# —NOTES—

## ON A FAMILY OF ROTATIONAL GAS FLOWS*

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Steady plane flows of an ideal gas (i.e., a thermodynamically perfect gas without viscosity or thermal conductivity) in the absence of body forces may be divided into three distinct classes:

(a) flows in which the velocity field **v** is irrotational,

(b) flows in which the **v** field is rotational, but is derivable from an irrotational flow by the substitution principle established by Munk and Prim[1], and

(c) all other flows.

For the study of rotational gas flows, the actual velocity field **v** does not play so basic a role as does the reduced velocity field **w** defined by[1,2]

$$\mathbf{w} \equiv \mathbf{v}/a,$$

where $a$ is the local value of the maximum or "ultimate" velocity magnitude attainable by expansion to zero pressure. A necessary condition for irrotationality of **v** is that $a$ be constant throughout the flow. Therefore the flows in Class (a) are characterized by irrotationality of both the **w** field and the **v** field. The flows in Class (b) have rotational **v** fields, but share their **w** field (and hence their geometrical properties) with flows of Class (a). The flows of Class (c) are truly rotational, i.e. they have rotational **w** fields as well as rotational **v** fields and will be, in general, geometrically distinct from the flows of Class (a).

Of the truly rotational flows of Class (c), those having a streamline pattern consisting of concentric circles or of parallel straight lines constitute a relatively degenerate subclass. Any value of velocity magnitude $v$ and of reduced velocity magnitude $w$ may be assigned to each individual streamline, and the resulting flow will satisfy all the governing equations. Except for this degenerate case, no formally simple solutions for this class of flows have been known heretofore.

This paper presents a formally simple infinite family of truly rotational flow solutions. This family includes the familiar irrotational Prandtl-Meyer corner flow as a special case.

**The new family of solutions.** Necessary and sufficient conditions that a vector field must satisfy in order that it may serve as the reduced velocity field of a steady flow of an ideal gas in the absence of body forces are the following set of equations[2]

$$\operatorname{div}[(1 - w^2)^{(\lambda-1)/2}\mathbf{w}] = 0, \tag{1}$$

$$\operatorname{curl}\left[\frac{\mathbf{w} \times \operatorname{curl} \mathbf{w}}{1 - w^2}\right] = 0. \tag{2}$$

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[1]M. Munk and R. Prim, *On the multiplicity of steady gas flows having the same streamline pattern*, Proc. Nat. Acad. Sci., **33**, 137-141 (1947).

[2]M. Munk and R. Prim, *On the canonical form of the equations of steady motion of a perfect gas*, Naval Ordnance Laboratory Memorandum 9169 (1947).

($\lambda \equiv (\gamma + 1)/(\gamma - 1)$, where $\gamma$ denotes the adiabatic exponent; for air, $\lambda = 6$.)

We shall refer these equations to a cylindrical coordinate system $r$, $\theta$, $z$ and consider flows of the following type:

$$w_r = u(\theta), \qquad w_\theta = v(\theta), \qquad w_z = 0.$$

Thus specialized, Eqs. (1) and (2) yield the following pair of ordinary differential equations restricting the functions $u(\theta)$ and $v(\theta)$

