## UNSTABLE SOLUTIONS OF A CLASS OF HILL DIFFERENTIAL EQUATIONS\*

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1. Introduction. Linear differential equations with periodic coefficients play an important role in problems of engineering and physics. The best-known of these equations is Mathieu's equation. A somewhat more complicated equation is

$$\frac{d^2v}{d\psi^2} + \left[\theta_{-2}e^{-2i\psi} + \theta_{-1}e^{-i\psi} + \theta_0 + \theta_1e^{i\psi} + \theta_2e^{2i\psi}\right]v = 0 \tag{1a}$$

which reduces to Mathieu's equation for

$$\theta_0^* = \theta_0, \quad \theta_{-1}^* = \theta_{-1} = \theta_1, \quad \theta_{-2} = \theta_2 = 0.$$

where the asterisk is used to denote the conjugate complex quantity.

This paper is concerned with the determination of the solutions

$$v(\psi) = e^{\sigma \psi} \sum_{-\infty}^{+\infty} c_k e^{ik\psi} \tag{2}$$

of Eq. (1a) subject to the restrictions

$$\theta_0^* = \theta_0, \quad \theta_{-1}^* = \theta_1, \quad \theta_{-2}^* = \theta_2$$
 (1b)

and

$$\theta_1 = O(\mu), \qquad \theta_2 = O(\mu^2), \tag{3}$$

where  $\mu$  is a small positive quantity. It will be seen that solution of the problem involves the determination of the "characteristic exponent"  $\sigma$  from the equation

$$\sin i\pi\sigma = \sqrt{D}\sin \pi\sqrt{\theta_0},\tag{4}$$

where D denotes the expansion

$$\mathcal{D} = 1 + C_{\delta}\delta + C_{\epsilon}\epsilon + C_{\eta}\eta + C_{\delta}^{2}\delta^{2} + C_{\delta\epsilon}\delta\epsilon + \cdots$$
 (5)

in the three real combinations

$$\delta = \theta_{-1}\theta_1, \qquad \epsilon = \theta_{-2}\theta_2, \qquad \eta = \frac{1}{2}(\theta_1^2\theta_{-2} + \theta_{-1}^2\theta_2)$$
 (6a)

of the four quantities, real and imaginary parts of  $\theta_1$  and  $\theta_2$ . D is a power series in  $\mu^2$  since

$$\delta = O(\mu^2), \qquad \epsilon = O(\mu^4), \qquad \eta = O(\mu^4). \tag{6b}$$

The coefficients C of the series depend on  $\theta_0$  alone.

The numerical evaluation of the coefficients of the expansion is the principal aim of this paper. This is best accomplished by first re-expressing the "doubly infinite"  $Hill\ determinant\ D$  in terms of its "simply infinite" principal subdeterminants  $D_n$ ,

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$$\mathcal{D} = f(D_0, D_1, D_2, \cdots), \tag{7}$$

and then expanding  $D_n$  into the series

$$D_n = 1 + A_{\delta}^n \delta + A_{\epsilon}^n \epsilon + A_{\eta}^n \eta + A_{\delta}^{\eta 2} \delta^2 + \cdots$$
 (8)

The coefficients of the expansions (7) and (8) are tabulated in Tables II and I respectively for a convenient range of  $\theta_0$ . For the sake of simpler printing the notation

$$A_{\delta^i \epsilon^j \pi^k}^n \equiv \left\{ n, \, \delta^i \epsilon^j \eta^k \right\} \tag{8'}$$

will be used whenever the subscript of A becomes excessively long.

The practical solution of Eq. (1) is carried out in four steps. First, the determinants  $D_n$ , Eqs. (8), are evaluated by means of Table I. Next  $\mathcal{D}$ , Eq. (7), is determined from Table II. The third step consists in solving Eq. (4) or one of its variants (13a, b, c) for  $\sigma$ , and the last step is the determination of the coefficients  $c_k$  of solution (2). A convenient method for carrying out this last step is discussed in Section 2. The derivations of the formulas for  $\{n, \delta^i \epsilon^j \eta^k\}$  and for the coefficients of (7) are presented in Section 3. A numerical example is given in Section 4.

The present paper is based on a study which was recently undertaken at the McDonnell Aircraft Corp. under the sponsorship of the Bureau of Aeronautics, U. S. Navy Department. The study was prompted by recent instances of control difficulties of some helicopters and rotor blade failures of others. As will be shown in a separate paper,<sup>1</sup> the natural modes in which hinged rotor blades flap can be represented by solutions of Eq. (1) multiplied by suitable damping factors. It will be found that the stability of the blade motion decreases as the speed of advance of the helicopter increases (as  $\mu$  increases). Nevertheless, instability does not set in, because an aerodynamic damping effect outweighs, at all feasable speeds, the tendency towards instability which results from the flapping motion.

The writer's thanks are due to his colleague, Elizabeth J. Spitzer, for checking the derivations and the numerical work. The writer also wishes to express his indebtedness to Messrs. W. R. Foote, H. Poritsky and J. J. Slade, who in their paper on rotational instability of shafts<sup>2</sup> applied a Laplace expansion to a doubly infinite determinant, and thus suggested the present approach.

2. Method of solution. The solution of Eq. (1a) is assumed in the standard form

$$v(\psi) = e^{\sigma \psi} \sum_{-\infty}^{+\infty} c_k e^{ik\psi}. \tag{2}$$

Substitution of expression (2) into Eq. (1a) leads to the infinite set of homogeneous equations for the coefficients  $c_k(\sigma)$ :

$$k = -2: \qquad \theta_{2}c_{-4} + \theta_{1}c_{-3} + \left[ (\sigma - 2i)^{2} + \theta_{0} \right]c_{-2} + \theta_{-1}c_{-1} + \theta_{-2}c_{0}$$

$$k = -1: \qquad \theta_{2}c_{-3} + \theta_{1}c_{-2} + \left[ (\sigma - i)^{2} + \theta_{0} \right]c_{-1} + \theta_{-1}c_{0} + \theta_{-2}c_{1}$$

$$k = 0: \qquad \theta_{2}c_{-2} + \theta_{1}c_{-1} + \left[ \sigma^{2} + \theta_{0} \right]c_{0} + \theta_{-1}c_{1} + \theta_{-2}c_{2}$$

$$= 0, (9)$$

$$k = 0: \qquad \theta_{2}c_{-2} + \theta_{1}c_{-1} + \left[ \sigma^{2} + \theta_{0} \right]c_{0} + \theta_{-1}c_{1} + \theta_{-2}c_{2}$$

$$= 0, (9)$$

<sup>&</sup>lt;sup>1</sup> G. Horvay, Rotor blade flapping motion, to be published soon.

