THE PROPAGATION OF A SPHERICAL OR A CYLINDRICAL WAVE OF FINITE AMPLITUDE AND THE PRODUCTION OF SHOCK WAVES*

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1. Introduction. When a mass of gas is set into motion by a sudden rise of pressure which possesses either a cylindrical symmetry or a spherical symmetry in the case of an explosion, pressure or density will be propagated into space as a cylindrical or spherical wave of finite amplitude in a manner different from that of the propagation of sound. The most conspicuous phenomenon of such a non-linear wave motion is perhaps the appearance of a shock wave. In the case of plane waves of finite amplitude, the problem was studied independently by B. Riemann[1] and S. Earnshaw.[2] It was shown that when a compressed slab of gas is released, two progressive waves are produced travelling in opposite directions, with constant deformation in the wave-form during the course of the propagation. Eventually both waves develop into shock waves.

With regard to the spherical or cylindrical compression waves, the situation is quite different because the amplitude of the wave falls off at a much greater rate than for plane waves, while the wave propagates from the center of disturbances. The question is whether this rapid diminution of amplitude would prevent the formation of a shock. J. J. Unwin[3] has calculated a specific example of motion produced by a sudden release of a compressed sphere of air, and concluded that there is no indication of the development of a shock wave. Inasmuch as he adopted a numerical method for one special case, the concludion reached cannot be regarded as general. In fact, W. Hantzsche and H. Wendt[4] considered a similar problem, where the sphere had a finite radius and expanded with the speed of sound into still air. The motion, in its early stage, is supposed to be continuous in pressure or density and velocity. But after a finite duration, the wave-front becomes a discontinuity surface characterized by an infinite velocity gradient in spite of the diminution of amplitude.

In view of these disagreeing results, it is felt that it is desirable to investigate this problem from a broad standpoint taking account of all initial boundary conditions. The problem of explosion such as the burst of a bomb is only one of many similar problems and, to be sure, the most interesting one. According to G. I. Taylor, the physical process taking place during an explosion can be treated, as a combination of two problems. The first problem is concerned with the effects produced in the atmos-

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[1] Riemann, B., *Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Abhandlungen d. Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Klase **8**, 43 (1860).

[2] Earnshaw, S., *On the mathematical theory of sound*, Phil. Trans. Roy. Soc. London, **150**, 133 (1860).

[3] Unwin, J. J., *The production of waves by a sudden release of a spherical distribution of compressed air in the atmosphere*. Proc. Roy. Soc. (A) **178**, 153 (1941).

[4] Hantzche, W. and Wendt, H., *Zum Verdichtungsstoss bei Zylinder- und Kugelwellen*, Jahrbuch 1940 der deutschen Luftfahrtforschung I, 536.

phere by a rapidly expanding spherical or cylindrical solid shell which compresses the surrounding air. In this case the motion of air in contact with the shell is completely prescribed by the motion of shell itself. The second problem deals with the motion produced by a compressed sphere or cylinder of air which is suddenly released. Each one of these constitutes a separate mathematical problem. To enlarge the scope of this discussion, the very meaning of the term explosion will be understood here as any process that is capable to create a pressure disturbance with spherical or cylindrical symmetry, propagating as a wave of finite amplitude.

An explosion is assumed to take place, during a short interval of time, in an infinite space which is filled only with air not abstructed by any solid bodies. Since the coefficients of viscosity and heat conduction for gases are generally very small, so long as the motion is continuous, the air may be regarded as non-viscous and non-conducting. The thermodynamic change of state of a fluid-particle along the path is then adiabatic; and if, initially, the entropy of the air is uniform throughout the space, the motion is isentropic. For the first problem stated above this condition is satisfied. Namely, at the moment the shell starts to expand, the outside air may certainly be assumed to be at the standard conditions. After the shell has started to expand, it compresses the air and, thereby, sets it into motion; but, during this process, no heat has been imparted to the air, its thermodynamic state must remain on the same adiabatic curve. In the case of a compressed sphere or cylinder of air, it is reasonable to assume that the pressure or density was built under adiabatic compression at all points. Hence as long as the motion is continuous, it will be isentropic.

