THE GENERAL VARIATIONAL PRINCIPLE OF THE THEORY OF STRUCTURAL STABILITY*

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1. Introduction. This paper is concerned with the general problem of structural stability in the elastic or plastic range. Two slightly different formulations of this problem are found in the literature. According to the first, one considers a deformable body which, initially, is free from stresses, and which is then subjected to a system of loads of gradually increasing intensity. As long as these loads are sufficiently small the equilibrium configuration which the body assumes under their influence will be stable; one asks for that intensity of the loads for which this equilibrium configuration first becomes unstable. According to the second formulation of the problem of structural stability, one considers a given configuration of a deformable body and an equilibrium system of body and surface stresses and asks whether, in the presence of these *initial* stresses, the given configuration is stable or not. This second point of view is adopted in this paper because:

(1) it clearly separates the stability problem from the problem of finding the stresses produced by the given loads, and

(2) the manner in which the initial stresses are produced is irrelevant for the solution of the stability problem. In particular, it is by no means necessary that the initial stresses are produced by loads which are applied to an otherwise stressfree body; they may be produced by temperature changes or may partly be due to previous overstraining of the body.

Once this second point of view is adopted, stress-strain relations enter into the discussion at one point only: we must be able to predict the infinitesimal changes in stress which correspond to the infinitesimal strains associated with a system of infinitesimal displacements from the considered equilibrium configuration. As the relations between these infinitesimal changes in the stresses and strains are essentially *linear*, the only difference between the elastic and plastic ranges consists in the fact that in the plastic range a different set of coefficients must be used in these linear relations according to whether the change of stress constitutes "loading" or "unloading," while no such distinction need be made in the elastic range.

In Section 2, the general problem of structural stability is reduced to an eigen-value problem for the displacements from a configuration of indifferent equilibrium to a neighbouring configuration of this type. Except for the consideration of plastic deformations, we follow Biezeno and Hencky[1] in this derivation, but simplify the discussion by the systematic use of tensors. In Section 3, a variational principle is derived which is equivalent to the eigen-value problem formulated in Section 2. As an example for the application of this principle, the lateral buckling of an unevenly heated lamina is treated in Section 4.

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[1] C. B. Biezeno and H. Hencky, Proc. Roy. Acad. Amsterdam, **31**, 569–592 (1928).

2. The eigen-value problem associated with the general problem of structural stability. We consider a *given configuration* of a deformable body and an *equilibrium system* of body and surface stresses which is given to within an arbitrary factor λ. *If λ is sufficiently small, this equilibrium configuration will be stable; we ask for that value of λ for which it becomes indifferent, assuming that the additional stresses which are produced by infinitesimal displacements from the given equilibrium configuration are linearly related to the corresponding infinitesimal strains.* This critical value of λ will be called the *safety factor* of the considered equilibrium configuration. With respect to a system of rectangular Cartesian coordinates x_i, let us denote the components of the given stresses by $\lambda\sigma_{ij}$ and the components of an infinitesimal displacement from the given equilibrium configuration by u_i. If the unit vector along the outward normal to the surface is denoted by n_i, the surface stresses are

$$\lambda T_j = \lambda\sigma_{ij}n_i. \tag{1}$$

The quantities σ_{ij} must satisfy the equilibrium conditions

$$\sigma_{ij,i} = 0, \tag{2}$$

where the subscript i after the comma denotes differentiation with respect to x_i, and the usual summation convention regarding repeated subscripts is adopted.