<sup>&</sup>lt;sup>2</sup> W. R. Foote, H. Poritsky and J. J. Slade, Critical speeds of a rotor with unequal shaft flexibilities, mounted in bearings of unequal flexibility, Journal of Applied Mechanics, 10, A77, 1943.

$$k = -1: \qquad \theta_2 c_{-1} + \theta_1 c_0 + \left[ (\sigma + i)^2 + \theta_0 \right] c_1 + \theta_{-1} c_2 + \theta_{-2} c_3 = 0,$$

$$k = -2: \qquad \theta_2 c_0 + \theta_1 c_1 + \left[ (\sigma + 2i)^2 + \theta_0 \right] c_2 + \theta_{-1} c_3 + \theta_{-2} c_4 = 0,$$

The equations are consistent if their determinant,  $\Delta(\sigma)$ , vanishes. The consistency criterion

$$\Delta(\sigma) = 0 \tag{10}$$

can be expressed in the much simpler form<sup>8,4</sup>

$$\sin i\pi\sigma = \pm \sqrt{D}\sin \pi\sqrt{\theta_0},\tag{4}$$

where

$$\mathfrak{D} \equiv \Delta(0) = \begin{vmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \theta_{-1}y_{2} & \theta_{-2}y_{2} & 0 & 0 & \cdot \\
\cdot & \theta_{1}y_{1} & 1 & \theta_{-1}y_{1} & \theta_{-2}y_{1} & 0 & \cdot \\
\cdot & \theta_{2}y_{0} & \theta_{1}y_{0} & 1 & \theta_{-1}y_{0} & \theta_{-2}y_{0} & \cdot \\
\cdot & 0 & \theta_{2}y_{1} & \theta_{1}y_{1} & 1 & \theta_{-1}y_{1} & \cdot \\
\cdot & 0 & 0 & \theta_{2}y_{2} & \theta_{1}y_{2} & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{vmatrix}$$
(11a)

is the determinant of system (9) for  $\sigma = 0$  when each equation is divided by the coefficient of the diagonal term, and

$$y_k = \frac{1}{\theta_0 - k^2} \,. \tag{11b}$$

 $\mathcal{D}$  is either positive or negative, and so is  $\theta_0$ . Thus the quantity  $\sqrt{\mathcal{D}} \sin \pi \sqrt{\bar{\theta}_0}$  is either real or pure imaginary. In the first case set

$$q = \sqrt{\mathcal{D}} \sin \pi \sqrt{\theta_0} = -\sqrt{-\mathcal{D}} \sinh \pi \sqrt{-\theta_0}. \tag{12a}$$

In the second case set

$$q' = q/i = -\sqrt{-D}\sin \pi \sqrt{\theta_0} = -\sqrt{D}\sinh \pi \sqrt{-\theta_0}.$$
 (12b)

Then the solution of the transcendental equation (4) is given by

$$\pm \sigma = \frac{1}{\pi} \log (q/i + \sqrt{1 - q^2}) + mi \qquad (m = 0, \pm 1, \pm 2, \cdots),$$

$$= \frac{i}{\pi} \arctan \frac{-q}{\sqrt{1 - q^2}} + mi \qquad \text{for } -1 \le q \le 1, \qquad (13a)$$

$$= \frac{1}{\pi} \log (q + \sqrt{q^2 - 1}) + (m - \frac{1}{2})i \text{ for } q \le -1, q \ge 1,$$

$$= \frac{1}{\pi} \log (q' + \sqrt{q'^2 + 1}) + mi \text{ for } q \text{ imaginary}.$$
(13b)

$$= \frac{1}{\pi} \log \left( q' + \sqrt{q'^2 + 1} \right) + mi \qquad \text{for } q \text{ imaginary.}$$
 (13c)

<sup>&</sup>lt;sup>3</sup> Whittaker and Watson, A course of modern analysis, Cambridge, 1927, p. 416.

M. J. O. Strutt, Lamésche, Mathieusche und verwondte Funktionen in Physik und Technik, Springer 1932 (Edward Bros., 1944), p. 22.

Once  $\mathcal{D}$  is known, the calculation of  $\sigma$  from (13) is simple matter. For any m there are two solutions  $\sigma_1$  and  $\sigma_2$  which differ only in sign

$$\sigma_2 = -\sigma_1. \tag{14a}$$

Let  $\sigma_1$  be the solution with the positive real part

$$\sigma_1 = \sigma_r + i\sigma_i, \qquad \sigma_r \ge 0.$$
 (14b)

The function  $v_1(\psi)$  (or  $v_2(\psi)$ ) associated with  $\sigma_1$  (or with  $\sigma_2$ ) is readily obtained by placing  $\sigma_1$  (or  $\sigma_2$ ) into the system (9) which is now limited to the equations k=-N, -N+1,  $\cdots$ , -1, +1,  $\cdots$ , +N. Assuming  $c_k=0$  for k<-N and k>+N, one can solve the 2N equations for  $c_{-N}$ ,  $c_{-N+1}$ ,  $\cdots$ ,  $c_{-1}$ ,  $c_1$ ,  $\cdots$ ,  $c_N$  in terms of the arbitrary constant  $c_0$ , and then use equation k=0 as a check. The greater is N, the more harmonics are taken into account, and the more accurate is the solution. In practice the calculations are most conveniently carried out by solving equations k=N and k=-N for  $c_N$  and  $c_{-N}$  in terms of the variables  $c_{N-1}$ ,  $c_{N-2}$  and  $c_{-N+1}$ ,  $c_{-N+2}$ , respectively. The results are then substituted into equations k=N-1 and k=-N+1; similarly  $c_{N-1}$  and  $c_{-N+1}$  are determined. Continuing the process one finally arrives at equations k=1 and k=1 involving the two variables  $c_1$  and  $c_1$  only, and the parameter  $c_0$  which can be assumed as 1. One eliminates one of the unknowns, say  $c_1$  determines from the real and imaginary parts of  $c_1$ , and then, retracing the steps, obtains in succession the numerical values of  $c_1$ ,  $c_2$ ,  $c_2$ ,  $\cdots$ ,  $c_N$ ,  $c_N$ .

