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CONJUGATE ACTION OF INVOLUTE HELICAL GEARS WITH PARALLEL OR INCLINED AXES*

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Introduction. A problem of interest in mechanics of machinery is the transformation of a uniform rotation about one axis A into a uniform rotation about another axis A'. This problem and the design of the gear surfaces give rise to interesting geometry. It is shown in the following (Theorems A and B) that such a transformation both for the case of parallel axes A, A' and non-parallel or skew axes A, A' may be accomplished by means of two involute helical surfaces to be described presently. The case of non-parallel axes is covered by Theorems A, B; that by parallel axes by Theorem C. It is further shown that where this transformation is possible, the ratio of velocities for any two involute helical surfaces is independent of the relative orientation of the two surfaces. This result is generalized in Theorems A′, B′, C′ by the inclusion of uniform translations; the ratio of rotational velocities, while constant, is then dependent also on the translational velocities. An alternative proof of Theorems A, B, C follows from Theorem D in which it is shown that by rotating an involute helicoid about its axis through an arbitrary angle one obtains a family of parallel surfaces. As a limiting case of Theorem C, if the axis A' is made to recede to infinity one obtains the mating action of a rotating involute helicoid and a plane which is undergoing a uniform translation; this is given in Theorem E.

Not only are these results of interest from point of view of *transmitting uniform rotation* by means of gears, (or transforming a rotation of a gear into a translation of a rack), but they are also of direct industrial interest from point of view of *generating* the *gear surfaces* in question. By designing the cutting tool (hob or shaving tool) so that the cutting edges lie on an involute helical surface, and providing proper rotation of the work and the tool, each about its axis, and proper translation of the tool parallel to the work axis, one can prove from the converse of the above theorems that the tool will generate an involute helical surface.†

The results of this paper will also serve to answer certain practical questions which constantly come up in gear manufacturing work. For instance, in hobbing helical involute gears the angle Σ between the axis of the hob and the axis of the gear is set to

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†It should be pointed out that if the generating tool is not a grinding wheel and possesses a finite number of cutting edges, it produces "scallops", and it is necessary from the practical standpoint to design and operate the tool so as to limit the size of scallops and thereby obtain a surface corresponding closely to the theoretical surfaces considered herein.

agree with the *design* lead angles of the two members. During the life of the hob, its outside diameter and lead angle change as its cutting surface is ground away in sharpening. Likewise in shaving, the tool changes size and its actual lead angle will vary from the design lead angle. Many gear engineers have questioned whether or not it was necessary to adjust the shaft angle in hobbing and shaving to agree with the actual lead angle of the tool. It has been thought that a discrepancy in the shaft angle of the cutting tool would produce a profile error in the part being machined.

In spiral gear drives, errors in machining the casings and gear parts will result in the gears being run at a shaft angle somewhat different than the design angle. Gear inspectors often question how accurately the shaft alignment must be held to maintain quiet running gears and adequate gear tooth contact.

These questions and many others can be answered by applying the results of this paper. Some of these practical applications to gears are discussed in Sec. 5.

While a general appreciation of the main properties of involute spur and helical gears is wide spread in industry, and some treatments of their properties may be found in the literature [see, for instance, the article by Nikola Trbojevich, "Problem of the Theoretically Correct Involute Hob", *Machinery*, **25**, 429–433 (1919)], the authors have not run across any existing comprehensive exposition of the inherently simple basic geometry and kinematics of these gears. Moreover, from personal industrial contacts, they are aware that the equations for finding the effect of changes in distance and angle on tooth shape, locus of contact, and backlash are either unknown, or not readily available. The present paper contains an exposition of the simple basic helical involute gear geometry and aims to provide equations from which the practical design and shop questions involving these gears may be answered.

1. Statement of results. By a *helical surface* is meant a surface generated by a curve C during a screw motion about an axis. Such a surface consists of helices generated by the individual points of C during the screw motion. By an *involute* is meant an involute of a proper circle known as the *base circle*, whose radius, the *base radius*, will be denoted by R_b. Finally, by an *involute helicoid* is meant a helical surface generated by an involute lying in a plane perpendicular to the axis of the screw motion, that is a helical surface such that a section of it by a plane perpendicular to the axis of the helical surface is an involute of a proper base circle.

In cylindrical coordinates r, θ, z, with the z-axis along the axis A of the screw motion and θ measured as a right-handed rotation from $y = 0$ about the positive z-axis, the equation of a helical surface is given by

$$r = f(z - C\theta), \tag{1.1}$$

where C represents the axial displacement corresponding to a rotation of one radian about the axis, and is positive for a right-handed screw, negative for a left-handed one.

To determine the equations of an involute helical surface, we start from the equations of an involute

$$\begin{aligned} x &= R_b\,[\cos\varphi_1 - (\varphi_2 - \varphi_1)\sin\varphi_1], \\ y &= R_b\,[\sin\varphi_1 + (\varphi_2 - \varphi_1)\cos\varphi_1], \end{aligned} \tag{1.2}$$

where φ_2 is the value of θ at the cusp of the involute on the *base circle* $r = R_b$, and φ_1 the value of θ at the point of contact of the tangent line with the base circle (Fig. 1)

By keeping φ_2 constant and varying φ_1 one obtains various involutes of the same base circle, the cusp corresponding to $\varphi_1 = \varphi_2$. For constant φ_1 and variable φ_2 the tangents to the base circle result. For $\varphi_1 > \varphi_2$, φ_1, φ_2 yield an orthogonal family of curves outside the base circle; similarly for $\varphi_1 < \varphi_2$. All involutes are congruent, any one being obtainable from any other one through a rotation about the origin of amount $\Delta\varphi_2$; in the

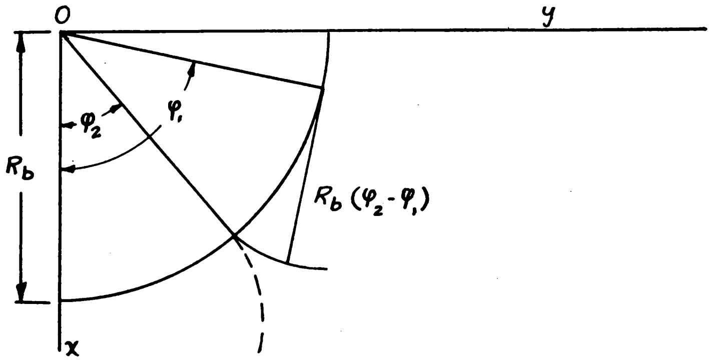

FIG. 1.

Eqs. (1.2) such a rotation corresponds to an increase of amount $\Delta\varphi_2$ both in φ_1 and φ_2.

In parametric form the equations of an involute helicoid S are given by

$$\begin{aligned} x &= R_b\,[\cos\varphi_1 - (\varphi_2 - \varphi_1)\sin\varphi_1], \\ y &= R_b\,[\sin\varphi_1 + (\varphi_2 - \varphi_1)\cos\varphi_1], \\ z &= C\;[\varphi_2 - \varphi_{20}], \end{aligned} \tag{1.3}$$

where φ_{20} is a constant. Indeed a constant value of z corresponds to a value of φ_2, and the increments $\Delta\varphi_2 = \varphi_2 - \varphi_{20}$, $\Delta\varphi_1 = \Delta\varphi_2$, $\Delta z = C\Delta\varphi_2$ correspond to a rotation $\Delta\theta$ in $\Delta\varphi_2$ and an axial translation $C\Delta\theta$ of the involute $\varphi_2 = \varphi_{20}$ in the plane $z = 0$.