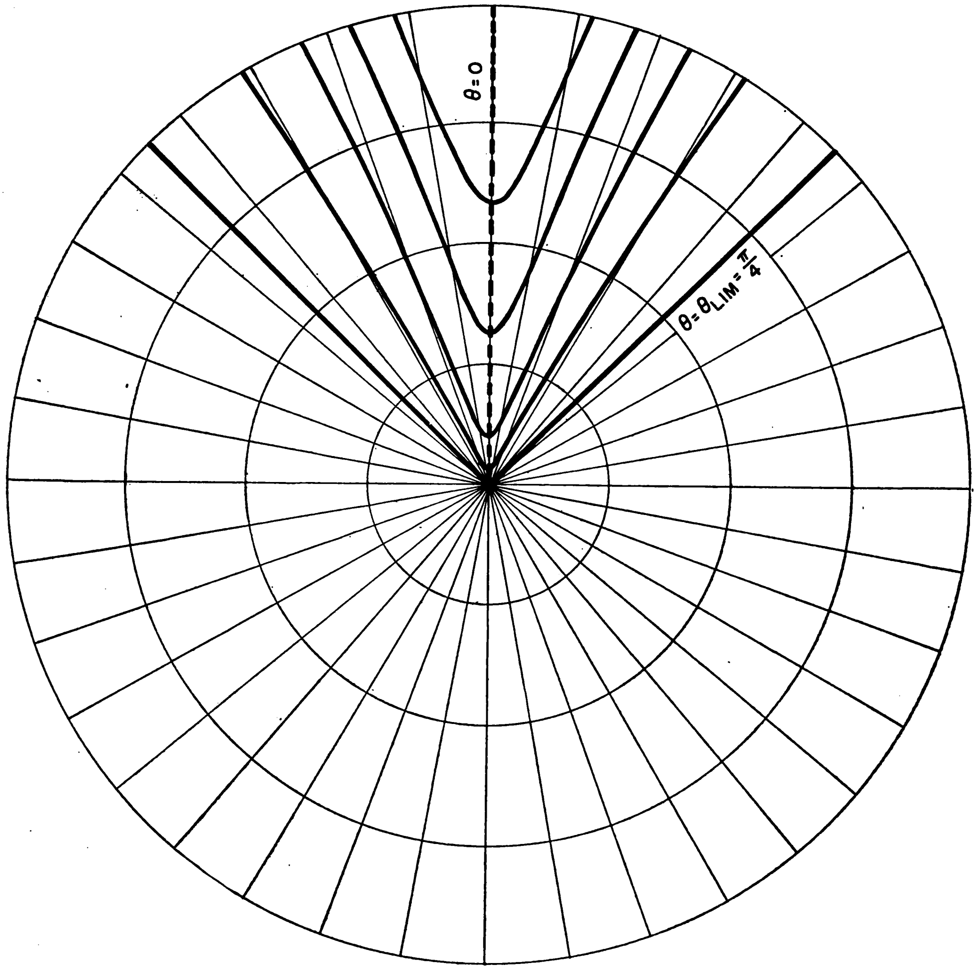


FIG. 1.

$$(u + v')(1 - u^2 - v^2) - (\lambda - 1)v(uu' + vv') = 0, \tag{3}$$

$$v(v - u') - B(1 - u^2 - v^2) = 0, \tag{4}$$

where $B$ is an arbitrary constant parameter. (These equations appear in more general form in a paper by Nemenyi and Prim.[3])

[3]P. Nemenyi and R. Prim, *Some patterns of vorticose flow of a perfect gas*, Naval Ordnance Laboratory Memorandum 9219 (1947).

The general solution of the set of equations (3) and (4) is formally forbidding and yields the functions $u(\theta)$ and $v(\theta)$ only implicitly. However, for the case $B = 0$ (for which the $\mathbf{w}$ field is irrotational) the solution is easy and yields the following two types of solutions:

$$v = \lambda^{-1/2} \cos [(\theta - \theta_0)/\lambda^{1/2}], \qquad u = \sin [(\theta - \theta_0)/\lambda^{1/2}], \tag{5}$$

and

$$v = A \cos (\theta - \theta_0), \qquad u = A \sin (\theta - \theta_0), \tag{6}$$