Evidently, in principle, it is immaterial what mi  $(m=0, \pm 1, \pm 2, \cdots)$  is used in the  $\sigma$  of Eqs. (9). For instance, the set of equations k=-N to +N with m=2, the set of equations k=-N-2 to +N-2 with m=4, and the set of equations k=-N+3 to +N+3 with m=-1 are identical. Thus, as one passes to the limit  $N\to\infty$ , any m and any 2N+1 adjoining equations will lead to the same function  $v_1(\psi)$  [or  $v_2(\psi)$ ]. In practice, where one is limited to a finite number of equations, 2N+1, it is best to use the centrally located equations k=-N to +N with an m which makes  $c_0$  the dominating term in the series (2).

In general  $v_1$ , associated with  $\sigma_1$ , and  $v_2$ , associated with  $\sigma_2$ , are linearly independent functions. An exceptional case arises when  $\sigma_r = 0$ , and  $\sigma_i$  is an integral multiple of  $\frac{1}{2}$ . Then substitution of  $\sigma_1$  and  $\sigma_2$  into the system (9) leads to the same function

$$v_1 = e^{i\nu\psi} \sum_{-\infty}^{+\infty} c_k e^{ik\psi}, \qquad (\nu = 0 \text{ or } \frac{1}{2}).$$
 (15a)

The second, linearly independent, solution is now a "quasiperiodic function":4a

$$v_2 = e^{i\nu\psi} \left[ \psi \sum_{-\infty}^{+\infty} c_k e^{ik\psi} + \sum_{-\infty}^{+\infty} d_k e^{ik\psi} \right], \qquad (\nu = 0, \frac{1}{2}).$$
 (15b)

For convenience the functions (15a) will be called "purely periodic" functions.

Determination of the purely periodic solutions (15a) forms the subject matter of most investigations on Mathieu and Hill differential equations. The purely periodic

<sup>&</sup>lt;sup>4n</sup> M. J. O. Strutt, *loc. cit.*, p. 23. As an exception there may be two purely periodic solutions. For instance, for  $\theta_0 = 4$ ,  $\theta_1 = \theta_2 = 0$ , one obtains  $v_1 = \cos 2\psi$ ,  $v_2 = \sin 2\psi$ .

solutions are usually of greatest interest, because they separate the  $\mu$ -regions of stability  $(\sigma_r = 0; v_1 \text{ and } v_2 \text{ are oscillatory})$  from the  $\mu$ -regions of instability  $(\sigma_r > 0, v_1 \to \infty \text{ as } \psi \to \infty)$ . A purely periodic solution can be obtained, cf. Eqs. (12), (13), only when q' = 0  $(\mathcal{D} = 0, \text{ or perhaps } \theta_0 = k^2)$ , or when  $q = \pm 1$   $(\mathcal{D} \text{ and } \theta_0 \text{ are such that } \sqrt{\mathcal{D}} \sin \pi \sqrt{\theta_0} = 1$  or  $\sqrt{-\mathcal{D}} \sinh \pi \sqrt{-\theta_0} = 1$ ). In general  $\theta_0 = k^2$  does not provide a purely periodic function.

In the present analysis the principal interest is attached to the unstable solutions

$$v_1(\psi) = e^{\sigma_r \psi} \sum_{-\infty}^{+\infty} c_k e^{i(\sigma_i + k)\psi}, \qquad (\sigma_r > 0)$$
 (16)

of (1) which, after multiplication by a damping factor  $e^{-n\psi/2}$ , are still stable. These solutions are in the "transition region" which extends from the  $\mu$ -value for which  $v(\psi)$  is purely periodic to the  $\mu$ -value for which  $e^{-n\psi/2}v(\psi)$  is purely periodic. It will be seen in Reference 1 that a rapidly advancing helicopter usually operates in the transition region.

3. Expansion of Hill's infinite determinant. It will be convenient to call the determinant  $\mathcal{D}$ , Eq. (11a), a doubly infinite determinant to indicate that it extends to infinity both upward and downward. Simply infinite determinant are the principal subdeterminants of  $\mathcal{D}$ ,

$$D_{n} = \begin{vmatrix} 1 & \theta_{-1}y_{n} & \theta_{-2}y_{n} & 0 & \cdot \\ \theta_{1}y_{n+1} & 1 & \theta_{-1}y_{n+1} & \theta_{-2}y_{n+1} & \cdot \\ \theta_{2}y_{n+2} & \theta_{1}y_{n+2} & 1 & \theta_{-1}y_{n+2} & \cdot \\ 0 & \theta_{2}y_{n+3} & \theta_{1}y_{n+3} & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}; \quad E_{n} = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \theta_{-1}y_{n+3} & \theta_{-2}y_{n+2} \\ \cdot & \theta_{1}y_{n+2} & 1 & \theta_{-1}y_{n+2} & \theta_{-1}y_{n+2} \\ \cdot & \theta_{2}y_{n+1} & \theta_{1}y_{n+1} & 1 & \theta_{-1}y_{n+1} \\ \cdot & 0 & \theta_{2}y_{n} & \theta_{1}y_{n} & 1 \end{vmatrix}.$$
 (17a, b)

The first extends to infinity downward, the second upward. Simply infinite determinants are also the auxiliary subdeterminants

$$S_{n} = \begin{vmatrix} \theta_{1}y_{n} & \theta_{-1}y_{n} & \theta_{-2}y_{n} & 0 & \cdot \\ \theta_{2}y_{n+1} & 1 & \theta_{-1}y_{n+1} & \theta_{-2}y_{n+1} & \cdot \\ 0 & \theta_{1}y_{n+2} & 1 & \theta_{-1}y_{n+2} & \cdot \\ 0 & \theta_{2}y_{n+3} & \theta_{1}y_{n+3} & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}; \quad T_{n} = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \theta_{-1}y_{n+3} & \theta_{-2}y_{n+3} & 0 \\ \cdot & \theta_{1}y_{n+2} & 1 & \theta_{-1}y_{n+2} & 0 \\ \cdot & \theta_{2}y_{n+1} & \theta_{1}y_{n+1} & 1 & \theta_{-2}y_{n+1} \\ \cdot & 0 & \theta_{2}y_{n} & \theta_{1}y_{n} & \theta_{-1}y_{n} \end{vmatrix}. \quad (18a, b)$$

 $S_n$  differs from  $D_n$  only in the first column;  $T_n$  differs from  $E_n$  only in the rightmost column.