In applying the above to the problem of transforming a rotation about one axis A into a rotation about a skew axis A', it will be convenient to set up two fixed systems of coordinates with origins at O, O', the feet of the common perpendicular between A and A': the x, y, z system with the z-axis along the axis A and the x-axis along OO'; and the x', y', z' system with the z'-axis along A' and the x'-axis is along $O'O$. The

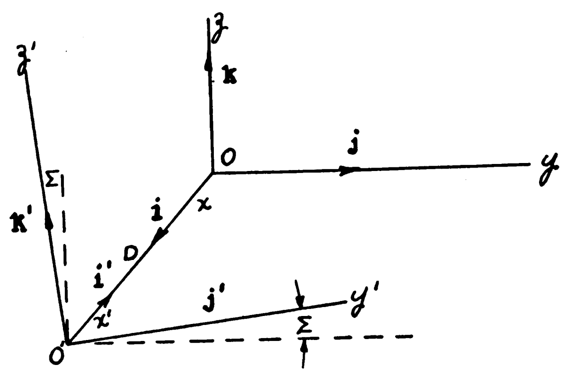

FIG. 2.

x'axis coincides with the x-axis but is reversed in direction and starts from a different origin. The relations between these coordinate systems are shown in Fig. 2, where a right-hand system is used for x, y, z, a left-hand one for x', y', z':

$$\begin{aligned} x' &= D - x, \qquad D = OO' \\ y' &= y \cos \Sigma + z \sin \Sigma, \\ z' &= -y \sin \Sigma + z \cos \Sigma. \end{aligned} \tag{1.4}$$

Here D is the mutual distance between A and A', and Σ is the angle between A and A', to be taken positive between 0 and π. The rotations ω, ω' are positive when right-handed relative to the z-, z'-axes. Unit vectors along the two sets of axes are denoted by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$; $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ respectively; one obtains

$$\begin{aligned} \mathbf{i}' &= -\mathbf{i}, \\ \mathbf{j}' &= \mathbf{j} \cos \Sigma + \mathbf{k} \cos \Sigma, \\ \mathbf{k}' &= -\mathbf{j} \sin \Sigma + \mathbf{k} \sin \Sigma. \end{aligned} \tag{1.5}$$

Denoting by R_b', C' the base radius and the axial displacement per unit rotation of θ' about the z'-axis of the involute helical surface S', one obtains for its equations

$$\begin{aligned} x' &= R_b' [\cos \varphi_1' - (\varphi_2' - \varphi_1') \sin \varphi_1'], \\ -y' &= R_b' [\sin \varphi_1' + (\varphi_2' - \varphi_1') \cos \varphi_1'], \\ z' &= C' [\varphi_2' - \varphi_{20}']. \end{aligned} \tag{1.6}$$

Of course, both equations (1.3) and (1.6) represent *fixed positions* of the respective helical surfaces S, S'.

For the sake of definiteness, it will be assumed that

$$| C/R_b | \geq | C'/R_b' |, \quad (1.7) \qquad R_b \geq R_b' . \quad (1.8)$$

The base radii R_b, R_b' will always be assumed to be positive; on the other hand, either C or C' may be positive or negative corresponding to right-handed or left-handed helical surfaces S, S'.

In Sec. 3 we shall prove the following theorem.

THEOREM A. *If the inequalities*

$$\Sigma < | \lambda_b | + | \lambda_b' |, \quad (1.9) \qquad | \lambda_b' | > | (\lambda_b) - \Sigma | \quad (1.10)$$

hold, it is possible to rotate the surface S' about its axis A' till it comes in point contact with the surface S at a point P_0. If S is then rotated with a uniform velocity ω about its axis A, and if S' is allowed to rotate about its axis A' so as to maintain contact with S, then S' will rotate with a uniform velocity ω' about its axis A', where the constant ratio of velocities is given by

$$\frac{\omega'}{\omega} = \frac{C \cos \lambda_b}{C' \cos \lambda_b'} = \frac{R_b \sin \lambda_b}{R_b' \sin \lambda_b'}, \tag{1.11}$$

$$\tan \lambda_b = C/R_b, \ \tan \lambda_b' = C'/R_b', \ [| \lambda_b | \geq | \lambda_b' | \text{ on account of (1.7)}], \tag{1.12}$$

and is independent both of the distance D between the axes and of their inclination. As the *rotation proceeds, the contact point P of S, S' describes a straight line l in space with constant linear velocity*

$$v_n = C\omega \cos \lambda_b = C'\omega' \cos \lambda_b' \; ; \tag{1.13}$$

this line l is the common normal to both surfaces at the initial contact point P_0, and remains normal to S, S' at P as the rotations progress.

For the limiting cases of the inequalities (1.9), (1.10) the following result is proved in Sec. 3.

THEOREM B. *Let one of the relations*

$$\Sigma = |\lambda_b| + |\lambda_b'|, \quad (1.14) \qquad \Sigma = |\lambda_b| - |\lambda_b'| \quad (1.15)$$

hold, as well as one of the relations

$$D = R_b + R_b', \quad (1.16) \qquad D = R_b - R_b'. \quad (1.17)$$

Then S' can be rotated about A' till it comes in contact with S, the contact being along a straight line g_0. If S is then rotated about A with a uniform velocity ω and S' is rotated about A' so as to maintain contact with S, then S' will rotate with a uniform velocity ω' given by (1.11). *Contact between S, S' is always along a straight line g, parallel to g_0; g moves normally to itself, in the direction of a straight line l normal to both surfaces S, S' at any point P_0 of their initial line of contact g_0, with constant velocity v_n given by* (1.13); *during this motion g describes a plane parallel to A, A' (and containing g_0 and l).*

A further limiting case of Theorem A is obtained by letting the axes A and A' become parallel. In this case the following result is obtained.

THEOREM C. *If*

$$\Sigma = 0, \qquad \lambda_b' = \pm\lambda_b, \qquad D > R_b + R_b', \tag{1.18}$$

the same conclusions hold as in Theorem B.

Two proofs of the above theorems are given in Secs. 3, 4. The proof of Sec. 4 is based upon the following theorem.

THEOREM D. *The Family $\mathfrak{F}$ of involute helicoids S with the same base cylinder and the same lead angle λ_b of the base helix forms a family of* parallel *or* normal *surfaces, that is a family of surfaces possessing the same normals, any two surfaces intersecting each normal in two points at a constant distance apart. These ∞^2 normals are all tangent to the base cylinder and make an angle λ_b with its axis. The members S of $\mathfrak{F}$ can be obtained by rotating S_0, any one member of $\mathfrak{F}$, about the cylinder axis, a rotation $\Delta\theta$ corresponding to a displacement*

$$C\Delta\theta \cos \lambda_b \tag{1.19}$$

of the intersection points along the normals.

In Sec. 3 the following theorems are also proved.

THEOREMS A′, B′, C′. *Under the same conditions as in Theorems A, B, C, after initial contact between S, S' has been secured, let S be simultaneously rotated about its axis A with a constant velocity $\bar{\omega}$ and translated in a direction parallel to A with a constant velocity V,* *and at the same time let S' undergo a simultaneous translation V' in direction of its axis A' and a proper rotation about A' to maintain contact with S. Then the latter rotation is a uniform rotation $\overline{\omega'}$ given by the equations*

$$\omega = \bar{\omega} - V/C, \quad (1.20) \qquad \omega' = \overline{\omega'} - V'/C', \quad (1.21)$$

where ω, ω' satisfy (1.11). *From these equations $\overline{\omega'}$ can be determined in terms of $\bar{\omega}$, V, V'. As the motion proceeds the contact is of the same nature as in the respective Theorems A, B, C for the case $V = V' = 0$, and rotations ω, ω'.*

Another limiting case of interest is obtained by allowing the base radius of one of the two contacting surfaces, say S' to become infinite, while keeping its normal at a fixed point of contact with the other surface. In the limit the surface S' flattens out into a plane, the rotation of S' about its axis degenerates into a translation of the plane S' and one obtains the following theorem.

Theorem E. *Let π be a plane tangent to S at a point P_0. Let S rotate about its axis with a uniform velocity ω, and let π, while remaining parallel to itself, be translated uniformly, normally to itself, with a velocity v_n*

$$v_n = C\omega \cos \lambda_b , \qquad (1.22)$$

then π will remain tangent to S, contact occurring at any time along a straight line g which moves normally to π (and to itself) with a velocity v_n given by (1.22).

The proof of Theorem E is mentioned in Sec. 4. Its physical applications relate to mating of involute helical gears with plane racks.

Applications of the above theorems to physical gears are discussed in Sec. 5. These involve limitations due to finite axial and radial extent of the tooth surfaces, simultaneous contact between corresponding surfaces of several teeth, as well as contact to both sides of the same teeth, and avoidance of interference. The results of Sec. 5 are directly helpful in answering practical questions arising in manufacture of gears and described in the introductory section.

2. Geometry of involute helicoids. Before proceeding with the proof of the results just stated, we shall obtain several useful properties of involute helicoids.