The present study reveals that such a continuous and isentropic motion generally does not exist in the whole field. This type of motion breaks down when a "limiting line" appears, which would make the solution multi-valued. This would be impossible unless the motion is discontinuous. Hence, the appearance of a "limiting line" serves to indicate the necessity of presence of a shock wave in the actual motion. After the shock is formed, the Rankine-Hugoniot theory asserts that the process through which a fluid-particle has undergone by crossing the shock-front is irreversible and, consequently, the entropy increases in a discontinuous manner. The jump in entropy is not constant, however. It varies as the shock wave propagates, because the conditions at the shock change with time. As a result the motion behind such a non-uniform shock cannot be isentropic. Therefore once the "limiting line" appears, isentropic flow cannot be maintained and the resultant flow cannot be analyzed by the present method.

The mathematical condition for the appearance of a "limiting line" in the case of a spherical or cylindrical isentropic motion is that one of the two families of characteristics admits an envelope, just as in the case of a plane wave. Along this envelope the accelerations of the fluid-particles are infinite. In fact, a closer examination indicates that the motion generally must break down even before the "limiting line" is reached. It then seems that any motion of a compressible fluid has a tendency to develop a shock wave and that the effect of the "spreading" in the case of a non-linear spherical or cylindrical wave plays but a minor role.

2. Differential equations of motion. The motion under consideration is supposed to be axially or spherically symmetric, i.e., at any instant the velocity u, pressure p and density ρ depend on the time and the radial distance x only. If the effects of viscosity and of body force are neglected, the equations governing the motion are

$$u_t + uu_x + \frac{p_x}{\rho} = 0, \tag{2.1}$$

$$\rho_t + u\rho_x + \rho\left(u_x + \frac{\alpha u}{x}\right) = 0. \tag{2.2}$$

Here the subscripts denote the partial derivatives with respect to the variable indicated by the subscript; $\alpha=1$ for a cylindrical and $\alpha=2$ for a spherical wave. In each case, the variable x will be interpreted differently. Furthermore, it is assumed that the motion is continuous and that the effects of viscosity and heat-transfer in the fluid can be ignored. If initially constant, throughout the fluid, the entropy then remains constant. In other words, for an ideal gas the relation between the pressure and density is

$$p = K\rho^{\gamma}, \tag{2.3}$$

where γ stands for the ratio of the specific heats and K is a constant. With a set of appropriate initial conditions the mathematical problem can then be solved, at least theoretically. However, we may understand the singular behavior of such a solution and the conditions for its existence without actually solving the differential equations.

By eliminating the pressure with the aid of Eq. (2.3) and by introducing the square of the sonic speed as a variable in the place of the density, we reduce Eqs. (2.1) and (2.2) to

$$u_t + uu_x + v_x = 0, \tag{2.4}$$

$$v_t + uv_x + \beta v\left(u_x + \frac{\alpha u}{x}\right) = 0, \tag{2.5}$$

where

$$\beta v = c^2, \qquad \beta = \gamma - 1,$$

and c is the speed of sound defined by $\sqrt{\gamma(p/\rho)}$. This system of differential equations is of the hyperbolic type, the two families of real characteristics C being determined by

$$(dx - udt)^2 - \beta v dt^2 = 0, \tag{2.6}$$

where v is positive.

As it stands, this system of equations can reveal but little information concerning the behavior of the solution. To expose such properties, one has to transform the differential equations to a new coordinate-system and then study the condition under which the transformation would be valid. In the case of a steady irrotational motion, this is well-known as the hodograph method which has been effectively and successfully applied by W. Tollmien[5] and H. S. Tsien[6] in investigating the two dimensional and three dimensional isentropic motion respectively. By a slight modification, it can also be applied to the present problem. To this end, the following one-one point-transformation is introduced

[5] Tollmien, W., *Grenzlinien adiabatischer Potentialströmungen*, Z. angew. Math. Mech. **21**, 140 (1941).