One readily establishes the recurrence relations

$$D_{n} = D_{n+1} - \theta_{-1} y_{n} S_{n+1} + \theta_{1} \theta_{-2} y_{n} y_{n+1} S_{n+2} - \epsilon y_{n} y_{n+2} D_{n+3}$$

$$+ \epsilon^{2} y_{n} y_{n+1} y_{n+2} y_{n+3} D_{n+4},$$
(19a)

$$S_n = \theta_1 y_n D_{n+1} - \theta_{-1} \theta_2 y_n y_{n+1} D_{n+2} + \epsilon y_n y_{n+1} S_{n+2}. \tag{19b}$$

A Laplace expansion of the doubly infinite determinant  $\mathcal{D}$  along a dividing line between row k=0 and k=-1 leads to the following expression involving only simply infinite determinants of type  $D_n$ ,  $E_n$ ,  $S_r$ ,  $T_n$ :

$$\mathcal{D} = D_0 E_1 - S_0 T_1 + y_0 y_1 (\theta_{-1} \theta_2 D_1 T_2 + \theta_1 \theta_{-2} S_1 E_2) - \epsilon (y_0 y_2 D_1 E_3 + y_1^2 D_2 E_2 + y_0 y_1 S_1 T_2) + \epsilon^2 (y_0 y_1 y_2 y_3 D_1 E_4 + y_0 y_1^2 y_2 E_2 D_3 + y_0 y_1^2 y_2 D_2 E_3).$$
 (20)

Since, by virtue of (1b)

$$E_n = D_n^* = D_n, (21a)$$

$$T_n = S_n^* \qquad (\neq S_n, \text{ when } \theta_1^* \neq \theta_1, \theta_2^* \neq \theta_2)$$
 (21b)

and by virtue of (2) and (19b)

$$S_n T_m = O(\mu^2), \tag{21c}$$

a replacement of  $E_n$  by  $D_n$  and a repeated insertion of (19b) into (20) gradually eliminates all but the  $D_n$  type of determinants from the expression for  $\mathcal{D}$ . It is also found that  $\theta_1$ ,  $\theta_{-1}$ ,  $\theta_2$ ,  $\theta_{-2}$  appear only in the combinations  $\delta$ ,  $\epsilon$ ,  $\eta$  given in (6a). Thus, by virtue of (6b), the expansion of  $\mathcal{D}$  progresses in powers of  $\mu^2$ . Using the notation

$$y_{0112} = y_0 y_1^2 y_2, (22a)$$

one finds that to  $\mu^{10}$  terms

$$D = D_0 D_1 - \delta y_{01} D_1 D_2 - \epsilon (y_{02} D_1 D_3 + y_{11} D_2^2) + \eta (2y_{011} D_2^2 + 2y_{012} D_1 D_3)$$

$$- \delta \epsilon (4y_{0112} D_2 D_3 + 2y_{0123} D_1 D_4) + \epsilon^2 (y_{0123} D_1 D_4 + 2y_{0112} D_2 D_3)$$

$$+ \epsilon \eta (4y_{01123} D_2 D_4 + 2y_{01234} D_1 D_5 + 2y_{01122} D_3^2)$$

$$- \delta \epsilon^2 (4y_{011234} D_2 D_5 + 4y_{011223} D_3 D_4 + 2y_{012345} D_1 D_6) + \cdots$$
(23)

The same process can also be carried out for the simply infinite determinant  $D_n$ . Disregarding the exceptional case  $\theta_0 = k^2$   $(y_k = \infty)$ , one finds that to  $\mu^{10}$  terms

$$D_0 = D_1 - \delta y_{01}D_2 + (-\epsilon y_{02} + 2\eta y_{012})D_3 + (-2\delta\epsilon + \epsilon^2)y_{0123}D_4 + 2\epsilon\eta y_{01234}D_5 - 2\delta\epsilon^2 y_{012345}D_6 + \cdots,$$
 (24a)

and  $D_n$  is obtained from  $D_0$  by increasing the subscripts in the latter's expression by n.

(24b)

It will be convenient to introduce at this point the notation

$$\sum y_{356} = y_{356} + y_{467} + y_{578} + \cdots, \qquad (22b)$$

$$\sum y_{12} \sum y_{356} = y_{12} \sum y_{356} + y_{23} \sum y_{467} + y_{34} \sum y_{578} - \cdots$$
 (22c)

Noting that

$$\lim D_n = 1, \tag{25}$$

one obtains, by repeated application of (24),

$$D_{0} = (1 - \delta y_{01})D_{2} + (-\delta y_{12} - \epsilon y_{02} + 2\eta y_{012})D_{3} + \cdots$$

$$= [1 - (y_{01} + y_{12})\delta - y_{02}\epsilon + 2y_{012}\eta]D_{3} + \cdots$$
(26a)

$$=1+A_{\delta}^{0}\delta+A_{\epsilon\epsilon}^{0}+A_{\eta\eta}^{0}+\cdots, \qquad (26b)$$

where

$$A_{\delta}^{0} = -\sum y_{01}, \quad A_{\epsilon}^{0} = -\sum y_{02}, \quad A_{\pi}^{0} = 2\sum y_{012}, \quad A_{\delta^{2}}^{0} = \sum y_{01} \sum y_{23}, \cdots$$
 (27)