It will be noticed that in (1.3) the coordinates are linear in φ_2. The curves $\varphi_1 =$ constant on S thus reduce to straight lines, and we conclude that an involute helicoid is a *ruled surface*.

The vectors

$$\frac{\partial \mathbf{r}}{\partial \varphi_1} = R_b (\varphi_2 - \varphi_1)(\mathbf{i} \cos \varphi_1 - \mathbf{j} \sin \varphi_1), \qquad (2.1)$$

$$\frac{\partial \mathbf{r}}{\partial \varphi_2} = -\mathbf{i}R_b \sin \varphi_1 + \mathbf{j}R_b \cos \varphi_1 + \mathbf{k}C \qquad (2.2)$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of S, are tangent to the parametric (φ_1, φ_2)-net on S; the former vector is tangent to the involutes $\varphi_2 =$ constant, $z =$ constant; the latter to the generators $\varphi_1 =$ constant. The latter vector, being dependent on φ_1 only, is constant along the generator g in question and gives the direction of g. Reduction of (2.1), (2.2) to unit vectors leads to

$$\mathbf{u} = \mathbf{i} \cos \varphi_1 - \mathbf{j} \sin \varphi_1 , \tag{2.3}$$

$$\mathbf{t} = -\mathbf{i} \sin \varphi_1 \cos \lambda_b + \mathbf{j} \cos \varphi_1 \cos \lambda_b + \mathbf{k} \sin \lambda_b , \tag{2.4}$$

where λ_b is given by (1.11).

It will be noted that

$$\mathbf{u} \cdot \mathbf{t} = 0 \tag{2.5}$$

hence the generators g are normal to the involutes $z =$ constant.

Since the first two equations (1.3) are identical with (1.2), the projections of g on $z = 0$ are identical with the straight lines which are tangent to the base circle $r = R_b$. Hence, the generators of S are tangent to the "base cylinder" $r = R_b$. The point of contact is obtained by putting $\varphi_2 = \varphi_1$, whereupon (1.3) yields

$$x = R_b \cos \varphi_1 , \qquad y = R_b \sin \varphi_1 , \qquad z = C(\varphi_1 - \varphi_{20}). \tag{2.6}$$

This is a helix lying on the base cylinder $r = R_b$ and belonging to the screw motion of the helicoid. Its "lead angle", that is the angle it makes with the planes $z =$ constant, is equal to arc tan C/R_b that is to λ_b. Taking differentials of (2.6) one obtains

$$d\mathbf{r} = R_b \left[-\mathbf{i} \sin \varphi_1 + \mathbf{j} \cos \varphi_1 + \mathbf{k}C/R_b\right] d\varphi_2 \tag{2.7}$$

for the tangent vector element to the helix 2(6); comparison of (2.7), (2.4) shows that both vectors are parallel, so that g is tangent to (2.6). Hence, all the generators are tangent to the helix (2.6); the latter is known as the "base helix." Thus S is a developable surface consisting of the tangents to the base helix. Therefore, S has zero (Gaussian) curvature. One of the directions of principal curvature on S corresponds to that of the generator g, with zero as the corresponding value of the principal curvature, the other principal curvature direction corresponds to the involutes sections $z =$ constant. The generators and the involutes form the lines of curvature on S.

The normal unit vector to S is given by the cross-product $\mathbf{t} \times \mathbf{u}$:

$$\mathbf{n} = \mathbf{t} \times \mathbf{u} = \mathbf{i} \sin \varphi_1 \sin \lambda_b - \mathbf{j} \cos \varphi_1 \sin \lambda_b + \mathbf{k} \cos \lambda_b . \tag{2.8}$$

Since $\mathbf{n}$ is independent of φ_2, it follows that along every point of a generator g the normal to S has the same direction—a characteristic property of developable ruled surfaces.

Since $\mathbf{t}$ is also the unit tangent vector to the base helix, its derivative

$$\frac{d\mathbf{t}}{d\varphi_1} = -(\mathbf{i} \cos \varphi_1 + \mathbf{j} \sin \varphi_1) \cos \lambda_b \tag{2.9}$$

is readily seen to yield the direction of the principal normal of the base helix. Since $\mathbf{n}$ is normal both to $\mathbf{t}$ and to the principal (2.9), it follows that $\mathbf{n}$ is parallel to the binormal to the base helix at its point of contact with g.*

From (2.8) it will be noted that

$$\mathbf{n} \cdot \mathbf{k} = \cos \lambda_b ; \tag{2.10}$$

hence the angle between $\mathbf{n}$ and the direction of the axis A is λ_b. Thus, if from a fixed point F the various unit normals $\mathbf{n}$ to S be drawn, they will form a cone C_n of semi-

*This property, too, is characteristic of developable ruled surfaces, the binormal to the base curve C being normal to each osculating plane, and therefore normal to the ruled surface; the latter consists of tangents to C which is the envelope of the osculating planes.

angle λ_b with axis parallel to A. Similarly the unit vectors **t** giving the directions of the generators, if drawn from F, will outline a similar cone C_g, of semi-angle $\pi/2 - \lambda_b$. It will be noted that the vectors **k**, **t**, **n** are coplanar; hence, corresponding vectors **n**, **t** on C_n, C_g are mutually perpendicular and lie in the same plane through the axis. Figure 3 shows these relations on a unit sphere.

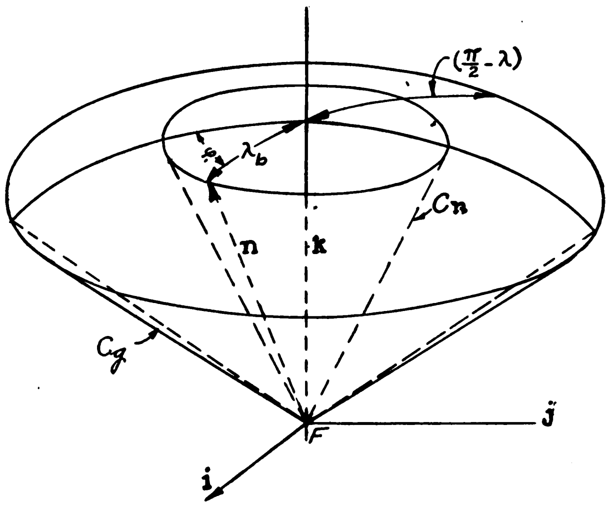

FIG. 3.

3. Contact between two involute helicoids. Contact between the two surfaces S, S' can only occur at a point at which the normal to the one surface is also normal to the other. As shown above, the unit normals **n** to S, when drawn from a fixed point F, form a cone C_n of semi-angle λ_b with axis parallel to the axis A. Likewise the unit normals to S',

$$\mathbf{n}' = \mathbf{i}' \sin \varphi_1' \sin \lambda_b' + \mathbf{j}' \cos \varphi_1' \sin \lambda_b' + \mathbf{k}' \cos \lambda_b', \tag{3.1}$$

when drawn from the same point F, will form a cone C_n' of semi-angle λ_b' with axis parallel to A' which is parallel to

$$\mathbf{k}' = \mathbf{i} \cos \Sigma - \mathbf{j} \sin \Sigma, \tag{3.2}$$

where λ_b' is the lead angle of the generating base helix of S'. If C_n, C_n' intersect, each intersection of the two cones furnishes a possible direction for the common normal, and contact between S, S' can only occur at points at which both surfaces are normal to that direction. Figure 4 shows the two cones C_n, C_n' and their intersections FE, FE.

The rotation of the surface S about A leaves the cone C_n in Fig. 4 unchanged as a whole, though the normals rotate along the cone; similarly for rotation of S' about A' and the cone C_n'. The direction of the common normal to both mating surfaces at the point of contact, if any, thus remains invariant with time.

Several cases present themselves in regard to the cones C_n, C_n'.