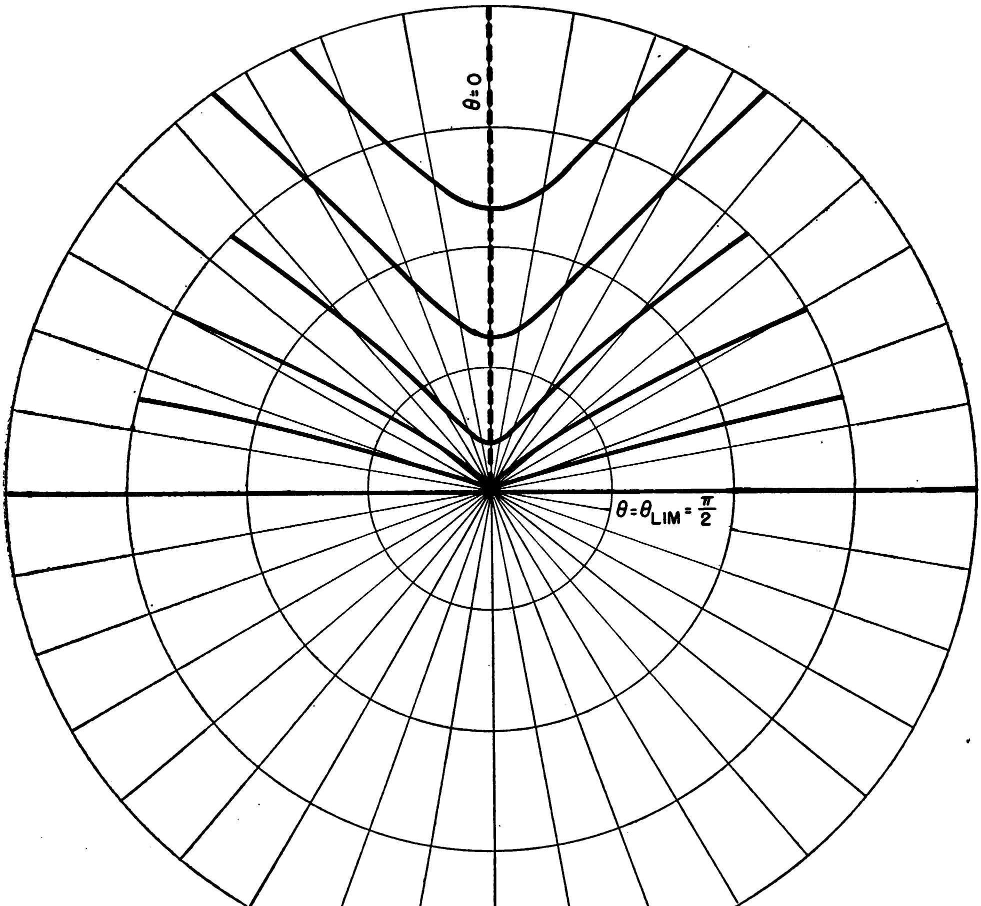


FIG. 2.

where $\theta_0$ and $A$ are arbitrary constants (except that $A^2 < 1$ physically). The solution (5) is the well-known Prandtl-Meyer irrotational corner flow, while solution (6) is the trivial case of uniform parallel flow with $w = A$.

Guided by the form of (5) and (6) we investigate the possibility of special solutions of the form

$$v = C \cos m(\theta - \theta_0), \qquad u = D \sin m(\theta - \theta_0), \tag{7}$$

when $B$ is not required to be zero. Substitution of (7) into (3) and (4) yields

$$\cos^2 m(\theta - \theta_0) = -\frac{(D - Cm)(1 - D^2)}{(D^2 - C^2)(D - \lambda Cm)} \tag{8}$$

and

$$\cos^2 m(\theta - \theta_0) = -\frac{B(1 - D^2)}{B(D^2 - C^2) - C(C - Dm)}. \tag{9}$$