[6] Tsien, H. S., *The "limiting line" in mixed subsonic and supersonic flows of compressible fluids*, N.A.C.A. Tech. Note 961 (1945).

$$u = u(t, x), \qquad v = v(t, x). \tag{2.7}$$

We have

$$u_t = \frac{x_v}{J}, \qquad u_x = -\frac{t_v}{J},$$

$$v_t = -\frac{x_u}{J}, \qquad v_x = \frac{t_u}{J},$$

provided the Jacobian $J(u, v) \equiv t_u x_v - t_v x_u \neq 0$. Equations (2.4) and (2.5) will then be transformed into

$$x_v - ut_v + t_u = 0, \tag{2.8}$$

$$x_u - ut_u + \beta v t_v - \frac{\alpha\beta uv}{x}(t_u x_v - t_v x_u) = 0. \tag{2.9}$$

This system of equations can be simplified considerably by introducing a function $\chi(u, v)$ defined by

$$x - ut = \chi_u, \qquad t = -\chi_v, \tag{2.10}$$

so that Eq. (2.8) is satisfied identically while Eq. (2.9) reduces to

$$\chi_{uu} - \beta v \chi_{vv} - \frac{\alpha\beta uv}{x}(\chi_{uu}\chi_{vv} - \chi_{uv}^2 - \chi_v \chi_{vv}) = \chi_v. \tag{2.11}$$

The corresponding characteristics Γ in the u, v-plane are determined by

$$\left(1 - \frac{\alpha\beta uv}{x}\chi_{vv}\right)dv^2 - \frac{2\alpha\beta uv}{x}\chi_{uv}du dv - \alpha\beta uv\left(\frac{1}{\alpha u} + \frac{\chi_{uu} - \chi_v}{x}\right)du^2 = 0. \tag{2.12}$$

3. Limiting line. The relationship between the characteristics C and Γ associated respectively with the differential equation in the t, x- and u, v-planes has an important bearing on the singular character of the solution and its elucidation often contributes much toward the understanding of the nature of the physical problem. For this purpose, we first transform the differential equation (2.6) by means of the following pair of relations:

$$dx = (\chi_{uu} - \chi_v - u\chi_{uv})du + (\chi_{uv} - u\chi_{vv})dv,$$
$$dt = -\chi_{uv}du - \chi_{vv}dv.$$

Substituting in Eq. (2.6) together with Eq. (2.11), we bring the equation of the characteristics C into the form

$$J\left[\left(1 - \frac{\alpha\beta uv}{x}\chi_{vv}\right)dv^2 - \frac{2\alpha\beta uv}{x}\chi_{uv}du dv - \alpha\beta uv\left(\frac{1}{\alpha u} + \frac{\chi_{uu} - \chi_v}{x}\right)du^2\right] = 0. \tag{3.1}$$

This shows that if $J \neq 0$, the characteristics C in the t, x-plane correspond to the characteristics Γ in the u, v-plane. However, circumstances may arise such that

$$J(u, v) \equiv \chi_{uu}\chi_{vv} - \chi_{uv}^2 - \chi_v \chi_{vv} = 0, \tag{3.2}$$

while

$$\left(1 - \frac{\alpha\beta uv}{x}\chi_{vv}\right)dv^2 - \frac{2\alpha\beta uv}{x}\chi_{uv}du dv - \alpha\beta uv\left(\frac{1}{\alpha u} + \frac{\chi_{uu} - \chi_v}{x}\right)du^2 \neq 0$$

and the characteristic equation (2.6) is again satisfied. This means that if a point moves along a line λ defined by Eq. (3.2), the corresponding point will describe a line l in the t, x-plane, having the same tantents as the characteristics C. It does not coincide, however, with any one of the characteristics C. This may be proved as follows.