The infinitesimal strain associated with the displacements u_i is given by

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{3}$$

Since the relation between this strain and the corresponding additional stress τ_{ij} is assumed to be linear, we have

$$\tau_{ij} = C_{ijkl}\epsilon_{kl}, \tag{4}$$

where C_{ijkl} is a fourth order tensor which is symmetric with respect to i and j and with respect to k and l. If, in particular, τ_{ij} and ϵ_{ij} are assumed to be related to each other by the generalized law of Hooke, we have

$$C_{ijkl} = 2G_0\left(\delta_{ik}\delta_{kl} - \frac{\nu}{1-2\nu}\,\delta_{ij}\delta_{kl}\right), \tag{5a}$$

where G_0 denotes the modulus of rigidity, ν Poisson's ratio, and δ_{ij} is the Kronecker delta. If the body under consideration can be expected to behave like an isotropic elastic solid for an infinitesimal displacement from the given equilibrium configuration,[2] i.e. if the stresses $\lambda\sigma_{ij}$ do nowhere exceed the elastic limit of the material, the expression (5a) may be used in connection with the stress-strain relation (4). On the other hand, where the stresses $\lambda\sigma_{ij}$ exceed the elastic limit, different expressions must be used for C_{ijkl} according to whether the stresses τ_{ij} associated with the strains ϵ_{ij} constitute "loading" or "unloading." We reserve the complete discussion of suitable stress-strain relations beyond the elastic limit for another paper and give but one example here. Defining the *stress deviation* as $s_{ij}=\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}$ and its *intensity* as $S=\frac{1}{2}s_{ij}s_{ij}$, we set

[2] M. A. Biot [J. Appl. Phys., **10,** 860–864 (1939)] and, more recently, F. D. Murnaghan [Proc. Nat. Acad. Sci., **30,** 244–247 (1944)] have pointed out that an elastic solid under initial stress can be *strictly* isotropic only if the initial stress is of the nature of a hydrostatic pressure. For the conventional structural materials, however, this small anisotropy caused by the initial stress can be disregarded as long as the initial stress does not exceed the elastic limit.

$$C_{ijkl} = 2G_0\left(\delta_{ik}\delta_{jl} - \frac{\nu}{1-2\nu}\,\delta_{ij}\delta_{kl}\right) - \frac{G_0 - G}{S}\,s_{ij}s_{kl} \quad \text{for} \quad s_{ij}\epsilon_{ij} > 0 \tag{5b}$$

and

$$C_{ijkl} = 2G_0\left(\delta_{ik}\delta_{jl} - \frac{\nu}{1-2\nu}\,\delta_{ij}\delta_{kl}\right) \quad \text{for} \quad s_{ij}\epsilon_{ij} < 0. \tag{5c}$$

Here G_0 denotes the value which the modulus of rigidity assumes in the elastic range, while $G=G(S)$ is the so-called *tangent modulus of rigidity*. In the elastic range $G=G_0$, and (5b) as well as (5c) reduce to (5a). The stress-strain relations which are obtained by substituting (5b) and (5c) into (4) were suggested by J. H. Laning in an unpublished paper (1942); they constitute a generalization of stress-strain relations which the present author had used in earlier papers.[3] We note that $C_{ijkl}=C_{klij}$, according to (5a), (5b), and (5c).

A generic particle with the coordinates x_i in the initial state has the coordinates $\bar{x}_i=x_i+u_i$ in the considered neighbouring state, and

$$d\bar{x}_i = (\delta_{ij} + u_{i,j})dx_j = (\delta_{ij} + \epsilon_{ij} + \omega_{ij})dx_j, \tag{6}$$

where the deformation ϵ_{ij} is defined by (3) and

$$\omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}) \tag{7}$$

is the rotation associated with the displacement u_i.

The infinitesimal force λdf_j which is transmitted across the surface element dS in the initial state equals

$$\lambda df_j = \lambda T_j dS = \lambda\sigma_{ij}n_i dS. \tag{8}$$

The force which is transmitted to the corresponding material element in the neighbouring state will be written in the form

$$\lambda d\bar{f}_j = \lambda\bar{\sigma}_{ij}n_i dS. \tag{9}$$

Note that the normal vector n_i and the area ***dS in the initial state*** are used in (9). This means that the stress tensor $\lambda\bar{\sigma}_{ij}$ is defined in the *Lagrangian* manner[4] with the initial state as the state of reference. Consequently, $\lambda\bar{\sigma}_{ij}$ is not a symmetric tensor; it will be written in the form

$$\lambda\bar{\sigma}_{ij} = \lambda\sigma_{ij} + \tau_{ij} + \tau'_{ij} + \tau''_{ij}, \tag{10}$$

where the terms τ_{ij}, τ'_{ij}, and τ''_{ij} are infinitesimal changes of stress defined in the following manner:

