m}}{2^{2m} m!(m+1)!} \left[\ln\left(M \left|\frac{u}{2}\right|\right) + C - \frac{1}{2(m+1)} - \sum_{n=1}^{m}\left(\frac{1}{n}\right) + \frac{i\pi}{2}\right]\Big\} du \tag{17}$$

where C is Euler's constant. It will generally be convenient to integrate each of the terms retained in the integrand repeatedly by parts until the only integral remaining in the kernel is $\int_{-\infty}^{y} u^{-1} \exp(iu)\, du$.

A solution (15) can be carried out by writing

$$K(M, y) = \sum_{n=0}^{\infty} M^{2n} K_{2n}(y) + \ln(M) \sum_{n=1}^{\infty} M^{2n} K'_{2n}, \tag{18}$$

$$\gamma^*(x) = \sum_{n=0}^{\infty} M^{2n} \gamma_{2n}(x) + \ln(M) \sum_{n=1}^{\infty} M^{2n} \gamma'_{2n}(x) \tag{19}$$

and proceeding, in a manner entirely analogous to that utilized by Schwarz, towards a result of the form

$$\gamma(\theta) = \frac{\mu}{\pi} \int_0^{\pi} \exp[i\kappa(\cos\theta - \cos\varphi)] G(k^*, \theta, \varphi) w(k, \varphi, t)\, d\varphi, \tag{20}$$

$$G(k^*, \theta, \varphi) = \sum_{n=0}^{\infty} M^{2n} G_{2n}(k^*, \theta, \varphi) + \ln(M) \sum_{n=1}^{\infty} M^{2n} G'_{2n}(k^*, \theta, \varphi). \tag{21}$$

The algebraic manipulations required to solve for the G_{2n} and G'_{2n} are rather lengthy, and it is probably not feasible to include more than a few terms. However, the solution can be improved by iteration. Thus, if $\gamma^{*(N)}$ denotes the solution to

$$w^* = \int_{-1}^{1} K^{(N)} \gamma^{*(N)}\, d\xi, \tag{22}$$

the remainder $[\gamma^* - \gamma^{*(N)}]$ is given by the solution to

$$w^{*(N)} = \int_{-1}^{1} [K^{(N)} - K] \gamma^{*(N)}\, d\xi = \int_{-1}^{1} K[\gamma^* - \gamma^{*(N)}]\, d\xi, \tag{23}$$

where the left side to (23) may be regarded as a new velocity distribution $w^{*(N)}$.

The author has carried out the solution for $N = 1$*. In this case G_0 is given by (9) (noting that k must be replaced by k^*), while G_2 and G'_2 are given by

$$G_2(k^*, \theta, \varphi) = k^* \tan\left(\frac{\theta}{2}\right)(1 + \cos\varphi)\left\{\left[i\left(\alpha + \frac{1}{4}\right) - \frac{1}{2k^*}\right]\left(\frac{1+T}{2}\right)^2 + \left(\frac{1}{2k^*} - i\right)\left(\frac{1+T}{2}\right) + \frac{i}{4}\right\} - k^*\left(\frac{1+T}{2}\right)\tan\left(\frac{\theta}{2}\right)\sin^2\varphi$$
$$- k^{*2} \sin\theta\left[\alpha\left(\frac{1+T}{2}\right)(1 + \cos\varphi) + ik^*\left(\alpha - \frac{1}{2}\right)\sin^2\varphi\right]$$

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*Details of the solution are available from the author.

$$+ \frac{k^{*2}}{4}\left[1 - \left(\frac{1+T}{2}\right)\right] \sin\theta\cos\theta$$

$$- k^{*2}\sin\varphi(\cos\theta - \cos\varphi)\left[1 + \frac{ik^*}{2}(\cos\theta - \cos\varphi)\right]\ln\left[\frac{1-\cos(\theta+\varphi)}{1-\cos(\theta-\varphi)}\right], \quad (24)$$

$$G_2'(k^*, \theta, \varphi) = \frac{k^*}{2}\left(\frac{1+T}{2}\right)\tan\left(\frac{\theta}{2}\right)\left[i\left(\frac{1+T}{2}\right)(1+\cos\varphi) - k^*\sin^2\varphi\right]$$

$$- \frac{k^{*2}}{2}\left(\frac{1+T}{2}\right)\sin\theta\left[\left(\frac{1+T}{2}\right)(1+\cos\varphi) + ik^*\sin^2\varphi\right], \quad (25)$$

$$\alpha = \left(\frac{2C - 1 + i\pi}{4}\right) + \frac{1}{2}\ln\left(\frac{k}{4}\right). \quad (26)$$

The function $[(1 + T(k^*))/2]$ is the $C(k^*)$ used by Theodorsen.[5]

Other solutions. Other solutions of (2) have been given by Possio[6,7], Schade[8], Dietze[9] and Eichler[10]. The only extensive numerical results available are those of Possio for the lift and moment on a flat plate airfoil due to plunging and pitching motions. The collocation technique used by Possio is probably not practical for control surface motion.

Due to the large expenditure of labor which will be required to calculate subsonic flutter derivatives over (usefully) complete ranges of M and k, it appears that a preliminary study of all available methods should be conducted, probably under the auspices of one of the government agencies. While it appears to the author that the work of Schade[8] is probably best suited to a large scale computing program, some calculations should probably be made using each of the methods mentioned.

Further discussion of the German work has been given by Biot, et al.[11]

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[5] T. Theodorsen, *General theory of aerodynamic instability and the mechanism of flutter*, NACA T.R. 496 (1935).

[6] C. Possio, *L'azione aerodinamica sul profilo oscillante in un fluido compressibile di velocita iposonora*, L'Aerotechnica **18**, 441-458 (1938).

[7] R. A. Frazer, *Possio's derivative theory for an infinite aerofoil moving at subsonic speeds*, A.R.C. 4932, 0.205, 1941.

[8] Schade, *The numerical solution of Possio's integral equation for an oscillating aerofoil in a two-dimensional subsonic stream*, Aero-Versuchsanstalt, Göttingen; Reports B 44/J/27, 31.8.44; B44/J/43 27.11.44; B 44/J/44, 1.12.44, Trans. by S. W. Skan, Aero Dept. NPL, 1946 (Brit.).

[9] Dietze, *Die Luftkraft des harmonisch schwingenden Flügels im kompressiblen Medium bei Unterschallgeschwindigkeit* (*Ebenes Problem*), Deutsche Versuchsanstalt für Luftfahrt, 1943 (AAF Tr. F-Ts-506-RE, Wright Field, 1946).

[10] M. Eichler, *Auflösung der Integralgleichung von Possio*, Forschungs-Bericht Nr 1681, Junkers Flugzeug und Motorenwerke A-G, Dessau, 1942 (photostat available through N.A.C.A.).

[11] M. A. Biot, S. N. Karp, S. S. Shu, H. Weil, *Aerodynamics of the oscillating airfoil in compressible flow*, AAF Tech. Report F-TR-1167-ND.