I. The cones C_n, C_n' lie outside each other.

II. One of the cones C_n, C_n' lies inside the other one; on account of (1.7) C_n' will lie inside C_n.

III. The cones are tangent externally.

IV. The cones are tangent internally.
V. The cones intersect in two distinct lines.
VI. The cones coincide.

In Case I the inequality (1.9) fails to be satisfied; in Case II the inequality (1.15) holds; finally, in Case V, (1.9), (1.10) are both satisfied. Only in Cases III, IV, V, VI

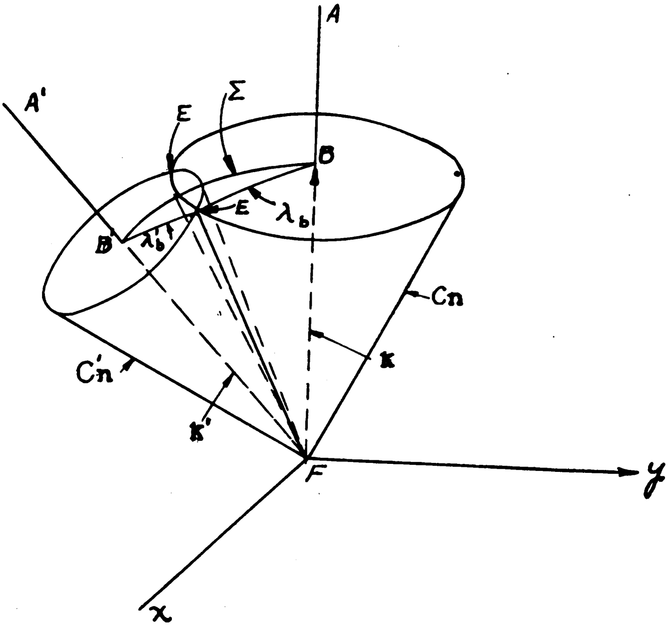

Fig. 4.

is contact between S, S' possible; III, IV lead to case discussed in Theorem B, Case V leads to the case covered in Theorem A, while VI leads to Theorem C.

The nature of contact between S and S' is exhibited in

Table I

Case	Theorem	Contacting Generators	Type of Contact
I		None	No contact
II		None	No contact
III	B	Parallel	Line contact
IV	B	Parallel	Line contact
V	A	Not parallel	Point contact
VI	C	Parallel	Line contact

Indeed, in Cases I, II, no common normal exists over S, S' — hence no contact between them is possible. In Case V, the common normal **n** (FE on Fig. 4) yields possible contacting generators g on S and g' on S' as the locus of points where the surface elements of S, S' are parallel. As indicated at the end of Sec. 2, the vector **t** giving the direct of g, is coplanar with **n**, **k** and 90° away from **n**. It therefore differs from **t**′, the unit vector along g', which vector is coplanar with **n**, **k**′ and 90° away from **n**. Hence, if contact can be obtained between g and g', this contact is at a point only. In Cases III, IV, as shown on Fig. 5, the vectors **n**, **k**, **k**′, are coplanar and **t**, **t**′ are coincident, leading to parallel (possible) contacting generators. If contact can be secured in these cases, it will be along a line. The same statement applies in Case VI, though now the two cones C_n, C_n' of Fig. 4 degenerate into the same cone; whatever common normal one picks along this cone, one obtains the same conclusion as in Cases III, IV.

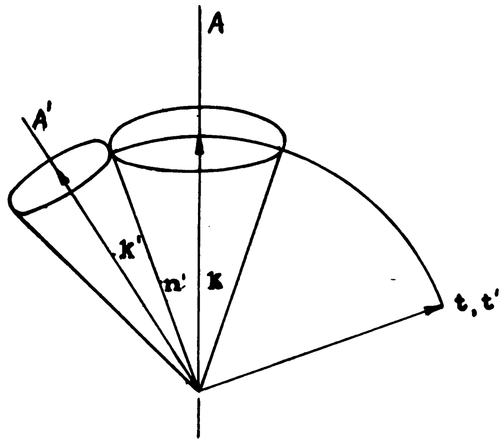

FIG. 5a.

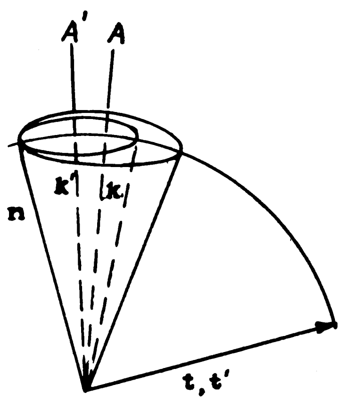

FIG. 5b.

So far it has been shown that *if* contact occurs it is of the type shown in Table I. It will now be shown that under the conditions stated in Theorems A, B, C contact between S and S' can always be secured by a proper rotation of S'.

From (1) follows that a rotation of a helical surface about its axis is equivalent to a proper translation parallel to that axis. Thus, a positive rotation of S about the z-axis through an angle $\Delta\theta$ is equivalent to a translation of S in the direction of (positive) z of amount

$$\Delta z = -C\Delta\theta. \tag{3.3}$$

Similarly, a rotation of S' about A' of angle $A{\cdot}\theta'$ is equivalent to a translation in the direction of positive z' of amount

$$\Delta z' = -C'\Delta\theta'. \tag{3.4}$$

Likewise, a continuous rotation of S about A, of angular velocity ω is equivalent to a translation in direction of positive z of magnitude

$$V = -C\omega, \tag{3.5}$$

and similarly a continuous rotation ω' of S' about A' is equivalent to a translation

$$V' = -C'\omega' \tag{3.6}$$

in the directions of positive z'.

Consider Case V first. Suppose that g_0, g_0', the initial possible contacting generators of S, S', fail to intersect. Since, as shown above, g_0' is not parallel to g_0, and since neither

g_0 nor g_0' is parallel to A'—this follows readily from Figs. 3, 4 and the equations of Sec. 2—it is always possible to translate S' and g_0' along with it till it intersects g_0 at a point P_0. Thus in Case V, it is possible to secure contact between S' and the initial position of S by properly translating S' about A', and hence also by rotating it through a proper angle about A'. Similar considerations show that contact can be maintained as rotation of S proceeds.

The directions of $\mathbf{n}$ and hence of g, g' remain invariant with time as S, S' rotate. Figure 6 shows the contacting generators g for S in the positions at which they make

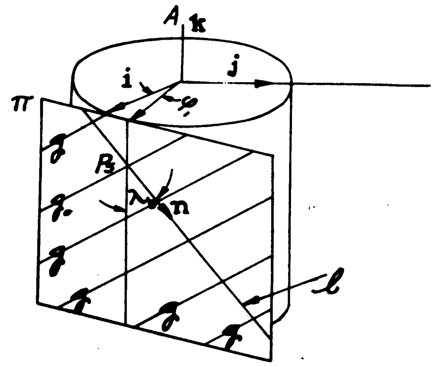

FIG. 6.

contact with corresponding generators g' or S'. If the position of the contacting generator g_0 of S is given at the time $t = 0$, the subsequent positions of g may be obtained by displacing it in the direction of the axis A with velocity (3.5). A similar statement applies to the contacting generator g' of S' where the displacement is in the direction of z'-axis with velocity (3.6).

As shown in Sec. 2, the generators of S are tangent to the base cylinder $r = R_b$. Therefore, as S rotates about its axis, the locus described by the contacting generator g is a plane π, through the initial position g_0 of g, parallel to A, and tangent to $r = R_b$. The plane π, in addition to being the locus of g, also contains the normals $\mathbf{n}$ to S at the points of contact P, since as shown on Fig. 3 $\mathbf{n}$, $\mathbf{k}$, $\mathbf{t}$ are coplanar. Likewise, the contacting generator g' of the surface S', as the latter rotates, describes a plane π' tangent to the base cylinder of S' and containing the directions of A', of g', and the normal $\mathbf{n}$. The line of intersection l of π, π' forms the locus of the contact point P. Since each of the two planes π, π' is parallel to $\mathbf{n}$, their line of intersection l is itself parallel to $\mathbf{n}$.

The line l is tangent to the base cylinder of S at P_S and normal to its base helix there. Being parallel to $\mathbf{n}$, l is normal to S at the contact point as S undergoes its rotation or equivalent displacement. The angle between l and the z-axis is the same as between $\mathbf{n}$ and $\mathbf{k}$, hence equal to λ_b. Hence, the translation (3.5) of the contacting generator g leads to a motion of the contact point P along l with velocity

$$V \cos \lambda_b = -C\omega \cos \lambda_b . \tag{3.7}$$

Similarly, the translation (3.6) of the contacting generator of S' leads to the motion of the contact point along l with velocity

$$V' \cos \lambda_b' = -C'\omega' \cos \lambda_b' . \tag{3.8}$$

Equating the two velocities and denoting their common value by v_n , one arrives at (1.13) and (1.11).

This essentially completes the proof of Theorem A.