The coefficient of a general term, like  $\delta \epsilon^2$ , is obtained as follows: Equation (24a) gives rise to the following symbolic products containing  $\delta \epsilon^2$ :

$$[-\delta y_{01}][-\epsilon y_{02}][-\epsilon y_{02}],$$
 (a)

$$[-\delta y_{01}][\epsilon^2 y_{0123}], \tag{b}$$

$$[-\epsilon y_{02}][-2\delta\epsilon y_{0123}], \tag{c}$$

$$[1][-2\delta\epsilon^2 y_{012345}]. \tag{d}$$

It is found that (a) contributes

$$-\sum y_{01} \sum y_{24} \sum y_{57} - \sum y_{02} \sum y_{34} \sum y_{57} - \sum y_{02} \sum y_{35} \sum y_{67}$$
 (28a)

to  $A_{\delta\epsilon}^0$ . (Note that no subscripts can be repeated, nor can any be skipped as one passes from one  $\sum$  to the next  $\sum$ ; furthermore (a) gives rise to three distinct summation expressions, because  $y_{k,k+1}$  can appear in the first place, in the second place, and in the third place.) The relation (b) yields

$$-\sum y_{01} \sum y_{2345} - \sum y_{0123} \sum y_{45}, \tag{28b}$$

which (c) yields

$$2\sum y_{02}\sum y_{3456} + 2\sum y_{0123}\sum y_{46} \tag{28c}$$

and (d) yields

$$-2\sum y_{012345}$$
 (28d)

The contributions (28a, b, c, d) sum up to  $\{0, \delta \epsilon^2\}$ . By increasing the subscripts of the expressions (28) by 2, one obtains

$$A_{\delta\epsilon^{2}}^{0} = -\sum y_{23} \sum y_{46} \sum y_{79} - \sum y_{24} \sum y_{56} \sum y_{79} - \sum y_{24} \sum y_{57} \sum y_{89} - \sum y_{23} \sum y_{4567} - \sum y_{2345} \sum y_{67} + 2 \sum y_{24} \sum y_{5678} + 2 \sum y_{2345} \sum y_{68} - 2 \sum y_{234567}.$$
(29)

The determination of the other  $\{n, \delta^i \epsilon^j \eta^k\}$  is similar.

The numerical values of the coefficients  $\{n, \delta^i \epsilon^j \eta^k\}$  are given in Table I to 5 decimal places, for  $\theta_0$  ranging from +0.9 to -1.0 (the interesting range in helicopter theory). In the evaluation of  $\{n, \delta^i \epsilon^j \eta^k\}$  the first 51  $y_m$  were taken into account. (The accuracy obtainable is thus equivalent to the use of a 101-row approximant to  $\mathcal{D}$ .)  $y_0$  to  $y_{20}$  were computed in some instances to 6, in some instances to 7 decimal places;  $y_{21}$  to  $y_{50}$  were computed to 7 decimal places. It is expected that the entries of Table I are in error by not more than 2 units in the fifth decimal place.

It is readily seen that the present method is not limited to Eq. (1), but can be extended to the general Hill differential equation where  $\theta_m e^{im\psi}$  form a convergent series.

For the special case of Mathieu's equation ( $\epsilon = \eta = 0$ ), one finds by (23) and (27) that

$$\mathcal{D} = D_1(D_0 - \delta y_{01}D_2) = 1 - 2\delta \sum_{k=0}^{\infty} \frac{1}{\theta_0 - k^2} \cdot \frac{1}{\theta_0 - (k+1)^2} + O(\delta^2)$$

$$= 1 - 2\delta \frac{\pi \cot \pi \sqrt{\theta_0}}{(4\theta_0 - 1)\sqrt{\theta_0}} + O(\delta^2). \tag{30}$$

<sup>&</sup>lt;sup>5</sup> An experienced computer can calculate a column of Table I in somewhat less than a day.

This formula was used by H. Bremekamp in 1926 in a study of the flow of electrons in metals.<sup>6</sup>

4. Example (a). Given  $\theta = 0.2$ ,  $\theta_1 = 0.19685 + 0.33465i$ ,  $\theta_2 = 0.03875 - 0.10258i$ . Determine  $v_1$ ,  $v_2$ . One finds  $\delta = 0.15074$ ,  $\epsilon = 0.01202$ ,  $\eta = -0.01635$ , and, by Table I,  $D_0 = 1.8422$ ,  $D_1 = 0.9434$ ,  $D_2 = 0.9934$ ,  $D_3 = 0.9980$ ,  $D_4 = 0.999$ ,  $D_5 = D_6 = 1.000$ . Likewise, by Table II,  $\mathcal{D} = 2.3291$ . Therefore,  $q = \sqrt{\mathcal{D}} \sin \pi \sqrt{\theta_0} = 1.5052$  and by (13b)  $\sigma_1 = 0.30782 + i/2$ ,  $\sigma_2 = -0.30782 - i/2$ . The associated functions  $v_1(\psi)$  and  $v_2(\psi)$  are determined from the equation system (9). Normalizing to  $c_0 = 1$ , and using the equations k = -4 to k = +4, one obtains<sup>7</sup>

$$v_{1} = (-0.1898 + 1.9817i)e^{+0.3078\psi} \left\{ -0.0958 \cos \frac{\psi}{2} + \sin \frac{\psi}{2} + 0.2076 \cos \frac{3\psi}{2} - 0.0083 \sin \frac{3\psi}{2} - 0.0125 \cos \frac{5\psi}{2} - 0.0038 \sin \frac{5\psi}{2} + 0.0008 \cos \frac{7\psi}{2} + 0.0019 \sin \frac{7\psi}{2} + \cdots \right\}$$

$$v_{2} = (1.6652 + 0.7467i)e^{-0.3078\psi} \left\{ \cos \frac{\psi}{2} + 0.4484 \sin \frac{\psi}{2} + 0.2107 \cos \frac{3\psi}{2} + 0.0188 \sin \frac{3\psi}{2} + 0.0041 \cos \frac{5\psi}{2} + 0.0101 \sin \frac{5\psi}{2} + 0.0005 \cos \frac{7\psi}{2} + 0.0020 \sin \frac{7\psi}{2} + \cdots \right\}.$$

5. Example (b). Given  $\theta_0 = 0$ ,  $\theta_1 = 0.37249 + 0.63323i$ ,  $\theta_2 = 0.13875 - 0.36728i$ . Determine  $\sigma$ . One finds

$$q = \pi \sqrt{D\theta_0} \cdot \sin \pi \sqrt{\theta_0} / \pi \sqrt{\theta_0} = \pi \sqrt{0.4392} = 2.082,$$
  

$$\sigma_1 = -\sigma_2 = 0.4339 + i/2.$$

<sup>&</sup>lt;sup>6</sup> M. J. O. Strutt, loc. cit., p. 26.