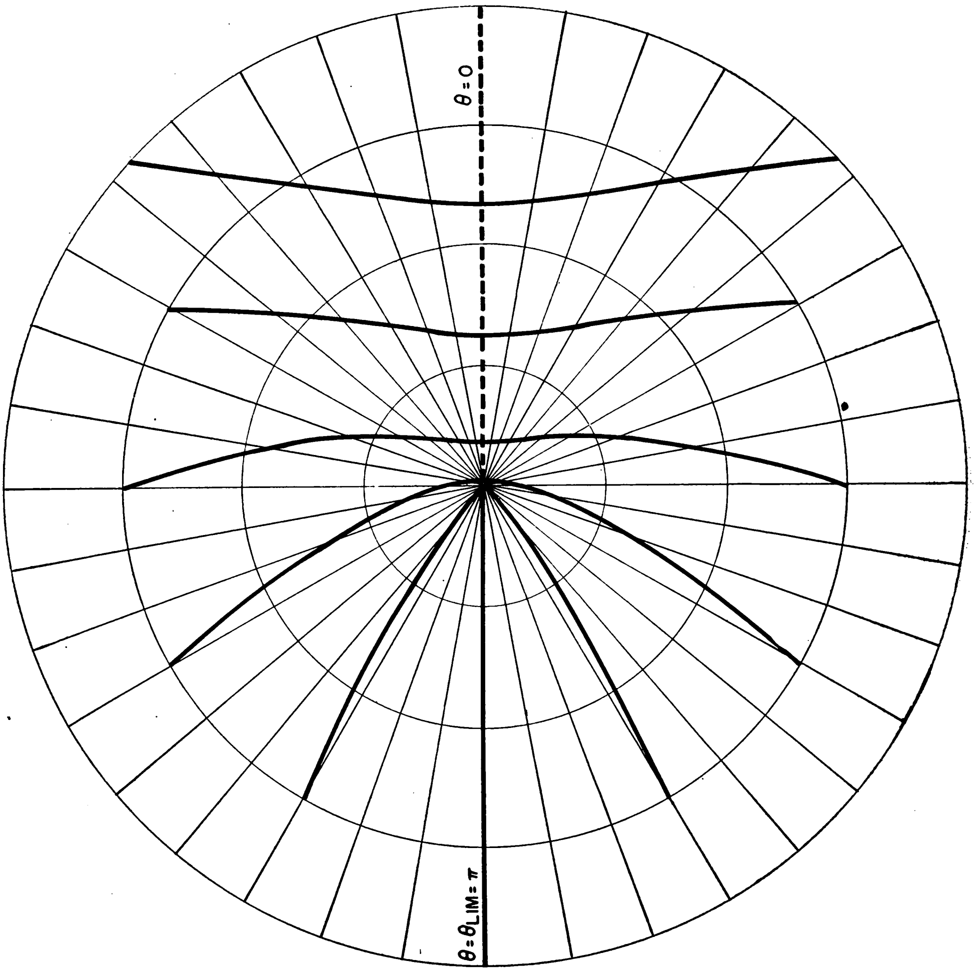


Fig. 3.

In order that (8) and (9) be valid for some finite range of $\theta - \theta_0$ it is necessary that the numerator and denominator of each right-hand member be identically zero. In addition to the trivial condition $C^2 = D^2 = m^2 = 1$, the following relationships between $B$, $C$, $D$ and $m$ satisfy these conditions

$$B = \frac{\lambda C^2 - 1}{\lambda(1 - C^2)}, \qquad D = \pm 1, \qquad m = \frac{D}{\lambda C}. \tag{10}$$

Replacing $C$ by $v_0$ and setting $\theta_0 = 0$ (this involves no real loss of generality), we have then the solution

$$v = v_0 \cos(\theta/\lambda v_0), \qquad u = \sin(\theta/\lambda v_0) \tag{11}$$