The cone C_n of unit normals $\mathbf{n}$ to S is given by (2.8) with variable φ_1 . The intersections of the two cones C_n , C_n' are obtained by imposing the condition

$$\mathbf{n}\cdot\mathbf{k}' = \cos\varphi_1 \sin\lambda_b \sin\Sigma + \cos\lambda_b \cos\Sigma = \cos\lambda_b' . \tag{3.9}$$

Solving for φ_1 and substituting in 2(8) yields the two possible common normals to S and S'.

Equating $\mathbf{n}'$ in 3(1) to $\mathbf{n}$ in 2(8) and expressing $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in terms of $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ one is led to the following relations:

$$\begin{aligned} \sin\varphi_1' \sin\lambda_b' &= -\sin\varphi_1 \sin\lambda_b , \\ \cos\varphi_1' \sin\lambda_b' &= -\cos\varphi_1 \sin\lambda_b \cos\Sigma + \cos\lambda_b \sin\Sigma, \\ \cos\lambda_b' &= \cos\varphi_1 \sin\lambda_b \sin\Sigma + \cos\lambda_b \cos\Sigma. \end{aligned} \tag{3.10}$$

The third of these relations is the same as (3.9), and will be recognized on the spherical triangle BEB' of Fig. 4 and Fig. 7 as the law of cosines for side $\lambda_b = EB$. The first equation (3.10) determines φ_1' in terms of φ_1 , λ_b , λ_b' , and forms the law of sines for the same triangle for the angles B, B' and the sides opposite.

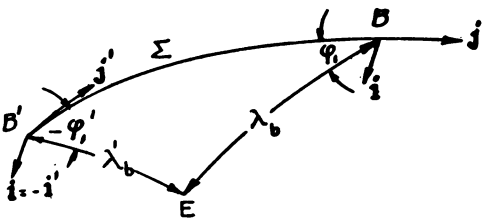

FIG. 7.

Substitution of the value of φ_1 obtained from (3.9) in (2.6) yields the point on the base helix of S in the plane π. Since this plane is tangent to $r = R_b$ at this point, its equation is

$$\pi: \quad x \cos\varphi_1 + y \sin\varphi_1 = R_b . \tag{3.11}$$

Both the contacting generator g and the line of contact l lie in π. The line l and g' lie in the plane π' whose equation is

$$\pi': \quad x' \cos\varphi_1' - y' \sin\varphi_1' = R_b' , \tag{3.12}$$

where φ_1' is determined by means of (3.10). Utilizing (1.4) one changes (3.12) into

$$x \cos\varphi_1' + y \cos\Sigma \sin\varphi_1' + z \sin\Sigma \sin\varphi_1' = D \cos\varphi_1' - R_b' . \tag{3.13}$$

Turn now to Cases III, IV. Referring to Figures 5, we now have for the common normal

$$\mathbf{n} = -\mathbf{j} \sin\lambda_b + \mathbf{k} \cos\lambda_b \tag{3.14}$$

corresponding to $\varphi_1 = 0$ in (2.8), and $\varphi_1' = 0$ in (3.1). All the vectors $\mathbf{k}$, $\mathbf{k}'$, $\mathbf{n}$, $\mathbf{t}_2 = \mathbf{t}_2'$

lie in the plane normal to $\mathbf{i}$. Hence both planes π and π', the locus of the possible contacting generators, are parallel; in fact their equations (3.11), (3.13) now reduce to

$$\pi: \; x = R_b \,, \quad (3.15) \qquad \pi': \; x = D \pm R_b' \,. \quad (3.16)$$

Clearly, unless π, π' coincide, no amount of translation of S' along A', or of equivalent rotation about A', will bring g' into contact with g. On the other hand, if π and π' do coincide, thus leading to (1.16) or (1.17), a proper translation of S' along A' will bring the possible contacting generator g_0' into coincidence with g_0. From this point on, the rest of Theorem B is proved similarly to Theorem A.

As regards Case VI and parallel axes A, A', one now finds that with coincident cones C_n, C_n' the degenerate Fig. 4 or equation (3.9) fails to pick common normals or possible contacting generators. Whatever normal $\mathbf{n}$ is assumed it leads to $\mathbf{t}_2 = \mathbf{t}_2'$, to parallel possible contacting generators g, g', and to parallel planes π, π'. For contact to occur, the planes π, π' must reduce to the common tangent planes of the two base cylinders:

$$\frac{R_b}{\cos \varphi_1} \pm \frac{R_b'}{\cos \varphi_1} = D. \tag{3.17}$$

With φ_1 determined from (3.17) the rest of the proof of Theorem C follows along the same lines as above. The last inequality (1.18) prevents the base cylinder $r = R_b$ from enclosing the other one, thus rendering common tangent planes non-existent. Contact occurs now not at a point but along a line g, and as rotation proceeds g describes the plane π which is identical with π' and is given by (3.17).

Theorems A', B', C' are proved by utilizing the equivalence of rotations and translations.

4. Alternative proof based on theorem D. To establish Theorem D, it will now be shown that the normal l to the surface S at a point P_0 on it is also normal to each involute helicoid of the family $\mathfrak{F}$, given by (3.1) for other values of the constant φ_{20}, at its point of intersection with that surface. The members of the family $\mathfrak{F}$ can be obtained by translating S along the z-axis through a variable distance Δz, or by rotating S about the z-axis through a variable angle $\Delta\theta$. They all have the same base cylinder and the same base lead angle λ_b, but different base helices.

Let P_0 be a point on S obtained by putting $\varphi_1 = \varphi_1^0$, $\varphi_2 = \varphi_2^0$ in (1.3):

$$\overrightarrow{OP}_0 = \mathbf{r}_0 = \begin{cases} R_b\,[\cos \varphi_1^0 - (\varphi_2^0 - \varphi_1^0) \sin \varphi_1^0], \\ R_b\,[\sin \varphi_1^0 + (\varphi_2^0 - \varphi_2^0) \cos \varphi_1^0], \\ C[\varphi_2^0 - \varphi_{20}]. \end{cases} \tag{4.1}$$

The unit normal along the line l, normal to S at P_0 is obtained from (2.8):

$$\mathbf{n} = (\sin \varphi_1^0 \sin \lambda_b \,, \qquad -\cos \varphi_1^0 \sin \lambda_b \,, \qquad \cos \lambda_b). \tag{4.2}$$

Now consider the point P on l a distance s from P_0:

$$\overrightarrow{0P} = \mathbf{r}_0 + s\mathbf{n}. \tag{4.3}$$

The coordinates of P can also be obtained from (1.3) by increasing φ_1, φ_2, φ_{20} from their values at P_0 by the amounts

$$\Delta\varphi_1 = 0, \qquad \Delta\varphi_2 = -(s/R_b) \sin \lambda_b , \qquad \Delta\varphi_{20} = -(s/C) \sec \lambda_b \tag{4.4}$$

and P thus lies on a surface of the family $\mathfrak{F}$. The normal to this surface at P, moreover, agrees with (4.2) since $\varphi_1 = \varphi_1^0$. Thus l is normal to the helical involute surface passing through P.

From the above it follows that the family $\mathfrak{F}$ is a family of *parallel* surfaces admitting the same normals, each surface being a constant distance apart from every other surface. The directions of principal curvatures along such surfaces at their points of intersection with the same normal are parallel and the principal radii of curvatures at these points differ by the constant normal distance.

The common normals l to this family $\mathfrak{F}$ of parallel surfaces are all tangent to the base cylinder and make an angle with the direction of its axis A equal to λ_b . Stated in another way, it has been shown that *the surfaces perpendicular to the lines which are tangent to the cylinder* $r = R_b$ *and make a fixed angle* λ_b *with its axis form a family of involute helicoids with their base helices on* $r = R_b$ *and of lead angle* λ_b .

Suppose next that S rotates about its axis A with uniform velocity ω. It will be shown that its intersection with the normal l at the point P_0 moves with uniform velocity

$$v = -C\omega \cos \lambda_b . \tag{4.5}$$

Indeed, putting

$$s = vt = -C\omega t \cos \lambda_b = -R_b\omega t \sin \lambda_b \tag{4.6}$$

in (4.3), (4.4) leads to the above point P on the normal; it is readily verified that the same point is obtained if in (1.3) we first put

$$\varphi_1 = \varphi_1^0 - \omega t, \tag{4.7}$$

$$\varphi_2 = \varphi_2^0 - \omega t \cos^2 \lambda_b \tag{4.8}$$

(while keeping φ_{20} fixed), then, corresponding to the rotation ωt, we increase φ_1 , φ_2 by ωt in the first two Eqs. (1.3), thus putting

$$\varphi_1 = \varphi_1^0 , \tag{4.9}$$

$$\varphi_2 = \varphi_2^0 + \omega t \sin^2 \lambda_b \tag{4.10}$$

there, while allowing (4.8) to hold in the third equation.