<sup>&</sup>lt;sup>7</sup> Note that the use of  $\sigma_1 = 0.30782 - i/2$  leads to the above expression of  $v_1$  when  $c_1$  is normalized to 1, and to the conjugate complex of the above when  $c_0$  is normalized to 1.

TABLE I.\* Numerical values of  $\{n, \delta^i e^j \eta^k\}$ .

		1 ABI	LE I. Numo	ericai vaiue	S OI \n, 0	εη- ζ.		
θ.	<b>= .9</b> .	.8 .	.7	.6	.5	.45	.4	.35
{ <b>0</b> , <b>δ</b> }	7.83180	4.63590	3.70199	3.38325	3.38203	3.48245	3.65865	3.92977
{0, €}	90665 6.36577	24871 3.51924	.00257 2.63695	.16467 2.270 <b>5</b> 0	.30899 2.14607	.38655 2.15140	.47423 2.20218	.57886 2.30622
$\{0, \eta\} \\ \{0, \delta^2\}$	55015	30117	22352	19037	17858	17822	18162	18937
(0, δε)	.52473	. 27858	. 20047	. 16578	. 15052	.14789	.14837	.15229
$\{0, \delta_{\eta}\}$	15952	08611	06307	05310	04908	04867	04928	05105
$\{0, \delta^2\}$ $\{0, \epsilon^2\}$	.00265	.00143 22658	.00104 16801	.00087 14320	.00081 13403	.00080 13372	.00081 13622	.00083 14199
(0, en)	00590	00363	00301	00278	00280	00292	00306	00330
$\{0, e\eta\} \\ \{0, \eta^2\} $	00638	00342	00249	00208	00191	00189	00190	00196
(U. 04)	00463 .00049	00248 .00025	00178 .00019	00150 .00015	00136 .00014	00133 .00014	00133 .00013	00136 .00014
$ \begin{cases} 0, & \delta^2 \eta \\ 0, & \delta e^2 \end{cases} $	.00317	.00170	.00123	.00103	.00095	.00093	.00013	.00096
{0, δεη}	00021	00012	00009	00008	00007	00007	00007	00007
{1, 8}	-3.27931 -1.26507	-1.61410	-1.05991	78342	61797	55795	50802	46584
[1, ε] [1, η]	80268	63934 38700	43033 24905	32553 18048	26241 13964	23943 12488	22021 11264	20392 10233
$\{1, \delta^2\}$	.04439	.02131	.01366	.00986	.00759	.00678	.00610	.00553
$\begin{array}{c} \{1,\ \delta\epsilon\} \\ \{1,\ \delta\eta\} \end{array}$	01646 .00739	00763 .00351	00471 .00223	00328 .00160	00243 .00122	00213 .00109	00187 .00098	00166 .00088
11. 83	00009	00004	00003	00002	00002	00001	00001	00001
{1, € <sup>2</sup> }	.03150	.01513	.00970	.00701	.00540	.00482	.00434	.00394
$ \begin{cases} 1, e\eta \\ 1, \eta^2 \end{cases} $	.00132	.00063	.00041	.00030	.00024	.00020	.00019	.00017
				.00004	.00002	.00002	.00002	
$\{2, \delta\}$	05351 03050	05160 02958	04981 02872	04813 02791	04654 02714	04579 02678	04505 02642	04434 02607
2. 7	00619	00591	00565	00541	00519	00508	02042 00498	00487
2. 62	.00025	.00024	.00023	.00021	.00021	.00020	.00020	.00019
2, 6e	.00001	.00001	.00001	.00001 .00002	.00001	.00001	.00002 .00002	.00002
2, δ 2, ε 2, η 2, δ <sup>2</sup> 2, δε 2, δη 2, ε <sup>2</sup> }	.00017	.00017	.00016	.00015	.00015	.00014	.00014	.00013
{3, 8}	01368	01349	01330	01311	01293	01284	01275	01267
{3. a}	00914	00903	00892	00881	00871	00866	00861	<b></b> .00856
3, η ) 3, δ² ) 3, δε )	00092 .00003	00090 .00003	00088 .00003	00086 .00003	00086 .00003	00084 .00003	00084 .00003	00083 .00003
3, åe	.00001	.00001	.00001	.00001	.00001	.00001	.00001	10000.
{3, €²}	.00002	.00002	.00002	.00002	.00002	.00002	.00002	.00002
{4, δ} {4, ∈}	00551	00546	00543	00538	00534	00532	00530	00528
}4, €}	00401 00024	00399 00024	00396 00023	00393 00023	00391 00023	00390 00023	00388 00023	00387 00023
$\{4, \eta\} \\ \{4, \delta e\}$	.00001	.00001	.00023	.00023	.00023	.00023	.00023	.00023
15. 81	00276	00275	00273	00272	00271	00270	00269	00269
{5, δ} {5, α} {5, η}	00213	00212	00211	00210	00209	00209	00208	00208
	00008	00008	00008	00008	00008	00008	00008	00008
<b>{6, 8}</b>	00158	00157	00157	00156	00155	00155	00155	00155
(6, ε) (6, η)	00126 00003	00126 00003	00126 00003	00125 00003	00125 00003	00125 00003	00125 00003	00125 00003
	1 .00005	00003	00003	00003	00003	0000	0000	00003
θ• =	.3	.25	.2	.15		.1	.05	0**
{0, 8}	4.33216	4.93480	5.87874	7.495	88 10.	78515	20.74569	1.000000
{0, €}	.71097	.88889	1.14867	1.573	91 2.	41481	4.92154	.250000
(U. 2)	2.48046 20279	2.75850 22456	3.21013 26021	4.000 322	81 5. 96	62957 <b>45</b> 255	10.59568 84825	.500000 039865
0, 82 0, 8e	.16054	.17498	.19957	.243	78 .	33621	.62017	.028683
(0, δη) (0, δ <sup>2</sup> ) (0, ε <sup>2</sup> )	05432	05977	06883	084	90 —.	11824	<b>22028</b>	010290
(0, 62)	.00089 15201	.00098 16827	.00112 19494	.001 241		00192 33884	.00357 63496	.000166 029835
(0, εη)	00364	00416	00496	006	30	00905	01736	000836
(0, εη) (0, η <sup>2</sup> ) (0, δ <sup>2</sup> ε)	00208 00144	00228 00158	00262 00180	003 002	22 —.	00448	- 00832	000388
0, 527	.000144	.00015	.00180	002 .000		00305 00032	00566 .00059	000262 .000028
$ \begin{array}{c c} (0, \delta^2 \eta) \\ (0, \delta e^2) \end{array} $	.00103	.00112	.00129	.001	60 .	00222	.00412	.000194
{0, ben}	00007	00007	00008	000	09	00012	00023	000012
{1, δ}	42975 18993	39853 17778	37126	347	26	32596	30694	28987
11. el	18993 09354	17778 08595	16712 07934	157 073	55 —.	14929 06844	14176 06388	13496 05980
$ \begin{cases} 1, & \eta \\ 1, & \delta^2 \end{cases} $	.00505	.00463	.00426	.003	95 .	00366	.00341	.00319
{1, δε}	00149 .00080	00134 .00073	00120 .00067	001	09 –.	00099	00091	00083
$ \begin{cases} 1, \delta \eta \\ 1, \delta^2 \end{cases} $	00080 00001	00073 00001	00001	.000 000		00057 00001	.00053 00001	.00050 00001
{1. e <sup>2</sup> }	.00359	.00330	.00303	.002	81 .	00261	.00243	.00227
$\{1, \epsilon \eta\} $ $\{1, \eta^2\}$	.00014	.00014	.00012	.000	12 .	00011	.00011	.00010
{1, ŋ²}	.00002	.00002	.00001	.000	V1 .	00001	.00001	.00001

