UPPER AND LOWER BOUNDS FOR THE SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM OF ELASTICITY*

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In another paper,[1] we have considered the boundary value problem

$$\Delta\Delta w = p, \qquad \text{in } R,$$
$$w = f, \qquad \frac{\partial w}{\partial n} = g, \qquad \text{on } C, \tag{1}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, R is a plane domain with boundary C, and $\partial/\partial n$ denotes differentiation in the direction of the outer normal to C. Assuming the existence and uniqueness of the solution of (1), we developed a method for obtaining upper and lower bounds for $w(x_0, y_0)$, the point x_0, y_0 of R being given in advance. The upper and lower bounds were given in terms of integrals of two classes of functions, one satisfying certain boundary conditions and the other certain partial differential equations.

In the present paper we shall be concerned with the application of the same method to a boundary value problem involving a system of partial differential equations. For definiteness we shall deal only with the first boundary value problem of three dimensional elasticity, i.e.

$$\mu w_{i,jj} + (\lambda + \mu) w_{j,ji} + X_i = 0, \qquad \text{in } R,$$
$$w_i = \text{a given function}, \qquad \text{on } S, \tag{2}$$

$(i, j = 1, 2, 3)$, where R is a domain in three dimensions bounded by a surface S, w_i is the i-th component of the displacement, λ and μ are Lamé's constants, X_i is the i-th component of the body force, commas indicate partial differentiation, and a repeated subscript indicates summation over the range 1, 2, 3. The differential equations in (2) are known as Navier's equations.

Given a point (ξ_1, ξ_2, ξ_3) of R, we seek to obtain upper and lower bounds for the numbers $w_1(\xi_1, \xi_2, \xi_3)$, $w_2(\xi_1, \xi_2, \xi_3)$, and $w_3(\xi_1, \xi_2, \xi_3)$. The "Green's formulas" needed in the discussion are included in the first section. The second section deals with some auxiliary inequalities and the third with the upper and lower bounds sought. A comparison with the paper quoted in footnote 1 reveals that the difficulty involved in the transition from one equation to a system of equations is principally one of notation.

1. Green's identities.[2] In deriving Green's identities for the system (2), we shall save space by using a convenient notation. We shall write $\phi = (\phi_1, \phi_2, \phi_3)$, for example, where the ϕ_i are real functions defined on R. Also, we shall need Green's theorem

$$\int_R A_{,j}\, dR = \int_S A n_j\, dS, \tag{3}$$

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[1]J. B. Diaz and H. J. Greenberg, *Upper and lower bounds for the first biharmonic boundary value problem*, to appear in J. Math. Phys.

[2]See O. D. Kellogg, *Potential theory*, 1929. The formulas (6), (7), and (9) of this paper correspond to those for Laplace's equation which appear on pages 212, 215, and 219, respectively, of Kellogg's book.

$j = 1, 2, 3$, where n_j is the j-th component of the outer normal to S.

Let $\phi = (\phi_1, \phi_2, \phi_3)$ and $\psi = (\psi_1, \psi_2, \psi_3)$. From Green's theorem we have at once that

$$\int_R [\mu(\phi_i \psi_{i,j})_{,j} + (\lambda + \mu)(\phi_i \psi_{j,j})_{,i}]\, dR = \int_S \phi_i[\mu \psi_{i,j} n_j + (\lambda + \mu)\psi_{j,j} n_i]\, dS. \tag{4}$$

Introducing the notations

$$(\phi, \psi) = \int_R [\mu \phi_{i,j} \psi_{i,j} + (\lambda + \mu)\phi_{i,i}\psi_{j,j}]\, dR,$$

$$(\phi, \psi)_R = \int_R \phi_i[\mu \psi_{i,jj} + (\lambda + \mu)\psi_{j,ji}]\, dR, \tag{5}$$

$$(\phi, \psi)_S = \int_S \phi_i[\mu \psi_{i,j} n_j + (\lambda + \mu)\psi_{j,j} n_i]\, dS,$$

we may rewrite Eq. (4) as follows:

$$(\phi, \psi)_R + (\phi, \psi) = (\phi, \psi)_S \tag{6}$$

The integral $(\phi, \psi) = (\psi, \phi)$ plays the role of the Dirichlet integral $\int_R (u_x v_x + u_y v_y + u_z v_z)\, dR$ in the theory of Laplace's equation. Interchanging ϕ and ψ in (6), and subtracting the resulting equation from (6), we obtain Green's reciprocal theorem