There results

$$\begin{aligned} \Delta x &= -R_b\omega t \sin^2 \lambda_b \sin \varphi_1^0 , \\ \Delta y &= R_b\omega t \sin^2 \lambda_b \cos \varphi_1^0 , \\ \Delta z &= -C\omega t \cos^2 \lambda_b \end{aligned} \tag{4.11}$$

and these agree with $s\mathbf{n}$ for s given by (4.6).

Thus a uniform rotation of S leads to a point of intersection with the normal l which moves with a uniform velocity along l.

By applying this result to two involute helical surfaces S, S' initially in contact at a point or a line, Theorems A, B, C follow.

Theorem E follows readily from D by considering a rotation of S, its intersection with a fixed normal line l and the tangent plane to S which is normal to l at that point of intersection.

5. Application to gears. In applying the results of the preceding sections to two mating gears G, G', certain limitations occur due to the finite dimensions of the gears, to non-interference of mating teeth, and to the integral number of teeth on each gear.

Let the gears G, G' possess an integer number of teeth, N, N' respectively, and suppose that the mating teeth are properly shaped so that a uniform rotation of G is transformed into a uniform rotation ω' of G'. After a time T corresponding to N' complete rotations of G, the number of teeth of G that have gone through mesh is NN'. During the same time T a like number NN' of teeth of G' has gone through mesh; hence G' has gone through N complete rotations. Thus, the number of complete rotations of G, G' in the time T varies inversely as the number of teeth, and hence

$$\frac{\omega'}{\omega} = \frac{N}{N'}. \tag{5.1}$$

For involute helicoid tooth shape, combining (5.1) with (1.11), one obtains

$$\frac{\omega'}{\omega} = \frac{N}{N'} = \frac{C \cos \lambda_b}{C' \cos \lambda_b'} = \frac{R_b \sin \lambda_b}{R_b' \sin \lambda_b'}. \tag{5.2}$$

This is a restriction on the design brought about by the integral numbers of teeth.

In actual gears the radii r, r' of the involute surfaces are limited by the dedendum and addendum radii, and continuity of rotations is assured by having *several teeth* in mesh part or all of the time. This is obtained by having the angle of contact exceed $2\pi/N$. Since, in accordance with Theorem A, the point of contact moves with velocity (1.13) along the line of contact l, this condition is assured by having a segment on the line of contact greater than

$$\frac{2\pi}{N} C \cos \lambda_b = d \tag{5.3}$$

lie both within the limits of r, r' corresponding to the mating teeth and within the z, z' limits corresponding to the face width. For involute helicoid gears satisfying (5.2), *no new* conditions are required for assuring correct *simultaneous contact* between corresponding surfaces of several teeth. Two involute surfaces S spaced $2\pi/N$ radians apart, according to Sec. 4, are at a constant (normal) distance apart, equal to d. Hence, condition (5.2) assures one that for two adjacent tooth surfaces on G' the distance d' agrees with the corresponding d for the mating surfaces of S; hence, simultaneous contact takes place without interference.

If the gears have no backlash, contact on *opposite sides* of various teeth occurs, and the question of interference between the mating action of opposite sides arises. These opposite sides on S are characterized by the respective inequalities

$$\varphi_2 > \varphi_1 , \tag{5.4}$$

$$\varphi_2 < \varphi_1 \tag{5.5}$$

in Eqs. (1.3) (see Fig. 8). This is evident on Fig. 1 where the cusp of the involute is seen to correspond to

$$\varphi_1 = \varphi_2 . \tag{5.6}$$

The theorems and proofs of Sec. 1–4 now have to be reviewed from point of view of the inequalities (5.4), (5.5) and the corresponding inequalities

$$\varphi_2' > \varphi_1' , \tag{5.7}$$

$$\varphi_2' < \varphi_1' . \tag{5.8}$$

in Eqs. (1.6) for the surfaces of the teeth of S'.

As stated in Sec. 1, only if one of the inequalities (5.4), (5.5) holds will the orthogonal net of involutes $\varphi_2 =$ const., and the tangents to the base circle, $\varphi_1 =$ const., cover

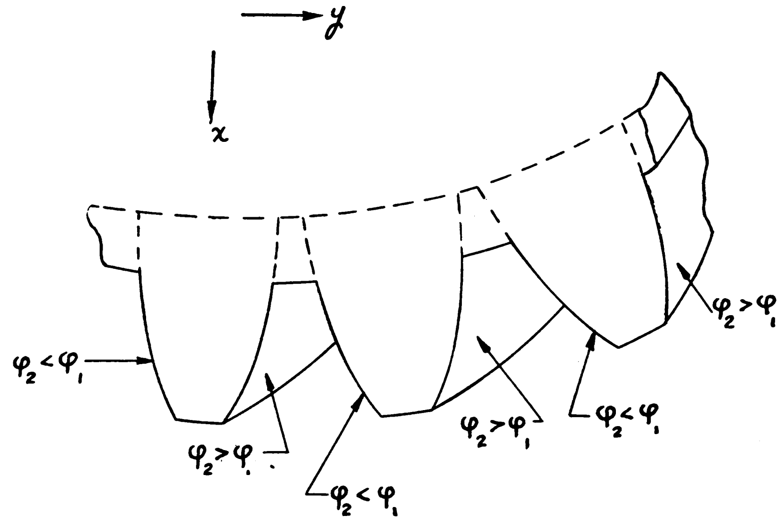

Fig. 8.

the region outside the base circle so that through each point there passes one and only one involute, and one and only one tangent line. Likewise, only if *one* of the inequalities (5.4), (5.5) holds does there result a family of involute helicoids $\mathfrak{F}$ such that through each point outside the base cylinder there passes one and only one member of this family. The two involute helicoids through a point satisfying (5.4), (5.5) respectively have distinct normals. In particular, the two intersections of the cones C_n, C_n' on Fig. 4 correspond respectively to upper and lower faces of gear teeth of G. These distinct normals are obtained from (3.9) by choosing the positive and the negative values of φ_1. Each one of the normals leads to a distinct line l and distinct planes π, π'. However, on account of the inequalities (5.4), (5.5), (5.7), (5.8), contact along each line l is restricted to a portion lying to the proper side of each of the two base cylinders. (In practice this segment is even further restricted by the limits of r, r'; z, z' of the teeth of G, G'.)

As an example, consider the technically important case where one of the two contact lines l cuts the x-axis, that is where the line of contact intersects the common perpendicular of the axes A, A'. Putting $y = 0$, $z = 0$ in (3.11), (3.13), one obtains for the resulting contact point

$$x = R_b/\cos \varphi_1 , \qquad y = 0, \qquad z = 0 \tag{5.9}$$

as well as

$$x' = (D - x) = R_b'/\cos \varphi_1' , \qquad y' = 0, \qquad z' = 0, \tag{5.10}$$

whence

$$D = \frac{R_b}{\cos \varphi_1} + \frac{R_b'}{\cos \varphi_1'} . \tag{5.11}$$

The last restriction on the distance D furnishes the condition for the intersection of l and the x-axis. It will be noted now that if (5.11) is satisfied, it will still be valid if the signs of φ_1, φ_1' are changed. Hence, *both* lines of contact l will intersect the x-axis if one of them does.

Substituting (5.9) in (1.3) one obtains

$$\varphi_2 = \varphi_1 - \tan \varphi_1 . \tag{5.12}$$

Hence the sign of φ_2 is opposite to that of φ_1, so that if $\varphi_1 > 0$ Eq. (5.5) holds, while for $\varphi_1 < 0$ Eq. (5.4) applies. If we suppose further that G, G' are right handed and $\Sigma < \pi/2$ it will be seen that the line l for which $\varphi_1 > 0$ corresponds to contact between an upper face of G and a lower face of G'. This checks with the first Eq. (3.10) according to which φ_1' is now negative. The line l with $\varphi_1 < 0$ yields the contact between a lower face of G and an upper face of G'. Both lines l pass through the point (5.9), which is now known as the "common pitch point" of G, G'—while their directions are given by (2.8) for the proper value of φ_1.