The differential equation for the path s of a fluid-particle in the t, x-plane is

$$\left(\frac{dx}{dt}\right)_s = u. \tag{3.3}$$

The corresponding path σ in the u, v-plane is given by

$$\left(\frac{dv}{du}\right)_\sigma = -\frac{\chi_{uu} - \chi_v}{\chi_{uv}}. \tag{3.4}$$

Now the differential equation for one family of characteristics, say Γ_+, is

$$\left(\frac{dv}{du}\right)_{\Gamma_+} = -\frac{\chi_{uu} - \chi_v + \sqrt{\beta v}\,\chi_{uv}}{\chi_{uv} + \sqrt{\beta v}\,\chi_{vv}}. \tag{3.5}$$

On the other hand, the vanishing of the Jacobian, when combined with Eq. (2.11), can be written as

$$(\chi_{uv} - \sqrt{\beta v}\,\chi_{vv})(\chi_{uv} + \sqrt{\beta v}\,\chi_{vv}) = 0. \tag{3.6}$$

It is easy to see that

$$\left(\frac{dv}{du}\right)_\sigma = \left(\frac{dv}{du}\right)_{\Gamma_+}, \quad \text{if} \quad \chi_{uv} = \sqrt{\beta v}\,\chi_{vv} \tag{3.7}$$

or

$$\left(\frac{dv}{du}\right)_\sigma = \left(\frac{dv}{du}\right)_{\Gamma_-}, \quad \text{if} \quad \chi_{uv} = -\sqrt{\beta v}\,\chi_{vv}. \tag{3.8}$$

The condition under which this result holds is both necessary and sufficient. This shows that the lines λ_+ and λ_- are respectively the locus of the points of tangency of the path σ with Γ_+ and σ with Γ_-. Furthermore, the paths σ do not have an envelope and that of Γ is

$$\beta v = 0$$

which corresponds to $\rho = 0$ and is, of course, uninteresting. Hence, it cannot belong to either family of the characteristics Γ. The only alternative is that it is an envelope of one family of the characteristics C in the t, x-plane. By analogy with the steady irrotational motion it is again called "limiting line," the justification will be found in the following section.

4. The properties of the "limiting line." Being the envelope of one family of real characteristics in the t, x-plane, the "limiting line" will be entirely in the field of motion. It is, therefore, paramount to investigate the behavior of the solution along this line.

Consider first the line element of a path s of a fluid-particle at the "limiting line" l. Generally, for any line element one obtains from Eq. (2.10)

$$dx = (\chi_{uu} - u\chi_{uv} - \chi_v)du + (\chi_{uv} - u\chi_{vv})dv,$$
$$dt = -\chi_{uv}du - \chi_{vv}dv.$$

Along a path s given by $dx/dt = u$, we have

$$(\chi_{uu} - \chi_v)du + \chi_{uv}dv = 0.$$

Using this relation to eliminate dv from dx and dt and by regarding u as a parameter, we obtain the following parametric equations for the path s:

$$dx = u\,\frac{\chi_{uu}\chi_{vv} - \chi_{uv}^2 - \chi_v\chi_{vv}}{\chi_{uv}}\,du, \tag{4.1}$$

$$dt = \frac{\chi_{uu}\chi_{vv} - \chi_{uv}^2 - \chi_v\chi_{vv}}{\chi_{uv}}\,du. \tag{4.2}$$

According to our previous findings, $J=0$ yields two lines λ_+ and λ_-, each of which associates with only one group of characteristics Γ in the u, v-plane. This shows that on the "limiting line" dx and dt both become differentials of higher order and will change sign on crossing the line λ. This agrees, of course, with the cuspidal nature of the singularity.

Dividing both sides by dx and dt respectively, we obtain the following expressions for the derivatives u_x and u_t along s:

$$(u_x)_s = \frac{\chi_{uv}}{u(\chi_{uu}\chi_{vv} - \chi_{uv}^2 - \chi_v\chi_{vv})}, \tag{4.3}$$

$$(u_t)_s = \frac{\chi_{uv}}{\chi_{uu}\chi_{vv} - \chi_{uv}^2 - \chi_v\chi_{vv}}. \tag{4.4}$$

Thus on the "limiting line" the acceleration of a fluid-particle becomes infinite as χ_{uv} is finite there. This implies also an infinite pressure gradient [see Eq. (2.1)].

