THE BOUNDARY LAYER IN A CORNER*

BY

G. F. CARRIER

Brown University

1. Introduction. The laminar flow of a relatively non-viscous fluid through a channel is characterized by the presence of a thin boundary layer along the walls. In straight channels, such boundary layers are usually assumed to have the velocity distribution determined by Blasius [1] for the flow past a flat plate, and the flow pattern in the neighborhood of any corner is not mentioned. It seems of interest to develop here the change in the Blasius flow implied by such a corner.

2. The boundary layer problem. We shall consider the laminar flow of an incompressible fluid which impinges with the uniform velocity V on the edges $x=0$ of the half planes $y=0$, $z=0$.

The Navier-Stokes equations and the continuity condition which govern such flows are

$$(\mathbf{v}\cdot\text{grad})\,\mathbf{v} + \rho^{-1}\,\text{grad}\,p = \nu\Delta\mathbf{v}, \tag{1}$$

$$\text{div}\,\mathbf{v} = 0. \tag{2}$$

Here $\mathbf{v}$ is the velocity with components u, v, w; p is the pressure, ν the kinematic viscosity, and ρ the density.

As v. Kármán has pointed out [2], the essence of the treatment of such equations in a boundary layer problem is to eliminate higher order terms (by a perturbation scheme or otherwise) in such a manner that the order of the equations is not decreased. In this way no boundary conditions need be relaxed. We may accomplish this by using what is essentially Prandtl's coordinate transformation [1], namely

$$\eta = y/(\nu x/V)^{1/2}, \qquad \zeta = z/(\nu x/V)^{1/2}. \tag{3}$$

We also define the parameter $\xi=(\nu/Vx)^{1/2}$.

Since the flow both within and outside the boundary layer may be expected to be essentially in the x direction and slowly varying in x, we may attempt to find a solution in the form

$$u = V[u_0(\eta,\zeta) + \xi u_1(\eta,\zeta) + \xi^2 u_2 + \cdots] \tag{4}$$

$$v = V(\xi v_1 + \xi^2 v_2 + \cdots) \tag{5}$$

$$w = V(\xi w_1 + \xi^2 w_2 + \cdots) \tag{6}$$

$$p = \rho V^2(p_0 + \xi p_1 + \cdots). \tag{7}$$

We commence the series for v and w with a term of order ξ, because we wish a solution for which v/V, w/V, are small. Furthermore, if we included terms v_0, w_0, the following set of equations would contain terms of order ξ^{-1} with no contribution from the viscous terms of Eqs. (1) and (2). Thus the solutions wherein v_0, w_0 were not identically zero would not provide results corresponding to the phenomenon under investigation.†

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† Actually, the fact that our results constitute a solution which obeys the differential equation and boundary conditions is sufficient justification for taking $v_0 \equiv w_0 \equiv 0$.

The substitution of Eqs. (4) to (7) into Eqs. (1) and (2) leads to the system

$$-\frac{u_0}{2}(\eta\partial u_0/\partial\eta+\zeta\partial u_0/\partial\zeta)+v_1\partial u_0/\partial\eta+w_1\partial u_0/\partial\beta-\frac{\eta}{2}\frac{\partial p_0}{\partial\eta}-\frac{\zeta}{2}\frac{\partial p_0}{\partial\zeta}$$
$$-\left(\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\zeta^2}\right)u_0+\xi(\cdots)+\cdots=0 \tag{8}$$

$$\frac{\partial p_0}{\partial\eta}+\xi\frac{\partial p_1}{\partial\eta}-\xi^2\left(\frac{u_0}{2}\left[\eta\frac{\partial v_1}{\partial\eta}+\zeta\frac{\partial v_1}{\partial\eta}\right]+\frac{\partial^2 v_1}{\partial\eta^2}+\cdots\right)+\cdots=0 \tag{9}$$

$$\frac{\partial p_0}{\partial\zeta}+\xi\frac{\partial p_1}{\partial\zeta}-\xi^2(\cdots)+\cdots=0 \tag{10}$$

$$\frac{\eta}{2}\frac{\partial u_0}{\partial\eta}+\frac{\zeta}{2}\frac{\partial u_0}{\partial\zeta}-\frac{\partial v_1}{\partial\eta}-\frac{\partial w_1}{\partial\zeta}+\xi(\cdots)+\cdots=0. \tag{11}$$

The solution of this system of equations requires that the coefficient of each power of ξ in each equation vanish. The first order approximation to the result is defined by the vanishing of the coefficients of ξ^0. The result can be expected to be valid only when the remaining terms of the series are negligible, that is when ξ is small. Thus the solution, like that for the flat plate, is valid only at sufficiently large distances from the leading edges of the planes.