The function $\tan \varphi - \varphi$ is sometimes denoted by "inv φ" (pronounce the "involute function of φ"), on account of its frequent occurrence in involute theory.

It is customary to introduce the "pitch radii"

$$R_p = R_b/\cos \varphi_1 , \quad (5.13) \qquad R_p' = R_b'/\cos \varphi_1' , \quad (5.14)$$

as the distances from the common pitch point to the axes A, A'. The helices of S, S' corresponding to the pitch radii are known as the "pitch lines." The lead angle of the pitch line of S is denoted by λ_p:

$$\tan \lambda_p = C/R_p = (C/R_b)(R_b/R_p) = \tan \lambda_b \cos \varphi_1 \tag{5.15}$$

and λ_p' is introduced similarly.

The normal projection of (either) common normal $\mathbf{n}$ on a plane x = constant, it will be noted from (2.8), makes an angle

$$\tan^{-1} (\sec \varphi_1 \tan \lambda_b) = \lambda_p \tag{5.16}$$

with the z-axis; it is thus normal to the pitch line of S; likewise, it is normal to the pitch line of S' through the same point. There is thus contact between the pitch lines of S, S' through the pitch point (5.9) at the instant of contact of S, S' at that point. This condition leads to

$$\Sigma = \lambda_p + \lambda_p' \tag{5.17}$$

—a relation that also follows from (3.10) by expressing λ_b, λ_b' in terms of λ_p, λ_p'. The angle $|\varphi_1|$ itself is now readily shown to be equal to the "pressure angle" at the pitch radius in the transverse section $z = 0$, that is to the angle between the radial direction and the involute through the pitch point. The pressure angles $|\varphi_1|$, $|\varphi_1'|$, the "lead angles" λ_p, λ_p', and the pitch radii R_p, R_p' are the quantities technically used in specifying helical gears.

For the case of pitch-point contact, the condition of simultaneous contact on both sides of the teeth of G, G' is given by

$$\frac{(\Delta\theta)_p}{2\pi/N} = 1 - \frac{(\Delta\theta')_p}{2\pi/N'} \tag{5.18}$$

where $(\Delta\theta)_p$, $(\Delta\theta')_p$ are the angular thicknesses of the teeth G, G' at their pitch radii; thus, $(\Delta\theta)_p$ is the increase in θ along $r = R_p$, $z = 0$ in passing from one side of a tooth of G to the other side. This equation is obtained by noting that since now the two lines of contact l both pass through the common pitch point (5.9), the time that it takes a tooth of G to pass over this point is equal to the time it takes the space between two adjacent teeth of G' to sweep over the same point.

To obtain a relation equivalent to (5.18) but involving angular tooth thickness at other radii than the pitch radius, the variation of tooth thickness with r is obtained from Eqs. (1.2), keeping in mind that along an involute φ_2 is constant. From these equations follow

$$r = R_b[1 + (\varphi_2 - \varphi_1)^2]^{1/2}, \tag{5.19}$$

$$\tan\theta = \tan\varphi_1 + (\varphi_2 - \varphi_1) = \varphi_2 + \text{inv}\,\varphi_1 \tag{5.20}$$

and after determining the change in φ_1 corresponding to a change in r, one may then obtain the corresponding change in θ from (5.20). Multiplication by 2 then leads to the change in tooth thickness.

To examine the case when (5.11) is not satisfied it is convenient first to shift the axis A', while keeping it parallel to itself, to a position for which (5.11) does hold, then move it back a distance

$$\Delta D = D - R_p - R_p' = D - R_b/\cos\varphi_1 - R_b'/\cos\varphi_1' \tag{5.21}$$

to its original position. Since the cones C_n, C_n' of Fig. 4 are unchanged by this displacement, the direction of $\mathbf{n}$, the values of φ_1, φ_1' which satisfy (3.9), as well as the directions of l and the possible contacting generators g, g' are not affected, except, of course, for the relative displacement of the latter. Comparing the solutions of (3.11), (3.12) for A' and for its shifted position, it will be noted that a solution is offered by the same x, y as for the pitch-point contact case, provided that z be increased by the amount

$$\Delta z = \Delta D \cot\varphi_1'/\sin\Sigma. \tag{5.22}$$

It will be noted that corresponding to a change of sign of φ_1 there is also a change of sign of Δz, that is that the *two* contact lines l are displaced in *opposite* directions from the point (5.10) as D is changed in value.

Let

$$n_x x + n_y y + n_z z = p \tag{5.23}$$

be the equation of the common tangent plane at a point of contact of the two gears G, G', where n_x, n_y, n_z are the components of the normal to the plane, and p the distance of the plane from the origin. Suppose now that G' undergoes a displacement $\Delta D\mathbf{i}$, thus throwing the gears out of contact, and displacing the tangent plane to G' from (5.24) to

$$n_x(x - \Delta D) + n_y y + n_z z = p, \tag{5.24}$$

or, transposing, to

$$n_x x + n_y y + n_z z = p + n_x \Delta D. \tag{5.25}$$

Since φ_1, φ_1' are not effected by the displacement $\Delta D\mathbf{i}$, the possible contacting generator g' undergoes the same displacement as G', while g remains unchanged. Now note that (5.25) may also be obtained by increasing z by the amount

$$\Delta z = \Delta D\, n_x/n_z\ ; \tag{5.26}$$

hence contact may be restored by displacing G in the direction of positive z by the amount (5.26). Substituting from (2.8) for n_x, n_z and recalling that a displacement of G in the direction of its axis A is equivalent to a proper rotation about A, it follows that

$$\frac{\Delta D}{C} \sin \varphi_1 \tan \lambda_b \tag{5.27}$$

is the rotation of G required to close up the gap due to the displacement ΔD of G, and restore contact with G'. Change of sign of φ_1 corresponding to contact on the other tooth side causes the rotation (5.27) to change in sign. Hence

$$\frac{2\Delta D}{C} \sin \varphi_1 \tan \lambda_b = \frac{2\Delta D}{R_b} \sin \varphi_1 \tag{5.28}$$

is the backlash between G, G' (measured as a rotation of G about its axis) produced by the displacement $\Delta D\mathbf{i}$ from initial zero backlash position. Again, (5.28) is the increased angular thickness of a tooth of G caused by a displacement $\Delta D\mathbf{i}$ of a generator tool (involute hob, involute shaving tool) having the shape of G'.

Similarly it is shown that the same backlash is measured by the rotation

$$\frac{2\Delta D}{C'} \sin \varphi_1' \tan \lambda_b' = \frac{2\Delta D}{R_b'} \sin \varphi_1' \tag{5.29}$$

of G' about its axis; this is the required thickening of teeth of G' to produce the same thickness tooth on G after the displacement $\Delta D\mathbf{i}$ has taken place.

These equations along with (5.18) and the general theory outlined above, suffice for the design of hobs and shaving tools for producing involute helical teeth and for investigating the effect of displacement of axes and angles, of resharpening dulled tools, etc.

As an example, suppose two external gears G, G', designed for pitch-point contact, are mounted so that their axes A, A' are at the correct distance D apart but with a wrong relative inclination, so that the angle $\overline{\Sigma}$ between A, A' is incorrect and (5.17) does not hold for $\overline{\Sigma}$. What effect does this have on the mating action of the gears?