<sup>\*</sup> Note 1.  $\{\pi, \delta^i e^j \eta^k\}$  which are less than .00001 in magnitude, or for which  $\pi > 6$ , are not shown. \*\* Note 2. For  $\theta_0 = 0$   $(y_0 = \infty)$  the coefficients  $\frac{A_0^0}{y_0}$ ,  $\frac{A_0^0}{y_0}$ ,  $\cdots$  of  $\frac{D_0}{y_0}$  are given.

TABLE I. (Continued)

θ ==  {2, δ} {2, e} {2, π} {2, π} {3, δ} {3, 6} {3, δ} {3, δ} {3, δ} {4, δ} {4, η} {5, δ} {6, ε} {6, η}  (0, δ) (0, ε) (0, δ) (0, δ) (0, δ) (0, δ] (	-19.32207 -5.06707 -9.46238 .75141 53194 .19281 00311	.25 042970254000468 .00018 .00002 .000013012500084600002 .00001 .00002005240038500022002680020700008012500012400124001240012400124001240012400124001240012424724	.2042320250700458 .00018 .00002 .00013012410084200002 .00001 .000020052200383000220020700207000080124400124001240003	0200 .00 .00 .00 .00 .000100 .00 .00 .00 .00	168	.1  04106 02445 00442 000017 000017 00002 00012  01225 00832 00078 000001 00002  00518 00381 00381 00002 00124 00003 3 7796388013	.05040450241500433 .00017 .00002 .00002 .00012012170082700078 .00002 .00001 .0000200516003800022002050020500123001230012335 -2.3228375802101322	0** 03 98023 80042500017 .00002 .000012012090082300070000200014003780002500205000080015300123000034
(2. e) (2. n) (2. n) (2. n) (2. n) (2. n) (2. e2) (3. e) (3. e) (3. e) (3. e) (3. e2) (4. e) (4. n) (5. e) (6. e) (6. n) (7. e) (9. e)	0257300477 .00019 .00002 .00002 .00013012580085100082 .00003 .00001 .00002005260038600207003680020700008001250012500124000030505050505050505050505050012400003	0254000468 .00018 .00002 .000013012500084600081 .00002 .00001 .000020052400385000220026800207000080012400124001380125400124	0250700458 .00018 .00002 .00002 .00013012410080 .00002 .0001 .00002005220038300022003830002200383000200052400124001240003	0200 .00 .00 .00 .00 .000100 .00 .00 .00 .00	475 450 018 018 018 018 018 018 018 018 019 01	.02445 .00442 .00017 .00002 .00002 .00012 .00125 .0026 .00001 .00002 .00518 .00381 .00002 .00266 .00266 .00008 .00154 .00124 .00003 3	0241500433 .00017 .00002 .00002 .00012012170082700078 .00002 .00001 .000020051600380003800025002050020500008001230012300003353528375802	0238
(2. e) (2. n) (2. n) (2. n) (2. n) (2. n) (2. e2) (3. e) (3. e) (3. e) (3. e) (3. e2) (4. e) (4. n) (5. e) (6. e) (6. n) (7. e) (9. e)	0257300477 .00019 .00002 .00002 .00013012580085100082 .00003 .00001 .00002005260038600207003680020700008001250012500124000030505050505050505050505050012400003	0254000468 .00018 .00002 .000013012500084600081 .00002 .00001 .000020052400385000220026800207000080012400124001380125400124	0250700458 .00018 .00002 .00002 .00013012410080 .00002 .0001 .00002005220038300022003830002200383000200052400124001240003	0200 .00 .00 .00 .00 .000100 .00 .00 .00 .00	475 450 018 018 018 018 018 018 018 018 019 01	.02445 .00442 .00017 .00002 .00002 .00012 .00125 .0026 .00001 .00002 .00518 .00381 .00002 .00266 .00266 .00008 .00154 .00124 .00003 3	0241500433 .00017 .00002 .00002 .00012012170082700078 .00002 .00001 .000020051600380003800025002050020500008001230012300003353528375802	0238
(2, η) (2, δ2) (2, δ2) (2, δ6) (2, 6π) (3, 6π) (3, 6π) (3, 6π) (3, 6π) (3, 6π) (4, 6π) (4, π) (5, δ) (5, π) (6, π) (7, δ) (0, 6π) (0, δπ) (0,	00477 .00019 .00002 .00002 .0000301258008510085100003 .00001 .00002005260038600022002680020700008001550012400003050501240000305050124090030505012400003	00468 .00018 .00002 .00002 .000013012500084600081 .00002 .00001 .00002005240038500022002070020800155001240000311351372.562204.48740 .3549124724	00458 .00018 .00002 .00002 .00003 01241 00842 00000 .00001 .00002 00522 00383 00022 00207 00207 00207 00207 00207 00154 