The physical state to which $J(u, v)=0$ corresponds can be readily deduced. It can be summarized in the statement that if the Jacobian vanishes, then the motion in the immediate neighborhood of the line $J=0$ is a compressive one. To prove this, let us consider the ratios v_t/u_x, u_t/v_x, u_t/u_x and v_t/v_x which, according to the relations obtined in Section 2, equal

$$\frac{v_t}{u_x} = -\beta v - u\frac{\chi_{uv}}{\chi_{vv}} - \frac{\alpha\beta uv}{x}\cdot\frac{J}{\chi_{vv}}, \qquad \frac{u_t}{u_x} = -u + \frac{\chi_{uv}}{\chi_{vv}},$$

$$\frac{u_t}{v_x} = u\frac{\chi_{vv}}{\chi_{uv}} - 1, \qquad \frac{v_t}{v_x} = -u + \beta v\frac{\chi_{vv}}{\chi_{uv}} + \frac{\alpha\beta uv}{x}\,\frac{J}{\chi_{uv}}.$$

In the u, v-plane, the expressions on the right-hand side are everywhere continuous. At the line λ_+ corresponding to $\chi_{uv} = \sqrt{\beta v}\chi_{vv}$, they become

$$\frac{v_t}{u_x} = -c^2\left(1-\frac{u}{c}\right) < 0, \qquad \frac{u_t}{u_x} = c\left(1-\frac{u}{c}\right) > 0,$$

$$\frac{u_t}{v_x} = -\left(1-\frac{u}{c}\right) < 0, \qquad \frac{v_t}{v_x} = c\left(1-\frac{u}{c}\right) > 0.$$

By continuity, the relative signs of the differential quotients hold in the neighborhood of the "limiting line." Thus, we conclude that either $v_t>0$, $v_x>0$ and $u_t<0$, $u_x<0$ or $v_t<0$, $v_x<0$ and $u_t>0$, $u_x>0$. The first case is exactly the condition for a compressive motion. Whereas the second case may either correspond to a rarefaction or to a change of sign of the Jacobian $J(u, v)$. As the rarefaction does not conform to the geometric properties of $J=0$, the second case corresponds to the second branch of the solution and hence can be disregarded.

5. Lost solution. In the previous sections, we assume that the Jacobian $J(u, v)$ does not vanish. Thus the one-to-one correspondence between the t, x- and u, v-planes is assured and the condition $J=0$ is restricted to the singular line l. In a special case the Jacobian may vanish identically, however. This vanishing of the Jacobian establishes a relation between v and u in the u, v-plane and, as a result, yields a class of solution not contained in the transformation (2.7). To study this form of solution, let us first set

$$v = v(u). \tag{5.1}$$

The differential equations (2.4) and (2.5) can then be rewritten as

$$u_t + \left(u + \frac{dv}{du}\right)u_x = 0, \tag{5.2}$$

$$u_t\frac{dv}{du} + \left(u\frac{dv}{du} + \beta v\right)u_x = -\frac{\alpha\beta uv}{x}\cdot \tag{5.3}$$

This type of solution has been discussed by K. Bechert[7] whose main result was as follows. By eliminating x and t the system of Eqs. (5.2) and (5.3) can be reduced to a second order non-linear total differential equation, based on the existence of a linear relation between t and x. By a slightly different procedure it can be shown that instead of a second order differential equation one can obtain a first order one of Abel's type being amenable to numerical integration. The main feature of the solution, however, can be discussed in the following manner.

Along $u=$const., i.e., along

$$du = u_x dx + u_t dt = 0,$$

the slope of the curve $u=$const. equals

$$\left(\frac{dx}{dt}\right)_u = -\frac{u_t}{u_x} = u + \frac{dv}{du}, \tag{5.4}$$

on account of Eq. (5.2). Since dv/du is a function of u alone, on $u=$const. $(dv/du)_u$ is

[7] Bechert, K., *Über die Ausbreitung von Zylinder- und Kugelwellen in reibungsfreien Gasen und Flüssigkeiten*, Ann. Phys. (5) **39**, 169 (1941).

constant. Therefore, the curve $u=\text{const.}$ is a straight line in the t, x-plane. In conformity to the assumption (5.1), there exists a parameter ξ defined by

$$\xi = \frac{x}{c_0(t+t_0)}, \tag{5.5}$$

where c_0 is the speed of sound at $u=0$, and t_0 a suitable constant. It is clear that $\xi=\text{const.}$ corresponds to $u=\text{const.}$ In other words, both v and u may be regarded as functions of ξ.