$$(\phi, \psi)_R - (\psi, \phi)_R = (\phi, \psi)_S - (\psi, \phi)_S\,. \tag{7}$$

In order to obtain the remaining formula of Green, we must employ the "singular" solutions of Navier's equations (2) with body forces zero, which were given by Lord Kelvin. These three solutions give the displacements produced throughout space by a concentrated unit force in the x_1, x_2, and x_3 directions respectively. The solutions $Z^{(k)}$, $k = 1, 2, 3$, of the homogeneous system (2), with (ξ_1, ξ_2, ξ_3) as singular point, are given by

$$Z_i^{(k)} = -\frac{\lambda + \mu}{8\pi\mu(\lambda + 2\mu)}\left[r_{,ki} - \frac{\lambda + 2\mu}{\lambda + \mu} r_{,ll}\delta_{ki}\right], \tag{8}$$

$i = 1, 2, 3$, where r is the distance from (ξ_1, ξ_2, ξ_3) to (x_1, x_2, x_3), and δ_{ki} is Kronecker's delta (unity if $i = k$ and zero if $i \neq k$). If we now employ a procedure common in potential theory, replacing ψ by $Z^{(k)}$ in (7), the integration being extended over the common part of R and the exterior of a sufficiently small sphere with center at (ξ_1, ξ_2, ξ_3), and pass to the limit as the sphere shrinks toward (ξ_1, ξ_2, ξ_3), we obtain[3]

$$\phi_k(\xi_1, \xi_2, \xi_3) = -(Z^{(k)}, \phi)_R + (Z^{(k)}, \phi)_S - (\phi, Z^{(k)})_S\,. \tag{9}$$

2. Preliminary inequalities. Starting with (ϕ, ψ), defined in (5), and writing $\alpha\phi = (\alpha\phi_1, \alpha\phi_2, \alpha\phi_3)$, for real α, we have

$$(\alpha\phi + \beta\psi, \alpha\phi + \beta\psi) = \alpha^2(\phi, \phi) + 2\alpha\beta(\phi, \psi) + \beta^2(\psi, \psi) \geqq 0,$$

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[3]See A. E. H. Love, *Elasticity*, New York, 1944, p. 245, and C. Somigliana, Annali di Matematica (2) **17**, 41 (1889).

for every real α and β, which yields Schwarz' inequality

$$[(\phi, \psi)]^2 \leqq (\phi, \phi) \cdot (\psi, \psi). \tag{10}$$

Let $k = 1, 2, 3$ be fixed, and consider the solution $W^{(k)} = (W_1^{(k)}, W_2^{(k)}, W_3^{(k)})$ of the boundary value problem

$$\begin{aligned} \mu W^{(k)}_{i,jj} + (\lambda + \mu) W^{(k)}_{j,ji} &= 0, \quad \text{in } R, \\ W_i^{(k)} &= -Z_i^{(k)}, \quad \text{on } S, \end{aligned} \tag{11}$$

$i = 1, 2, 3$. ($W^{(k)}$ is the "regular part" of a certain Green's function). Together with $W^{(k)}$ we shall consider "vectors" $V = (V_1, V_2, V_3)$ and $U^{(k)} = (U_1^{(k)}, U_2^{(k)}, U_3^{(k)})$ such that

$$\mu V_{i,jj} + (\lambda + \mu) V_{j,ji} = 0, \quad \text{in } R, \tag{12}$$

$i = 1, 2, 3$, and

$$U_i^{(k)} = W_i^{(k)} = -Z_i^{(k)}, \quad \text{on } S, \tag{13}$$

$i = 1, 2, 3$, respectively. In connection with the vector w of (2) we shall introduce the vectors v and u such that[4]

$$\mu v_{i,jj} + (\lambda + \mu) v_{j,ji} + X_i = 0, \quad \text{in } R, \tag{14}$$

$i = 1, 2, 3$, and

$$u_i = w_i, \quad \text{on } S. \tag{15}$$

It is now easy to verify that

$$\left.\begin{aligned} (u - w, u - w) \\ (v - w, v - w) \end{aligned}\right\} \leqq (u - v, u - v). \tag{16}$$

Recalling the definitions of the vectors involved, and using (5) and (6), we have

$$\begin{aligned} (u - v, u - v) &= ((u - w) + (w - v), (u - w) + (w - v)) \\ &= (u - w, u - w) + (v - w, v - w), \end{aligned} \tag{17}$$

since $u - w$ vanishes on S and $v - w$ satisfies the homogeneous system of equations in R.

