AN ALTERNATE PROOF OF THE CONSTANCY OF CIRCULATION*

By WALLACE D. HAYES (*Brown University*)

In a non-viscous, barotropic fluid with only conservative body forces the circulation about any circuit that moves with the fluid is constant. The direct proof of this theorem due to Kelvin is the one customarily given but other proofs due to Cauchy and Helmholtz are also available. By the application of Stokes' theorem the statement may be changed to the constancy of the flux of vorticity through a surface bounded by a contour, both of which move with the fluid. The theorem in this latter statement is here proven directly, using vector language.

Let **A** be a vector which satisfies the equation

$$\frac{D\mathbf{A}}{Dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{A} = (\mathbf{A}\cdot\nabla)\mathbf{q} - \mathbf{A}(\nabla\cdot\mathbf{q}), \tag{1}$$

where $\mathbf{q}$ is the velocity vector and D/Dt the total time derivative.

A differential surface element $\mathbf{dS}$ satisfies the kinematic condition

$$\frac{D\mathbf{dS}}{Dt} = -(\mathbf{dS}\cdot\nabla)\mathbf{q} + \mathbf{dS}(\nabla\cdot\mathbf{q}) - \mathbf{dS}\times(\nabla\times\mathbf{q}). \tag{2}$$

This condition may be derived by considering $\mathbf{dS}$ to be the vector product $(\mathbf{a}\times\mathbf{b})$ of two differential distance elements lying in the surface. The total derivative may be expressed

$$\frac{D(\mathbf{a}\times\mathbf{b})}{Dt} = (\mathbf{a}\cdot\nabla)\mathbf{q}\times\mathbf{b} + \mathbf{a}\times(\mathbf{b}\cdot\nabla)\mathbf{q}$$

$$= -[(\mathbf{a}\times\mathbf{b})\times\nabla]\times\mathbf{q}$$

which leads to relation (2). Taking the scalar product of Eq. (1) with $\mathbf{dS}$ and of Eq. (2) with $\mathbf{A}$ and adding yields

$$\frac{D(\mathbf{A}\cdot\mathbf{dS})}{Dt} = 0, \tag{3}$$

the terms on the right cancelling because of vector identities. If the integral of this quantity be taken over a surface which moves with the fluid and its bounding contour, the result is obtained:

$$\frac{D}{Dt}\iint \mathbf{A}\cdot\mathbf{dS} = 0, \tag{4}$$

i.e., the flux of the vector through the surface is constant. It may be noted that $\nabla\cdot\mathbf{A}$ necessarily satisfies the same continuity equation as does the density.

If a vector field **B** satisfies the condition

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{q}\times(\nabla\times\mathbf{B}) + \nabla\psi, \tag{5}$$

*Received Oct. 21, 1948.

where ψ is an arbitrary scalar, its curl satisfies condition (1). The result (4) together with Stokes' theorem gives

$$\frac{D}{Dt}\oint \mathbf{B}\cdot d\mathbf{l} = 0, \tag{6}$$

i.e., that the line integral of **B** about a circuit which moves with the fluid is constant. This may also be proven directly.

The velocity vector **q** satisfies (5) as a dynamical condition and the vorticity vector $\nabla \times \mathbf{q}$ satisfies (1). Thus (4) gives the theorem for the constancy of flux of vorticity and (6), the constancy of circulation.

A GENERAL APPROXIMATION METHOD IN THE THEORY OF PLATES OF SMALL DEFLECTION

A REMARK ON MY PAPER

QUARTERLY OF APPLIED MATHEMATICS, **6**, 31-52 (1948)

By M. Z. KRZYWOBLOCKI (*University of Illinois*)

The author wishes to apologize to Professor R. V. Southwell and Dr. A. Weinstein for not mentioning their methods in his paper. As Professor Southwell wrote in one of his papers, it is unlikely, in present circumstances, that any review of numerous methods which appeared lately in the theory of plates could be made complete and any judgement regarding the merits of different methods is premature.

So let us add just two more methods to those mentioned in the Introduction (p. 31) and on the bottom of p. 42 of the author's paper. Weinstein's variational method allows one to solve problems of plates of arbitrary shapes governed by the differential equation $\Delta\Delta w = q$, i.e. with a transverse load only (see, A. Weinstein, Mémorial des sciences mathématiques, No. 88, 1937; N. Aronszajn and A. Weinstein, Am. Jour. Math. **64**, 623-43 (1942); A. Weinstein and D. H. Rock, Quart. App. Math., **2**, 262-6 (1944); A. Weinstein and J. A. Jenkins, Trans. R. S. Canada, Sec. III, 60-67 (1946). Southwell's relaxation method, from its nature is applicable to all cases as may be seen from the examples in Southwell's "Relaxation Methods Applied to Engineering Problems" and from the following papers: L. Fox and R. V. Southwell, Trans. Roy. Soc. (A) **239**, 419-460 (1945); D. G. Christopherson, L. Fox, J. R. Green, F. S. Shaw, and R. V. Southwell, *ibid.* **239**, 461-487 (1945).

BOOK REVIEWS

Tables of Bessel functions of fractional order. Prepared by the Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards, Volume I. Columbia University Press, New York, 1948. XLII + 413 pp. $7.50.

The first part (pp. 1-272) contains tables for $J_\nu(x)$ for $\nu = \pm 1/4, \pm 1/3, \pm 2/3, \pm 3/4$ and x ranging from 0 to 25, at intervals of 0.001 for small values of x(up to 0.9 for $\nu = -3/4, -2/3$, to 0.8 for $\nu = -1/3, -1/4$, to 0.6 for $\nu = 1/4, 1/3$, and to 0.5 for $\nu = 2/3, 3/4$) and at intervals of 0.01 for larger