Since Eqs. (5.3), (5.3) are not affected by the angle between the axes, correct mating action with the proper ratio of speeds ω/ω' is still possible for *either* one side of the teeth or the other, but the matter of contact on *both* sides or of backlash or interference due to insufficient space between the teeth still remains to be examined. In the spherical triangle $BB'E$ of Figs. 4, 7 the sides λ_b, λ_b' are unaffected, but with Σ now changed to $\overline{\Sigma}$ new values $\overline{\varphi}_1$, $\overline{\varphi}_1'$ for φ_1, φ_1' result; these may be determined from the last and the first equations (3.10). For these values $\overline{\varphi}_1$, $\overline{\varphi}_1'$ and $\overline{\Sigma}$ it is possible to replace the nominal pitch radii R_p, R_p' by new pitch radii $\overline{R}_p$, $\overline{R}_p'$ determined from (5.13), (5.14) and to consider a new pitch-point contact condition with a distance $\overline{D}$ between the axes determined from (5.11). Assuming a zero backlash condition, and supposing that, say, G' has the correct tooth thickness, one determines its value $(\Delta\bar{\theta}')_p$ at $\bar{R}'_p$, and applying (5.18), one obtains tooth thickness $\Delta\bar{\theta}_p$ at $\bar{R}_p$. Finally, allowing the gear G' to undergo a displacement $\Delta D\mathbf{i} = (D - \bar{D})\mathbf{i}$ till the distance between A, A' has its correct initial value D, one adds (5.28) to $\Delta\bar{\theta}_p$ to obtain the tooth thickness of G at $\bar{R}_p$ for the zero-backlash condition corresponding to D, $\bar{\Sigma}$, and this may be compared with the design tooth thickness of G. If G' corresponds to a hob or to a shaving tool of involute profile, the above analysis shows that if it is set at an incorrect angle $\bar{\Sigma}$ or at a wrong distance D, (or both), a correct involute gear shape will be produced, but possibly a wrong tooth thickness. (There is also, of course, the effect of varying depth of penetration.)

The analysis of Secs. 3, 4 for obtaining contact between mating gears can be regarded as the equivalent of the *first necessary condition* for such contact and avoidance of interference; it is analogous to the vanishing of the first derivative $f'(x)$ for conditions of obtaining a maximum of a function $f(x)$, or more precisely, to the vanishing of the two partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ for a maximum of a function $f(x, y)$ of two variables x, y. In such maxima problems, in addition to the first necessary conditions, there also exist other conditions, for instance, sufficient conditions for a local maximum involving second derivatives. Similarly, in the problem of contact of two physical gears G, G' further conditions have to be examined in addition to the tangency of the tooth surfaces, to avoid interference and assure that at no time is any point inside the gear G trying to occupy a position inside G'.

A case of great technical importance is the case of contact of two external gears under the conditions of Theorem A and Case V, if

$$D > R_b + R'_b, \tag{5.30}$$

and if contact is of the common pitch-point type or "near it", that is reducible to pitch-point contact by a continuous change of D and Σ. It is with this important case in mind that Fig. 2 was drawn, and the directions of axes on it were chosen, the positive directions of x, x' pointing toward the region of contact. For this case the second-order conditions are easily verified by considering the curvatures of the two involute helical surfaces at the contact points. As will be recalled from Sec. 2, an involute helical surface is a ruled surface. For each of the contacting helical surfaces at the point of contact, the lines of zero curvature are the generators g, g', and they lie in the common tangent plane. For external gears in contact under the conditions stated, the centers of curvature for the second principal direction of curvature lie to opposite sides of the common tangent plane. This insures avoidance of interference near the contact point.

Under the condition (1.16) of Theorem B it turns out that the zero curvature directions are coincident (along the coincident rulings g, g'), but that the centers of curvature for the second principal curvature for S, S' lie on the *same* side of the tangent plane. It is therefore impossible to realize the type of contact envisioned in Theorem B between two external gears G, G'. This statement applies quite aside from the interpenetration of the teeth of one gear into the base circle of the other gear near the region of the common perpendicular to their axes (the x-axis).* Even if the active face widths (the limits of z and z' for the involute teeth) for G, G' are restricted to lie sufficiently far from the x-axis to avoid this obvious interference, it still turns out that the contact considered

*To a certain extent this may be avoided by proper undercutting of the teeth.

in Theorem B cannot be realized for external gears. Any attempt to run a pair of external gears under these conditions would merely result in contact of the edge of the teeth of one against the tooth surfaces of the other. Similar statements apply to Theorem A for external gears when the inequality

$$D < R_b + R_b' \tag{5.31}$$

holds.

If (5.2), (5.30) holds, under the conditions (1.18), and if contact is of the common pitch-point type or near it the conditions considered in Theorem C *can* be realized between external gears G, G', and contact always *is* of the common pitch-point type, irrespective of the value of D.

If G' is an external gear and G is an internal one, then the contact conditions of Theorem A generally cannot be realized physically due to interference.†

Thus, in shaving internal involute gears with involute helical cutters, a condition of theoretical interference is obtained. In some cases the amount of this interference is slight enough not to cause any trouble. In other cases the shaving cutter teeth are "relieved" at the ends to avoid interference. This relieving operation, of course, tends to destroy the involute helical shape.

If G is internal and G' external, the conditions of contact of Theorems B, C may be realized in certain cases, though, of course, the condition (5.30) has to be dispensed with for Theorem C.

Even for external gears with (5.30) holding other types of contact than of the "near common pitch-point contact" type are possible. This is not evident from the proofs of Secs. 3, 5, from which it would appear that in case of Theorem A *only two* lines of contact l are possible in the mating action of two gears G, G'. Indeed, Fig. 4 and Eq. (3.10) determine two possible directions of the normal and the values of φ_1, φ_1' in terms of λ_b, λ_b', Σ, while Eqs. (3.11), (3.13) then fix a line l uniquely for each pair φ_1, φ_1'. Actually, however, further contact lines l may exist due to the fact that on Fig. 4 *complete cones* and *not half-cones* C_n, C_n' should be used, that is cones proceeding *both ways* from the vertex F. Indeed, for a given φ_1, λ_b, not only is a possible unit normal $\mathbf{n}$ given by (2.8), but also by its negative. If the signs of both $\mathbf{n}$ and $\mathbf{n}'$ are reversed, two new circles of intersection of C_n, C_n' with the unit sphere are obtained on Fig. 4. The new circle for C_n may intersect the (old) circle for C_n' in two new points, thus giving rise to two further possible directions for the common normals. To obtain these (when they exist) it is necessary to modify Eqs. (3.10) by changing the signs of their right-hand members. With the new values of φ_1, φ_1' so obtained, Eqs. (3.11), (3.13) lead to two new contact lines l.

As a special case, if with the resulting new values of φ_1, φ_1', D is given by

$$D = \left| \frac{R_b}{\cos \varphi_1} - \frac{R_b'}{\cos \varphi_1'} \right|, \tag{5.32}$$

then the two contact lines l pass through a point Q on the line $00'$ lying outside the segment $00'$. The point Q is analogous to the common pitch point P which exists when

†It must be borne in mind that due to elastic yielding at contacting areas (in accordance with the Hertz theory) the purely geometric conclusions regarding what type of contact is or is not physically possible may be softened somewhat.

(5.11) holds. When (5.32) does not hold the new contact may be determined by displacing the lines l through Q by amounts (5.22) in the direction of the z-axis.

For the special case of parallel axes A, A', as pointed out above, the cones C_n, C'_n of Fig. 4 degenerate into coincident cones. Now contact is always of the pitch point type and contact lines passing through both P and Q are possible. On Fig. 9 are shown the projections of the lines l on the plane $z = 0$.

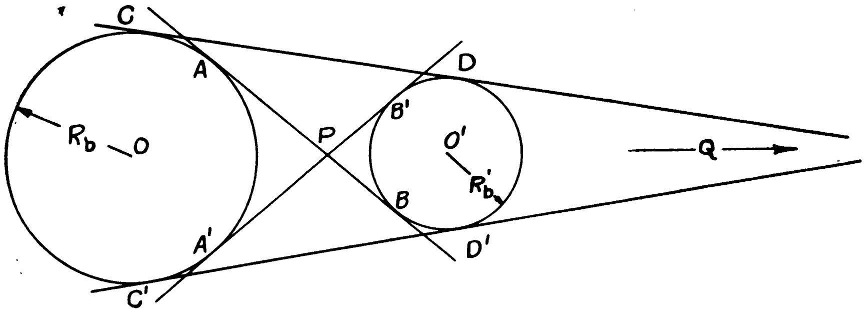

FIG. 9.

The point Q leads to a negative ratio ω/ω'; any attempt to realize contact along the tangent lines CD, $C'D'$ through it by means of external gears is necessarily confined to a portion between the points of tangency with the base circles, and therefore does not include the point Q itself as a contact point. It will be recognized that, due to interference and other difficulties, the angle of rotation will be limited.

Similarly, for helical involute gears and skew axes it is possible to have contact lines l which intersect the common perpendicular between the two axes outside $00'$. While the Theorem A can be applied to this type of contact, it again does not lead to generally useful results.