We now note that the ξ^0 terms of Eqs. (9) and (10) vanish only if $p_0=\text{const}$; the ξ^0 term of Eq. (11) vanishes if we write

$$u_0=g_{\eta\zeta}(\eta,\zeta),\qquad v_1=\tfrac{1}{2}(\eta g_{\eta\zeta}-g_\zeta),\qquad w_1=\tfrac{1}{2}(\zeta g_{\eta\zeta}-g_\eta).$$

Thus it remains to find $g(\eta,\zeta)$ such that,

$$g(0,\zeta)=g_\eta(0,\zeta)=g(\eta,0)=g_\zeta(\eta,0)=0$$

and

$$\lim_{\eta,\zeta\to\infty}g_{\eta\zeta}(\eta,\zeta)=1,$$

the implied symmetry condition

$$g(a,b)=g(b,a),$$

and the differential equation implied by Eq. (8)

$$g_{\eta\eta\eta\zeta}+g_{\eta\zeta\zeta\zeta}+\tfrac{1}{2}\{g_\zeta g_{\eta\eta\zeta}+g_\eta g_{\eta\zeta\zeta}\}=0. \tag{12}$$

We may expect that far from the corner the solution will be essentially that for the flat plate. Hence, we write

$$g(\eta,\zeta)=f_0(\eta)f_0(\zeta)+h(\eta,\zeta) \tag{13}$$

where f_0 is that solution of

$$2f'''+ff''=0$$

such that $f(0)=f'(0)=0$; $f'_{\alpha\to\infty}(\alpha)=1$. This function is tabulated in [1].

Equations (12) and (13) lead to the equation

$$h_{\eta\eta\eta\zeta} + h_{\eta\zeta\zeta\zeta} + 2a(\eta, \zeta)h_{\eta\eta\zeta} + 2a(\zeta, \eta)h_{\eta\zeta\zeta} + b(\eta, \zeta)h_{\eta} + b(\zeta, \eta)h_{\zeta} + \tfrac{1}{2}(h_{\zeta}h_{\eta\eta\zeta} + h_{\eta}h_{\eta\zeta\zeta}) = \tfrac{1}{25}A(\eta, \zeta), \tag{14}$$

where

$$a(\eta, \zeta) = \tfrac{1}{4}f_0(\eta)f_0'(\zeta), \qquad b(\eta, \zeta) = \tfrac{1}{2}f_0''(\eta)f_0'(\zeta)$$

$$A(\eta, \zeta) = \tfrac{1}{2}\{f_0(\eta)f_0''(\eta)f_0'(\zeta)[1 - f_0'(\zeta)] + f_0(\zeta)f_0''(\zeta)f_0'(\eta)[1 - f_0'(\eta)]\}.$$

This equation may be integrated once each over η and ζ taking account of the boundary conditions to yield (when $\varphi = -25h_{\eta\zeta}$)

$$\Delta\varphi + 2a(\eta, \zeta)\partial\varphi/\partial\eta + 2a(\zeta, \eta)\partial\varphi/\partial\zeta + b(\eta, \zeta)\int_0^{\eta}\varphi d\eta + b(\zeta, \eta)\int_0^{\zeta}\varphi d\zeta + \frac{1}{50}\left[\varphi_{\eta}\int_0^{\eta}\varphi d\eta + \varphi_{\zeta}\int_0^{\zeta}\varphi d\zeta\right] = \tfrac{1}{25}A(\eta, \zeta). \tag{15}$$

The boundary conditions are

$$\varphi(0, \zeta) = \varphi(\eta, 0) = \lim_{\eta,\zeta\to\infty}\varphi(\eta, \zeta) = 0.$$