(1) the tensor τ_{ij} is symmetric; it represents the change of stress associated with the infinitesimal strain ϵ_{ij} and is given by Eq. (4);

(2) the tensor τ'_{ij}, too, depends on the strain ϵ_{ij}; it is antisymmetric and represents the change of stress necessary to restore the moment equilibrium which is expressed by the symmetry of σ_{ij} in the initial state and which is disturbed by the deformation;

[3] W. Prager, Proc. 5th Internat. Congr. Appl. Mech. Cambridge, Mass., 1938, pp. 234–237; Prikladnaia Matematika i Mekhanika **5**, 419–430 (1941); Duke Math. J. **9**, 228–233 (1942).

[4] H. Jeffreys has recently given a similar analysis using the Eulerian approach [Proc. Cambridge Phil. Soc. **38**, 125–128 (1942)]. The Lagrangian approach seems more suitable, however, for the problem under consideration.

(3) the term τ''_{ij}, finally, depends on the rotation ω_{ij}; it represents the change of stress, with respect to the *fixed* coordinate axes, which is produced by this rotation.

Since only first order terms in ϵ_{ij} and ω_{ij} need be considered in the following analysis, the order in which the deformation ϵ_{ij} and the rotation ω_{ij} are applied is immaterial.

The antisymmetric tensor τ'_{ij} depends only on ϵ_{ij}. To find its mathematical expression, it is therefore sufficient to consider a *pure homogeneous deformation*, i.e., a deformation for which $u_{i,j}$ is independent of the coordinates and $u_{i,j}=u_{j,i}=\epsilon_{ij}$. On account of (9), the equations of equilibrium for the deformed body are

$$\int \bar{\sigma}_{ij}n_i dS = 0, \qquad \int (\bar{\sigma}_{ij}\bar{x}_k - \bar{\sigma}_{ik}\bar{x}_j)n_i dS = 0$$

or

$$\int \bar{\sigma}_{ij,i} dv = 0, \qquad \int (\bar{\sigma}_{ij}\bar{x}_k - \bar{\sigma}_{ik}\bar{x}_j)_{,i} dv = 0.$$

Since these equations must hold not only for the entire body, but also for an arbitrary portion of it, we must have

$$\bar{\sigma}_{ij,i} = 0, \quad (11) \qquad (\bar{\sigma}_{ij}\bar{x}_k - \bar{\sigma}_{ik}\bar{x}_j)_{,i} = 0. \quad (12)$$

For the considered *pure* deformation, $\tau''_{ij}=0$ and

$$\bar{x}_{i,j} = \delta_{ij} + u_{i,j} = \delta_{ij} + \epsilon_{ij}.$$

Using the symmetry of the tensors σ_{ij} and τ_{ij} in addition to the Eqs. (10), (11), (2), and neglecting higher order terms, we may therefore write (12) in the form

$$\tau'_{ij} - \tau'_{ji} = 2\tau'_{ij} = \lambda(\sigma_{ik}\epsilon_{kj} - \sigma_{jk}\epsilon_{ki}). \tag{13}$$

The tensor τ''_{ij} depends only on ω_{ij}. To find its mathematical expression, it is sufficient to consider a rigid body rotation, i.e., a system of displacements u_i which depend linearly on the coordinates x_i and satisfy $u_{i,j}=-u_{j,i}=\omega_{ij}$. By this rotation the components of the infinitesimal force transmitted across a given surface element are transformed according to

$$d\bar{f}_i = (\delta_{ij} + u_{i,j})df_j = (\delta_{ij} + \omega_{ij})df_j = df_i + \omega_{ij}df_j. \tag{14}$$