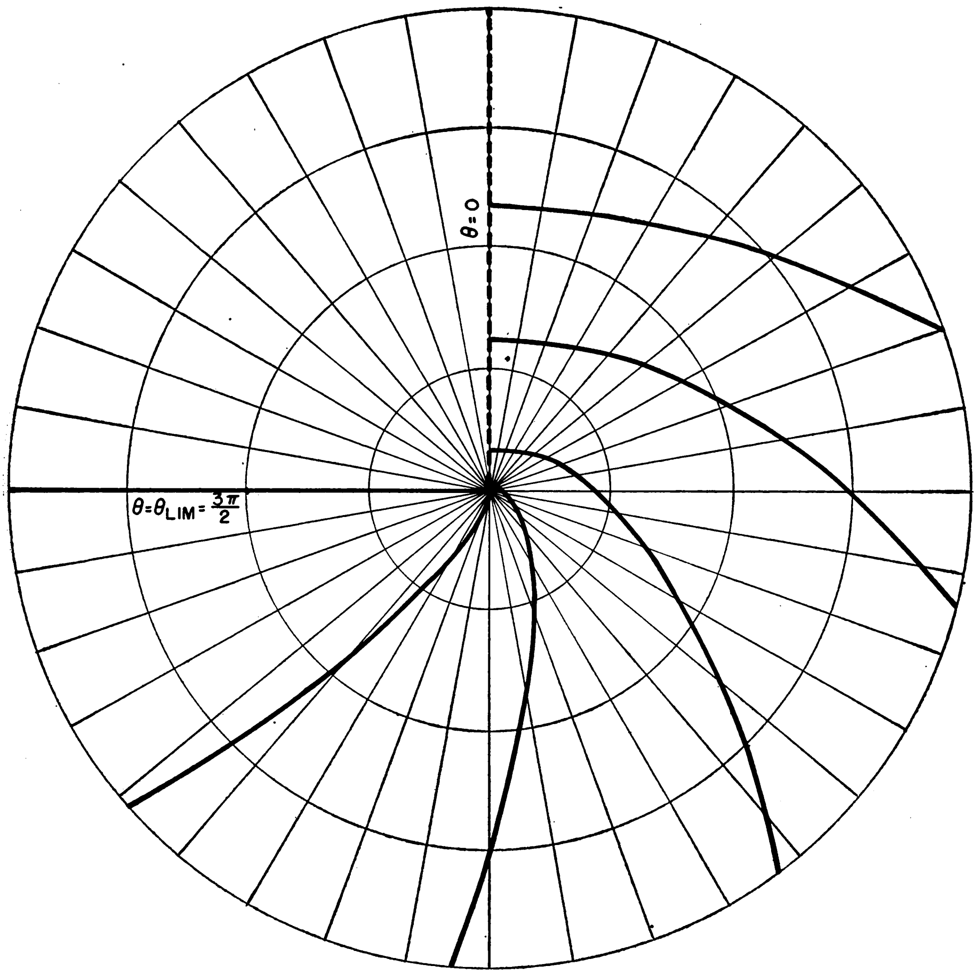


FIG. 4.

where $v_0$ is an arbitrary parameter (limited physically to the range $0 < v_0^2 < 1$).

This family of solutions is as simple formally as the Prandtl-Meyer solution (5) to which it reduces for the special case $v_0^2 = 1/\lambda$. For all other values of $v_0$ the solutions are truly rotational flows of Class (c).

**Some properties of the solutions.** The *equation of the streamlines* of these flows is readily obtained by integration of the defining equation

$$\frac{dr}{r d\theta} = \frac{u}{v} = \frac{1}{v_0} \tan(\theta/\lambda v_0)$$

giving

$$r = r_0 \, [\sec(\theta/\lambda v_0)]^{\lambda} \tag{12}$$

The *rotation* of the reduced velocity field is given by

$$\Omega \equiv \frac{1}{2} \mid \nabla \times \mathbf{w} \mid = \frac{v - u'}{2r}$$

$$\Omega = \frac{\lambda v_0^2 - 1}{2\lambda v_0} \frac{\cos(\theta/\lambda v_0)}{r} \tag{13}$$

or, along a particular streamline,

$$\Omega = \frac{\lambda v_0^2 - 1}{2\lambda v_0 r_0} [\cos(\theta/\lambda v_0)]^{\lambda+1}. \tag{14}$$

It will be noted that for $v_0 > 0$ the rotation is positive, zero, or negative near $\theta = 0$ depending on whether $v_0^2$ is respectively greater than, equal to, or less than $1/\lambda$, where $1/\lambda$ is the square of the "critical velocity" at which the flow velocity and sound velocity are equal.

The *pressure distribution* can be computed from the $\mathbf{w}$ field using the equation[2]

$$\text{grad} \ln p = -(\lambda + 1) \frac{(\mathbf{w} \cdot \text{grad})\mathbf{w}}{1 - w^2}$$

the result being

$$p = k\{r^{(\lambda v_0^2 - 1)/\lambda(1 - v_0^2)} [\cos(\theta/\lambda v_0)]^{(\lambda-1) v_0^2/(1 - v_0^2)}\}^{\lambda+1} \tag{15}$$

from which is readily obtained the *equation of the isobars*

$$r = L[\sec(\theta/\lambda v_0)]^{\lambda(\lambda-1) v_0^2/(\lambda v_0^2 - 1)} \tag{16}$$

where $K$ and $L$ are arbitrary positive constants. It should be noted that only for the irrotational case ($v_0^2 = 1/\lambda$) are the lines, $\theta =$ constant, lines of constant pressure. From (15) it is also seen that the nature of the pressure distribution is quite different depending on whether the initial velocity is supersonic ($v_0^2 > 1/\lambda$) or subsonic ($v_0^2 < 1/\lambda$). For $v_0^2 > 1/\lambda$, the pressure is zero at the origin and increases with increasing $r$; for $v_0^2 < 1/\lambda$ the pressure is infinite at the origin and decreases with increasing $r$.