If the determinant $v'^2-\beta v\neq 0$, u_x and u_t can be expressed in terms of u. We have

$$u_x = \frac{\alpha\beta uv}{x}\frac{1}{v'^2-\beta v}, \tag{5.6}$$

$$u_t = -\frac{\alpha\beta uv}{x}\frac{u+v'}{v'^2-\beta v}, \tag{5.7}$$

where the prime denotes the total differentiation with respect to u. Like in the general case, here again the solution possesses a singular line on which the partial derivatives generally become infinite. Its other properties will be studied presently. From Eq. (5.4) it is found that

$$\left(\frac{dx}{dt}\right)_u = u + v',$$

while the characteristics are

$$\left(\frac{dx}{dt}\right)_C = u \pm \sqrt{\beta v}.$$

On the other hand, where the singular line λ, i.e. the line

$$v'^2 - \beta v = 0, \tag{5.8}$$

intersects the integral-curve $v(u)$, we have

$$\left(\frac{dx}{dt}\right)_u = u \pm \sqrt{\beta v} = \left(\frac{dx}{dt}\right)_C. \tag{5.9}$$

This shows that at the singular point of the solution $v(u)$, the $u=\text{const.}$ line becomes the envelope of one family of characteristics C. Hence the envelope is a straight line. Furthermore, according to Eqs. (4.1) and (4.2) the parametric equations of the path s are

$$dx = -\frac{x}{\alpha\beta vv'}(v'+\sqrt{\beta v})(v'-\sqrt{\beta v})du, \tag{5.10}$$

$$dt = -\frac{x}{\alpha\beta uvv'}(v'+\sqrt{\beta v})(v'-\sqrt{\beta v})du. \tag{5.11}$$

Since each factor on the right-hand side corresponds to a group of the characteristics C, on crossing the line λ, where this factor vanishes, the elements dx and dt change their signs. This proves that the line l, the image of λ, possesses all the characteristics of a "limiting line."

It is interesting to note the difference between plane and spherical waves. In the former case, Eq. (5.8) would be satisfied identically. This lets the lines $u=$const. degenerate into the characteristics. Indeed, it is also possible for one family of the characteristics which are straight lines to have an envelope; the differential quotients u_x, u_t are finite, however. Consequently, we have no "limiting line," in the strict sense. This does not mean, of course, that the solution is regular. As a matter of fact, the solution already becomes many-valued before this line is reached.

6. Lost solution: a special problem. From the foregoing conclusions, a compressive spherical or cylindrical wave always becomes indeterminate when a singular line is reached. As an illustration the following special problem is considered.

Suppose there is a divergent spherical or cylindrical wave propagating with velocity c_0 into still air. On the wave-front, where the motion agrees with the outside conditions, the state-variables ρ, p become equal to those of the still air and the velocity is zero. The path of the wave-front is then described by

$$x = c_0(t + t_0). \tag{6.1}$$

The mathematical problem can thus be formulated in the following way:

$$\left.\begin{aligned} u &= 0, \quad \text{when} \quad x \geqq c_0(t + t_0), \\ u &\neq 0, \quad \text{when} \quad x < c_0(t + t_0). \end{aligned}\right\} \tag{6.2}$$

A particularly simple case will be the one where both the pressure and the velocity are propagated with constant speed. In other words, these quantities depend only on a common parameter.