For the considered rigid body rotation $\tau_{ij}=\tau'_{ij}=0$. Using (8), (9), and (10), we may therefore write (14) in the form

$$\tau''_{ij} = -\lambda\sigma_{ik}\omega_{kj}. \tag{15}$$

Returning now to the consideration of arbitrary infinitesimal displacements u_i, we write in accordance with (10), (13), and (15):

$$\lambda\bar{\sigma}_{ij} = \lambda\sigma_{ij} + \tau_{ij} + \tfrac{1}{2}\lambda(\sigma_{ik}\epsilon_{kj} - \sigma_{jk}\epsilon_{ki}) - \lambda\sigma_{ik}\omega_{kj}. \tag{16}$$

On account of (2), the equilibrium condition (11) furnishes therefore

$$[\tau_{ij} + \tfrac{1}{2}\lambda(\sigma_{ik}\epsilon_{kj} - \sigma_{jk}\epsilon_{ki}) - \lambda\sigma_{ik}\omega_{kj}]_{,i} = 0, \tag{17}$$

and the condition $d\bar{f}_i=df_i$ furnishes

$$[\tau_{ij} + \tfrac{1}{2}\lambda(\sigma_{ik}\epsilon_{kj} - \sigma_{jk}\epsilon_{ki}) - \lambda\sigma_{ik}\omega_{kj}]n_i = 0. \tag{18}$$

Except for our more general definition of the tensor τ_{ij}, Eqs. (17) and (18) agree with those derived by Biezeno and Hencky. Biot[5] obtained the same relations from his non-linear theory of elasticity, and Neuber[6] has recently discussed the formal relation of the differential equations (17) to the fundamental equations of elasticity. As was already pointed out by Biot, Eqs. (17) differ somewhat from the equations which Trefftz[7] derived using an unconventional definition of stress. If the given state of stress, $\lambda\sigma_{ij}$, is homogeneous and if the coordinate axes have the directions of the principal axes of this state of stress, Eqs. (17) reduce to the form given by Southwell.[8]

By means of (3), (4), and (7), the quantities ϵ_{ij}, τ_{ij}, and ω_{ij} can be expressed in terms of the first derivatives of the displacement u_i. In this manner an eigen-value problem for the displacement u_i is obtained. The smallest eigen-value λ is the desired safety factor for the given distribution of initial stresses. We refrain from formulating this eigen-value problem explicitly, because in all but the most simple cases its exact solution would hardly seem possible.

3. The variational principle associated with the general problem of structural stability. The form of Eqs. (17) and (18) suggests the existence of an equivalent variational principle from which approximate solutions of stability problems can be obtained. Indeed, let us establish the Euler equations and natural boundary conditions of the variational problem

$$\delta \int \left[C_{pqrs}\epsilon_{pq}\epsilon_{rs} + \lambda\sigma_{pq}(u_{r,p}u_{r,q} - \epsilon_{rp}\epsilon_{rq})\right]dv = 0, \tag{19}$$

where only the displacements u_p and hence strains ϵ_{pq} are to be varied, but not the stresses σ_{pq} and the coefficients C_{pqrs} which depend on the stresses. If the integrand of the left-hand side of (19) is denoted by F, the Euler equations and natural boundary conditions are

$$\frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial u_{j,i}}\right) = 0, \quad (20) \qquad \frac{\partial F}{\partial u_{j,i}} n_i = 0. \quad (21)$$

Since

$$\frac{\partial \epsilon_{pq}}{\partial u_{j,i}} = \tfrac{1}{2}(\delta_{jp}\delta_{iq} + \delta_{ip}\delta_{jq}),$$

we have

$$\begin{aligned}\frac{\partial F}{\partial u_{j,i}} &= 2C_{ijkl}\epsilon_{kl} + \lambda\left[2\sigma_{ik}u_{j,k} - \sigma_{ik}\epsilon_{jk} - \sigma_{jk}\epsilon_{ik}\right] \\ &= 2\tau_{ij} + \lambda\left[\sigma_{ik}\epsilon_{kj} - \sigma_{jk}\epsilon_{ki} - 2\sigma_{ik}\omega_{kj}\right].\end{aligned}$$

Equations (20) and (21) thus are indeed identical with (17) and (18).