A familiar characteristic of the Prandtl-Meyer solution is the existence of a *limiting angle* ($\theta_{\text{lim}} = \pi\lambda^{1/2}/2$) at which the flow velocity is of ultimate magnitude and radial in direction ($u = 1$, $v = 0$). From (11) it is seen that for each member of our family of solutions there exists such a limiting angle, its magnitude depending on the choice of $v_0$ :

$$\theta_{\text{lim}} = v_0 \lambda \pi/2. \tag{17}$$

By varying the choice of $v_0$ from zero to one, we may vary the limiting angle continuously from zero to $\lambda\pi/2$, ($3\pi$ for air).

Another familiar characteristic of the Prandtl-Meyer solution is the fact that the tangential velocity component $v$ is always equal to the local value of the velocity of sound $c$, i.e., $v/c = \pm 1$. We shall find that our entire family of solutions (11) has a similar, but more general, property which reduces to $v/c = \pm 1$ when $v_0^2 = 1/\lambda$. The square of the velocity of sound (referred to the ultimate velocity, of course) is given by

$$c^2 = \frac{1 - w^2}{\lambda - 1} \tag{18}$$

Making use of (11) we then have

$$\frac{v^2}{c^2} = v_0^2 \cos^2 \frac{\theta}{\lambda v_0} \left[ \frac{1 - v_0^2}{\lambda - 1} \cos^2 \frac{\theta}{\lambda v_0} \right]^{-1} = \frac{(\lambda - 1)v_0^2}{1 - v_0^2}$$

or, since for steady flows of ideal gas in absence of body forces, the Mach number $M$ is related to $w$ through

$$M^2 = \frac{(\lambda - 1)w^2}{1 - w^2} \tag{19}$$

$$\frac{v}{c} = \pm M_0 , \tag{20}$$

where $M_0$ is the Mach number at $\theta = 0$. (For the Prandtl-Meyer solution, $M_0 = 1$).

It is noteworthy that only for the irrotational case ($v_0^2 = 1/\lambda$, $M_0 = 1$) are the radial lines characteristic lines of the differential equations. Therefore, the useful "patching" properties of the Prandtl-Meyer solution are not shared by the other members of our family of solutions. However, sufficiently restricted regimes of our flows could be obtained by passage of an initially irrotational flow through a curved shock front.

The *Mach number* at any point in the flow is, from (11) and (19) given by

$$M = \left[ \frac{(\lambda - 1)[1 - (1 - v_0^2) \cos^2 (\theta/\lambda v_0)]}{(1 - v_0^2) \cos^2 (\theta/\lambda v_0)} \right]^{1/2} \tag{21}$$

By virtue of the substitution principle[1] each member of the family of solutions (11) in terms of the reduced velocity field $\mathbf{w}$ represents a vast variety of *actual velocity fields* $\mathbf{v}$. Denoting the tangential and radial components of $\mathbf{v}$ respectively by $v^*$ and $u^*$, and using $r_0$ of (12) to parametrize the streamlines, we have

$$\begin{aligned} v^* &= a(r_0) v_0 \cos (\theta/\lambda v_0) \\ u^* &= a(r_0) \sin (\theta/\lambda v_0), \end{aligned} \tag{22}$$

where $a(r_0)$ [$>0$] denotes the ultimate velocity magnitude assignable arbitrarily upon and constant along each individual streamline. The *density distribution* $\rho$, can be computed by recourse to the basic relation (valid for ideal gases in absence of body forces)

$$\rho = \frac{\gamma p M^2}{V^2} = \frac{\gamma p}{a^2(r_0) c^2},$$

where $p$ and $c^2$ are given by (15) and (18).

**Examples of flow patterns.** The general nature of the flow patterns associated with several choices of $v_0$ is indicated in Figs. 1 to 4. These patterns were computed from equation (12) using $\lambda = 6$ (the value for air) and values for $v_0$ chosen to give limiting values of $\theta$ of $\pi/4$, $\pi/2$, $\pi$, and $3\pi/2$ (that is, flows turning from $\theta = 0$ through $-\pi/4$, 0, $\pi/2$, and $\pi$). The flow of Fig. 3 is of particular interest, representing a flow around the edge of an infinitely thin flat plate. This flow is similar to the irrotational Ringleb flow, but lacks the limiting lines which are characteristic of the latter.