To simplify the amount of mathematical work involved, the differential equations (2.4) and (2.5) will be put into the following equivalent form:

$$(c^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_{xt} - \phi_{tt} + \frac{2c^2\phi_x}{x} = 0 \tag{6.3}$$

by introducing a potential-function $\phi(t, x)$:

$$u = \phi_x, \qquad c^2 - c_0^2 + \frac{\beta}{2}\phi_x^2 = -\beta\phi_t. \tag{6.4}$$

In the case of a lost solution, there exists a parameter ξ defined by (5.5) such that $\xi=1$ corresponds to the initial curve (6.1). Then,

$$\phi(t, x) = c_0^2(t + t_0)f(\xi) \tag{6.5}$$

and hence

$$u(t, x) = c_0 f'(\xi), \tag{6.6}$$

$$c^2 = c_0^2\left[1 - \frac{\beta}{2}f'^2 - \beta(f - \xi f')\right], \tag{6.7}$$

where the prime indicates the total differentiation with respect to ξ, and the function $f(\xi)$ satisfies

$$[c^2 - c_0^2(f' - \xi)^2]\xi f'' + 2c_0^2 f' = 0 \tag{6.8}$$

subject to the initial conditions

$$f(1) = 0, \qquad f'(1) = 0. \tag{6.9}$$

The first condition, namely $f(1)=0$, is necessary to make $c=c_0$ on $\xi=1$. When the conditions (6.9) are substituted in Eq. (6.8), it appears that $f''(1)$ is arbitrary. We need not be alarmed by this situation, but recall that in this particular type of initial value problem, the "support" is a characteristic. Physically, this means that the initial conditions prescribed in this manner do not "know" the internal structure of the motion, because they propagate ahead with larger speed. It is only natural, then, that such an arbitrariness should arise which enables us to fit properly the physical conditions specified. This arbitrariness is only a partial one, however, since for a compressive motion the sign of $f''(1)$ is necessarily negative; for on $\xi=1$

$$(\rho_t)_1 + \rho_0(u_x)_1 = 0,$$

according to Eq. (2.2). In a compressive motion $(\rho_t)_1>0$, it follows that

$$(u_x)_1 = \frac{c_0}{x} f''(1) < 0. \tag{6.10}$$

Thus, for any compressive motion the absolute value of $f''(1)$ is determined in consistence with the physical process.

The differential equation (6.8) which determines the interior motion of a mass of air, has two singular points in the ξ, f-plane given by the vanishing of the coefficient of $f''(\xi)$. The geometrical interpretation is evident, when (6.8) is written as

$$(c + u - c_0\xi)(c - u + c_0\xi)$$
$$= -\left[\left(\frac{dx}{dt}\right)_{C_+} - \left(\frac{dx}{dt}\right)_\xi\right]\left[\left(\frac{dx}{dt}\right)_{C_-} - \left(\frac{dx}{dt}\right)_\xi\right] \tag{6.11}$$

that is, when one family of characteristics become tangent to a line $\xi=$const., an infinite curvature would occur if u is finite there. According to what has been said in the last section, this characterizes the "limiting line" of the solution.

Let us push the discussion a step further. For this purpose only the first order terms need be retained. Taking β as a small parameter, one has accordingly

$$f(\xi) = f_0(\xi) + \beta f_1(\xi) + \cdots. \tag{6.12}$$

Substituting in Eq. (6.8) we obtain

$$[1 - (f_0' - \xi)^2]\xi f_0'' + 2f_0' = 0. \tag{6.13}$$

This equation is free from f_0; letting $w=f'$ we find

$$\frac{dw}{d\xi} = \frac{w}{\xi}\,\frac{2}{(w - \xi)^2 - 1}, \qquad 0 \leqq \xi \leqq 1. \tag{6.14}$$

Aside from the two singular lines

$$w = \xi + 1, \tag{6.15}$$ $$w = \xi - 1, \tag{6.16}$$

where the slope of w is infinite, there are two additional singularities (1, 0) and (0, 0) where the slope is indeterminate. The point (1, 0) acts as a sort of nodal point which makes the initial condition insufficient. The point (0, 0) is a saddle point as locally the equation behaves like

$$\frac{dw}{d\xi} = -\frac{2w}{\xi}, \tag{6.17}$$

which form is obtained by neglecting $(w-\xi)^2$ as compared with 1.

The situation can now be summarized. The integral curve starting from (1, 0) rises as ξ decreases and eventually intersects with the line (6.15) where it will have a vertical tangent at $\xi<1$. After it crosses this line its slope changes sign. This causes the curve to bend backward again. Thus, ξ is seen to assume a minimum value. Owing to the fact that the origin is a saddle point, no integral curve could possibly cross the line $\xi=0$. This fact makes the continuation of the solution as far as $\xi=0$ impossible.