The variational principle (19) can be used in very much the same manner in which the principles of minimum potential energy and minimum complementary energy are used in elasticity:[9] by reasonable assumptions concerning the displacements u_i

[5] M. A. Biot, Phil. Mag. (7), **27**, 468–489 (1939).

[6] H. Neuber, Z. angew. Math. Mech. **23**, 321–330 (1943). The author is indebted to Professor E. Reissner for the reference to this paper.

[7] E. Trefftz, Z. angew. Math. Mech. **13**, 160–165 (1933).

[8] R. V. Southwell, Phil. Trans. Roy. Soc. London (A), **213**, 187–244 (1913).

[9] See, for instance, E. Volterra, Atti Accad. Lincei, Rend. (6), **20**, 424–428, 463–467 (1934); **21**, 14–19 (1935); **23**, 329–332 (1936).

the class of admitted functions is restricted and the variational problem simplified. In using this technique, we must see to it that the restrictions imposed on the displacements u_i do not rule out the possibility of fulfilling the boundary conditions (17).

4. An example. To illustrate the manner of application of the variational principle formulated in Section 3, let us discuss the lateral buckling of an elastic, prismatic beam of the length l which is built in at both ends. We assume that the cross section of this beam is doubly symmetric. Taking the origin of the coordinates at one end of the beam, we let the axis of x_1 coincide with the axis of the beam and the axes of x_2 and x_3 with the axes of symmetry of the cross section $x_1=0$. To simplify the expression (5a) for the coefficients C_{ijkl}, we shall assume that $\nu=0$. This assumption is in conformity with the spirit of the engineering theory of the bending of beams; in using it we must keep in mind that Young's modulus E_0 equals twice the modulus of rigidity G_0 if $\nu=0$.

As to the initial state of stress, let us consider the case where

$$\sigma_{11} = cx_2, \tag{22}$$

while all other components of σ_{ij} vanish. The constant c in (22) obviously has the dimension of a stress divided by a length. In an originally unstressed beam with built-in ends a stress distribution of the type (22) can be produced by changes of temperature which vary linearly with x_2. If the width of the beam (measured in the direction of x_3) is small in comparison to its height, (measured in the direction of x_2) the stresses (22) may produce lateral buckling. The infinitesimal displacements associated with this type of instability may be described in the following manner: a generic cross section x_1 of the beam undergoes a translation $u(x_1)$ in the direction of the x_3-axis, a rotation $-u'(x_1)$ about the x_2-axis which makes the cross section remain normal to the bent centerline of the beam, and, simultaneously, a rotation $-\theta(x_1)$ about the x_1-axis; in addition to this rigid body displacement the cross section undergoes a warping $-w(x_2, x_3)\theta'(x_1)$ which is associated with the twist $-\theta'(x_1)$. The corresponding displacement components are

$$u_1 = -\ x_3u'(x_1) - w(x_2, x_3)\theta'(x_1), \qquad u_2 = x_3\theta(x_1), \qquad u_3 = u(x_1) - x_2\theta(x_1). \tag{23}$$

Note that on account of the assumption $\nu=0$ the longitudinal extension $\partial u_1/\partial x_1$ is not accompanied by any lateral contraction. Particularly simple expressions for u_2 and u_3 are thus obtained. The matrices of the derivatives $u_{i,j}$ and of the strains ϵ_{ij} therefore are

$$u_{i,j} = \begin{bmatrix} -\ x_3u'' - w\theta'' & -\ \theta'\partial w/\partial x_2 & -\ u' - \theta'\partial w/\partial x_3 \\ x_3\theta' & 0 & -\ \theta \\ u' - x_2\theta' & \theta & 0 \end{bmatrix}, \tag{24}$$

$$\epsilon_{ij} = \begin{bmatrix} -\ x_3u'' - w\theta'' & \frac{1}{2}\theta'(x_3 - \partial w/\partial N_2) & -\ \frac{1}{2}\theta'(x_2 + \partial w/\partial x_3) \\ \frac{1}{2}\theta'(x_3 - \partial w/\partial x_2) & 0 & 0 \\ -\ \frac{1}{2}\theta'(x_2 + \partial w/\partial x_3) & 0 & 0 \end{bmatrix}. \tag{25}$$