The two known minimum principles for the solution of (2) can be immediately deduced from the inequalities (16), as will be seen in Sec. 4.

3. Upper and lower bounds for w at a given point. Applying (9) to the vector w we obtain

$$w_k(\xi_1, \xi_2, \xi_3) = -(Z^{(k)}, w)_R + (Z^{(k)}, w)_S - (w, Z^{(k)})_S, \tag{18}$$

$k = 1, 2, 3$, and we are seeking bounds on the left hand members of (18). Of the three integrals in (18), only $(Z^{(k)}, w)_S$ cannot be computed directly from the conditions (2)

[4] C. Somigliana, Annali di Matematica, Ser. 2, vol. 17, 1889, p. 39, gives a method for obtaining solutions of (12) and (14). Let the functions $F_l (l = 1, 2, 3,)$ satisfy $\Delta\Delta F_l = -X_l/\mu$. Then the functions $v_i = -(\lambda + \mu)/(\lambda + 2\mu) F_{l,li} + F_{i,ll}$ $(i = 1, 2, 3)$, satisfy (14). In particular, the singular solution $Z^{(k)}$, given in (8), is obtained by taking $F_l = r\delta_{kl}$.

on w, and hence it is only this integral for which we have to find bounds. For each k, Schwarz' inequality, (10), yields

$$[(u - w, U^{(k)} - W^{(k)})]^2 \leqq (u - w, u - w)\cdot(U^{(k)} - W^{(k)}, U^{(k)} - W^{(k)}). \tag{19}$$

This last equation, together with (16), implies that

$$[(u - w, U^{(k)} - W^{(k)})]^2 \leqq (u - v, u - v)\cdot(U^{(k)} - V, U^{(k)} - V). \tag{20}$$

But, from (6) we have that

$$\begin{aligned}(u - w, U^{(k)} - W^{(k)}) &= (u, U^{(k)}) - (U^{(k)}, w) - (u - w, W^{(k)}) \\ &= (u, U^{(k)}) + (U^{(k)}, w)_R - (U^{(k)}, w)_S .\end{aligned} \tag{21}$$

From (13),

$$-(U^{(k)}, w)_S = (Z^{(k)}, w)_S ,$$

the "unknown" surface integral of (18). Substituting for this surface integral from (18), and combining the resulting equation with (20), we obtain

$$(w_k(\xi_1 , \xi_2 , \xi_3) - b_k)^2 \leqq a\cdot c_k , \tag{22}$$

where

$$\left.\begin{aligned} a &= (u - v, u - v), \\ c_k &= (U^{(k)} - V, U^{(k)} - V), \\ b_k &= -(u, U^{(k)}) - (U^{(k)}, w)_R \\ &\quad -(Z^{(k)}, w)_R - (w, Z^{(k)})_S . \end{aligned}\right\} \tag{23}$$

Given $k = 1, 2,$ or 3, and four vectors u, v, $U^{(k)}$, and V, satisfying conditions (15), (14), (13), and (12), respectively, equations (22) and (23) yield upper and lower bounds for $w_k(\xi_1 , \xi_2 , \xi_3)$.

Another set of bounds can be obtained by starting with

$$[(v - w, V - W^{(k)})]^2 \leqq (v - w, v - w)\cdot(V - W^{(k)}, V - W^{(k)}), \tag{24}$$

instead of (19). From (16) we have that

$$[(v - w, V - W^{(k)})]^2 \leqq (u - v, u - v)\cdot(U^{(k)} - V, U^{(k)} - V), \tag{25}$$

and from (6) it follows that

$$\begin{aligned}(v - w, V - W^{(k)}) &= (v, V) + (W^{(k)}, w - v) - (w, V) \\ &= (v, V) + (W^{(k)}, w - v)_S - (w, V)_S .\end{aligned} \tag{26}$$

The "unknown" surface integral of (18) appears, with a minus sign, on the right hand side of (26). Substituting for this surface integral from (18), and combining the resulting equation with (25), we obtain

$$(w_k(\xi_1 , \xi_2 , \xi_3) - b_k')^2 \leqq a\cdot c_k , \tag{27}$$

where

$$\left.\begin{aligned} a &= (u - v, u - v), \\ c_k &= (U^{(k)} - V, U^{(k)} - V), \\ b_k' &= (v, V) - (W^{(k)}, v)_S - (w, V)_S \\ &\quad -(Z^{(k)}, w)_R - (w, Z^{(k)})_S . \end{aligned}\right\} \tag{28}$$

Again, given $k = 1, 2$, or 3, and four vectors u, v, $U^{(k)}$, and V, satisfying conditions (15), (14), (13), and (12), respectively, equations (27) and (28) yield upper and lower bounds for $w_k(\xi_1, \xi_2, \xi_3)$. The relations (22), (23), (27), and (28) are the inequalities we wished to derive.