7. Continuation of the solution. The results obtained in the previous sections show that, in the case of the propagation of a spherical or cylindrical wave, a continuous solution does not exist throughout the domain considered and can be constructed, at most, as far as a singular line l in the t, x-plane from a suitably chosen initial data. The line l thus acts as a sort of "frontier" into which no solution can enter and at which the solution is turned back as a second branch. The domain then is doubly covered. Physically, this is impossible and hence must be rejected as a solution. The question is: is it possible to connect it with a different solution beyond this line?

First, consider the line λ as a "support" with a given set of initial data and then solve the initial value problem[8] for a Monge-Ampère equation. Regarding λ as a parameter, we have along the line λ

$$\frac{d}{d\lambda}\chi_u = \chi_{uu}\frac{du}{d\lambda} + \chi_{uv}\frac{dv}{d\lambda}, \tag{7.1}$$

$$\frac{d}{d\lambda}\chi_v = \chi_{uv}\frac{du}{d\lambda} + \chi_{vv}\frac{dv}{d\lambda}, \tag{7.2}$$

and hence

$$(\chi_{uu}\chi_{vv} - \chi_{uv}^2)\frac{du}{d\lambda} = \chi_{vv}\frac{d}{d\lambda}\chi_u - \chi_{uv}\frac{d}{d\lambda}\chi_v.$$

Substituting this into Eq. (2.11) we obtain a linear relation between the partial derivatives:

$$\chi_{uu} + \left[\frac{\alpha\beta uv}{x}\left\{\frac{d\chi_u/d\lambda}{du/d\lambda} - \chi_v\right\} - \beta v\right]\chi_{vv} - \frac{\alpha\beta uv}{x}\frac{d\chi_v/d\lambda}{du/d\lambda}\chi_{uv} = \chi_v. \tag{7.3}$$

Since λ is not a characteristic, Eqs. (7.1), (7.2) and (7.3) are sufficient for a unique determination of χ_{uu}, χ_{uv} and χ_{vv}; and consequently a unique integral surface. The uniqueness of the solution is sufficient to show that the solution, when transformed

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[8] Courant, R. and Hilbert, D., *Methoden der math. Physik*, vol. 2, J. Springer, Berlin, 1937, p. 344.

back to the t, x-plane, will correspond to the very one that doubles back at the "limiting line." A continuous solution is thus out of the question.

The alternative procedure would be to continue it by joining it smoothly at the line λ to the lost solution. This is also impossible. Indeed, if this were possible, the line λ would have to coincide with the integral curve $v(u)$ in order to provide a continuous solution. This is contradictory, because it is easy to show that the line λ does not satisfy the differential equation for $v(u)$.

The other possibility which remains to be investigated is to identify the "limiting line" as a shock wave so as to construct a discontinuous solution. This would require the continued solution to satisfy the shock conditions. Since, in general, the "limiting line" l is curved, as a result there would be a non-uniform shock wave in the motion, for which both the speed and the strength are no longer constant and therefore the entropy would be constantly changing across the shock. This very fact makes the original assumption untenable. Hence to continue discontinuously a solution with entropy constant everywhere is also impossible.

The problem might be solved, however, if the original hypothesis of isentropic motion is abandoned. To include the possibility that a shock wave may exist within the motion, the continued solution must satisfy the following more general set of equations:

$$u_t + uu_x + \frac{p_x}{\rho} = 0, \tag{7.4}$$

$$\rho_t + u\rho_x + \rho\left(u_x + \frac{\alpha u}{x}\right) = 0, \tag{7.5}$$

$$(p\rho^{-\gamma})_t + u(p\rho^{-\gamma})_x = 0. \tag{7.6}$$

The task then is to construct a solution which should satisfy both the initial and the shock conditions in a region bounded by the initial curve, the shock line and a characteristic drawn to the initial curve through the point where the envelope first appears. The shock line, however, is not given, it should be chosen in such a way that it yields a solution fulfilling all the prescribed conditions. The mathematical problem thus turns out to be extremely difficult.

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