Since $\sigma_{ij}=0$ unless $i=j=1$, we need only $u_{k1}u_{k1}-\epsilon_{k1}\epsilon_{k1}$ for the evaluation of the term with the factor λ in (19). Now, for a doubly symmetric cross section the warping function w is odd in x_2 as well as in x_3. Taking account of this fact, and keeping in mind that σ_{11} is odd in x_2 and even in x_3, we find that

$$\int \sigma_{11}(u_{k1}u_{k1} - \epsilon_{k1}\epsilon_{k1})dv = -2c\int u'\theta' x_2^2 dv = -2cI_3\int_0^l u'\theta' dx_1, \tag{26}$$

where I_3 denotes the moment of inertia of the cross section with respect to the x_3-axis.

We now proceed to the evaluation of term $C_{pqrs}\epsilon_{pq}\epsilon_{rs}$ in (19). With $\nu=0$, Eq. (5a) takes the form $C_{ijkl}=2G_0\delta_{ik}\delta_{jl}$ and the stress-strain relation (4) reduces to

$$\tau_{ij} = 2G_0\epsilon_{ij}. \tag{27}$$

In applying this, we shall replace $2G_0$ by E_0 whenever $i=j$. In view of (25), we have

$$C_{pqrs}\epsilon_{pq}\epsilon_{rs} = \tau_{pq}\epsilon_{pq} = E_0(x_3u'' + w\theta'')^2 + 4G_0(\epsilon_{12}^2 + \epsilon_{13}^2), \tag{28}$$

where ϵ_{12} and ϵ_{13} depend on the twist θ' and on the warping w per unit twist in precisely the same manner as in the case of pure torsion. In this case, however, the integral of $4G_0(\epsilon_{12}^2+\epsilon_{13}^2)$ over the cross section equals $G_0C\theta'^2$, where G_0C denotes the torsional stiffness of the beam. Adopting the warping w per unit twist found in the case of pure torsion, and setting*

$$\Gamma = \int w^2 dA, \tag{29}$$

where dA denotes the area element of the cross section, we obtain

$$\int C_{pqrs}\epsilon_{pq}\epsilon_{rs}dv = E_0I_2\int_0^l u''^2dx_1 + E_0\Gamma\int_0^l \theta''^2 dx_1 + G_0C\int_0^l \theta'^2 dx_1, \tag{30}$$

where I_2 is the moment of inertia of the cross section with respect to the x_3-axis.

Substituting the expressions (26) and (30) into (19), we obtain

$$E_0I_2u^{IV} + \lambda cI_3\theta'' = 0, \qquad E_0\Gamma\theta^{IV} - G_0C\theta'' + \lambda cI_3u'' = 0 \tag{31}$$

as the Euler equations for our problem, and

$$\theta'' = 0 \quad \text{for} \quad x_1 = 0 \quad \text{and} \quad x_1 = l \tag{32}$$

as the natural boundary conditions. In addition to these natural boundary conditions, we have the imposed boundary conditions

$$\theta = u = u' = 0 \quad \text{at} \quad x_1 = 0 \quad \text{and} \quad x_1 = l. \tag{33}$$

The safety factor λ is found as the lowest eigen-value of the problem formulated by Eqs. (31), (32) and (33).

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* Note that for the doubly symmetric section considered here the point x_1, 0, 0 is the shear center of the cross section x_1. Since w is odd with respect to x_2 and x_3, we have $w=0$ at this point. These remarks identify the definition (29) with that given by J. N. Goodier, Eng. Exp. Station, Cornell University, Bulletin No. 27 (1941), p. 9.