This last form of the equation seems best suited for numerical evaluation. The relaxation method [3] appears to be the most appropriate for the determination of φ so we form the difference equation derived from Eq. (15) by taking points spaced unity apart in η and ζ. The subscripts m and n are used to index these point positions. The difference equation is

$$\varphi_{m+1,n} + \varphi_{m-1,n} + \varphi_{m,n-1} + \varphi_{m,n+1} - 4\varphi_{mn} + a_{mn}(\varphi_{m+1,n} - \varphi_{m-1,n}) + a_{nm}(\varphi_{m,n+1} - \varphi_{m,n-1}) + b_{mn}\int_0^{n}\varphi d\eta + b_{nm}\int_0^{m}\varphi d\zeta + .01\left[(\varphi_{m,n+1} - \varphi_{m,n-1})\int_0^{m}\varphi d\zeta + (\varphi_{m+1,n} - \varphi_{m-1,n})\int_0^{n}\varphi d\eta\right] + A_{mn} = 0. \tag{16}$$

In this equation the integrals may be evaluated by the simple trapezoidal rule since the function φ is very "smooth" although if more accuracy is desired a simple graphical method is conveniently employed.

TABLE I

η \ ζ	0	1	2	3	$\phi(\eta, \zeta)$ 4	5	6	7	8	$f_0'(\eta)$
0	0	0	0	0	0	0	0	0	0	0
1	0	.58	1.00	1.00	.64	.25	.08	.02	.00	.330
2	0	1.00	1.60	1.46	.86	.28	.08	.02	.00	.630
3	0	1.00	1.46	1.23	.61	.16	.04	.01	.00	.846
4	0	.64	.86	.61	.24	.03	.01	.00		.955
5	0	.25	.29	.16	.03	.01	.00			.992
6	0	.08	.08	.04	.01	.00	.00			.999
7	0	.02	.02	.01	.00					1.000
8	0	00	.00	.00						1.000

The numerical procedure is this: guess values for φ at all points $m, n \leqq 8$. Replace the zero on the right side of Eq. (12) by Q_{mn} and compute each Q_{mn} (the residuals). Then revise the guesses for the φ_{mn} in such a way as to decrease the Q_{mn}, disregarding the changes in the values of the terms containing integrals. When considerable improvement has been made, recompute the Q_{mn} using the complete equation (12) and

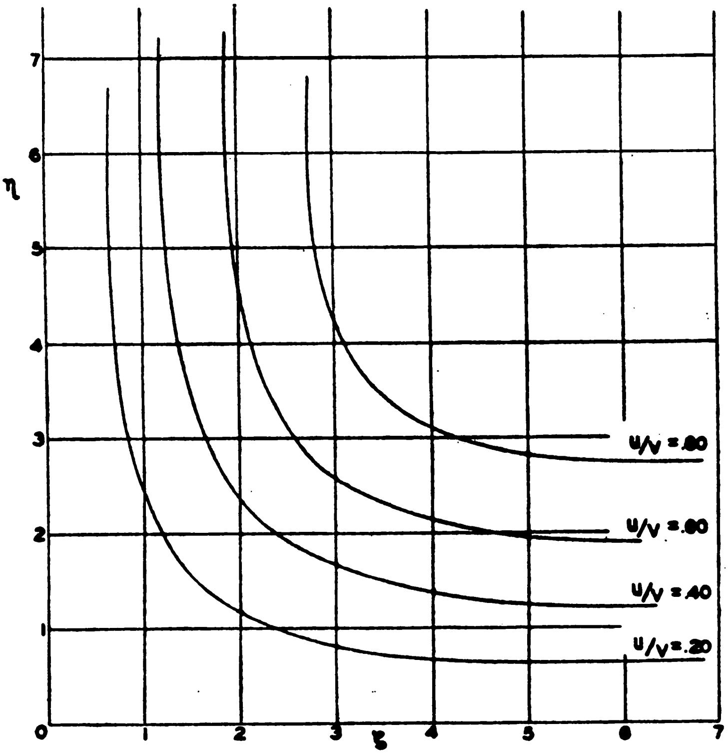

FIG. 1. Contours of constant U in corner boundary layer.

repeat the foregoing procedure. It is not necessary to get extremely accurate values of φ (especially since a, b, A are not known too finely) because the velocity $u_0 = f_0'(\eta) f_0'(\zeta) + \frac{1}{15}\varphi(\eta, \zeta)$ will be accurate to three places when φ is known to the one hundredths digit. The functions f_0' and φ are tabulated in Table I and contours of constant u_0 are shown in Fig. 1.

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