It is clear[5] that an iteration procedure for improving the upper and lower bounds already obtained may be given in terms of sequences u_i and w_i of vectors satisfying the homogeneous boundary conditions

$$u_{i1} = u_{i2} = u_{i3} = 0, \qquad \text{on } S,$$

and the homogeneous system of equations (12) respectively.

4. Minimum principles for w. It is interesting to observe that it is possible to deduce from (16) two known minimum principles associated with the solution w of (2). One principle picks out w from among the u's (the vectors satisfying the boundary conditions of (2)), and the other principle singles out w from among the v's (the vectors satisfying the system of partial differential equations of (2)).

Let us derive the first minimum principle ("principle of minimum potential energy"). From (5), $(\phi, \phi) = 0$ if and only if ϕ is a constant vector, i.e., each of ϕ_1, ϕ_2, ϕ_3 reduces to a constant. In view of this remark, equations (16) and (17) imply that, for any fixed v,

$$(v - w, v - w) \leqq (u - v, u - v), \tag{29}$$

the equality sign holding if and only if $u = w$. Expanding (29), employing (6) and (15), yields

$$(w, w) + 2(w, v)_R \leqq (u, u) + 2(u, v)_R . \tag{30}$$

Taking (5) and (14) into account, this result may be stated as follows: the functional

$$\int_R [\mu u_{i,j} u_{i,j} + (\lambda + \mu) u_{i,i} u_{j,j} - 2u_i X_i] \, dR \tag{31}$$

is minimized, over the class of vectors u satisfying (15), by the solution w of (2).

The second minimum principle ("principle of minimum complementary energy", or "Castigliano's principle") may be derived in a similar manner. Equations (16) and (17) imply that, for any fixed u,

$$(u - w, u - w) \leqq (u - v, u - v), \tag{32}$$

the equality sign holding if and only if $v - w$ is a constant vector. With the aid of (6) and (14), Eq. (32) yields

[5]See, for instance, section 3 of the paper quoted in footnote 1.

$$(w, w) - 2(u, w)_S \leqq (v, v) - 2(u, v)_S . \tag{33}$$

Taking (5) and (15) into account, this result may be stated as follows: the functional

$$\int_R [\mu v_{i,j} v_{i,j} + (\lambda + \mu) v_{i,i} v_{j,j}] \, dR - 2 \int_S w_i [\mu v_{i,j} n_j + (\lambda + \mu) v_{j,j} n_i] \, dS, \tag{34}$$

is minimized, over the class of all vectors v satisfying (14), by $w + c$, where w is the solution of (2) and c is any constant vector.

ON SOME SINGULAR SOLUTIONS OF THE TRICOMI EQUATION*

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1. Introduction. In this note we intend to develop briefly some singular solutions of the Tricomi equation. These solutions have application in the hodograph techniques for the theory of compressible fluids. The flows to which they apply will be discussed in a later work but it is felt that the singular solutions are of sufficient interest to merit this presentation.

2. The singular solutions. We shall consider the equations

$$\varphi_x = \psi_y , \tag{1}$$

$$\psi_x = -\varphi_y / y \tag{2}$$

which imply

$$\psi_{yy} + y\psi_{xx} = 0 \tag{3}$$

and

$$\varphi_{xx} + (\varphi_y / y)_y = 0. \tag{4}$$

Equation (3) is the Tricomi equation. The solution of primary interest has the following properties:

$$\psi_x \sim 1/y \qquad \text{on } x = 0,\ y \to 0;$$

$$\varphi_y \equiv 0 \qquad \text{on } y = 0, \text{ for } x \neq 0;$$

$$\psi, \varphi, \text{ regular in } y > 0 \text{ and on } x = 0,\ y \neq 0.$$

It is evident that if φ, ψ, are not regular in $y < 0$, the branch lines will occur along the characteristics $x^2 + 4y^3/9 = 0$. The development that follows is strictly formal and the proof that the solutions are those sought is readily found by substituting them into the original equations. To find them, we first replace y and φ by

$$s = (2/3)y^{3/2}, \qquad \varphi = (2/3)^{2/3} y \chi(x, s) = s^{2/3} \chi(x, s).$$

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