00124 00154 00124 00154 00124 00003	00 .00 .00 .00 .00 .00 .00 .00 .00 .00	450 018 002 002 002 013 233 837 079 002 001 002 520 382 0022 266 206 206 206 207 25 25 342538 1.05015	.00442 .00017 .00017 .00002 .00002 .00012 .01225 .00832 .00078 .00002 .00001 .00002 .00018 .00022 .00206 .00008 .00154 .00124 .00003 .00154 .00124 .00003 .00154 .00003 .00154 .00003	00433 .00017 .00002 .00002 .000012 01217 00827 00078 .00002 .00001 .00002 00380 00380 00022 00003 00123 00123 00123 0003 35	00425 .00017 .00002 .00012 .00012 01209 00823 0007 .00002 .00001 00514 00378 00022 00025 00008 00123 0003
(2, \delta\) [2, \delta\] [2, \delta\] [3, \delta\] [3, \delta\] [3, \delta\] [3, \delta\] [3, \delta\] [3, \delta\] [4, \delta\] [4, \delta\] [4, \delta\] [4, \delta\] [5, \delta\] [5, \delta\] [6, \delta\] [6, \delta\] [6, \delta\] [0, \		.00018 .00002 .00002 .00013012500084600081 .00002 .00001 .000020052400385000220026800207000080012400124001031 -9.35137 -2.56220 -4.48740 .3549124724	.00018 .00002 .00002 .00003 .00013 .000842 .000842 .000802 .00001 .00002 .00002 .000267 .00267 .00207 .00008 .00124 .00124 .00124 .00103 .15	00	018	.00017 .00002 .00002 .00002 .00012 .01225 .00832 .00078 .00002 .00001 .00002 .00518 .00381 .00381 .00381 .00266 .00206 .00008 .00154 .00124 .00003 .00124 .00003 .00124 .00003	.00017 .00002 .00002 .00012 01217 00827 00078 .00002 .00001 .00002 00516 00205 00205 00205 00205 00123 00123 00123 35	.00017 .00002 .00002 .00012 01209 00823 00077 .00002 00514 00378 00022 00264 00205 00103 00123 00103
(2, \delta\) [2, \delta\] [2, \delta\] [3, \delta\] [3, \delta\] [3, \delta\] [3, \delta\] [3, \delta\] [3, \delta\] [4, \delta\] [4, \delta\] [4, \delta\] [4, \delta\] [5, \delta\] [5, \delta\] [6, \delta\] [6, \delta\] [6, \delta\] [0, \			.00002 .00013 01241 00842 00080 .00002 .00001 .00002 00522 00383 00022 00207 00207 00207 00124 00124 00124 00124 00124 00124 00227	0000000000000000	002 003 003 233 837 -079 002 001 002 520 -382 -022 -2266 -008 -154 -25 -34258 -1.05015	.00002 .00002 .000012 .001225 .00832 .00078 .00002 .00001 .00002 .00518 .00381 .00002 .00022 .00206 .00206 .00124 .00124 .00124 .00003 3	.00002 .00002 .00012 01217 00827 00078 .00002 .00001 .00002 00516 00380 00022 00205 00205 00205 00123 00123 00123 35	.00002 .00002 .00012 01209 00823 00077 .00002 .00001 00514 00378 00205 00008 00123 00123 00003
(2, 6η   (2, e²)   (2, e²)   (2, e²)   (2, e²)   (3, e²)   (3, e²)   (3, 6²)   (3, e²)   (4, e²)   (4, e²)   (4, η)   (5, 6²)   (5, η)   (6, 6²)   (6, η)   (7, 6²)   (9, 6²)		.00002 .00013 01250 00846 00081 .00002 .00001 .00002 00524 00385 00022 00268 00207 00008 00155 00124 00003	.00002 .00013 01241 00842 00080 .00002 .00001 .00002 00522 00383 00022 00207 00207 00207 00154 00154 00154 00124 00154 00124 00154 00124 00154 00124 00154 0	0000000000000000	002 013 233 	.00002 .00012 .01225 .00032 .00078 .00002 .00001 .00002 .00381 .00022 .00266 .00206 .00154 .00124 .00003 3	.00002 .00012 01217 00827 00078 .00002 .00001 .00002 00516 00380 00205 00205 00205 00205 00205 00123 00123 00123 35	.00002 .00012 01209 00823 00077 .00002 .00001 .00002 00514 00378 00022 00264 00205 00008 00123 00123 00003
(2, e <sup>2</sup> ) (3, δ) (3, η) (3, δ) (3, η) (3, δ) (3, δ) (4, η) (4, η) (5, δ) (5, η) (6, δ) (6, σ) (7, η) (9, δ) (9,	.00013012580085100852 .00003 .00001 .00002005260038600207000080012500124000030519.322075.067079.46238 .7514153194 .19281 .00311	.00013012500084600081 .00002 .00001 .000020052400385000220026800207000080015500124000031 -9.35137 -2.56220 -4.48740 .3549124724	.00013012410084200080 .00002 .00001 .000020052200383000220026700207000080015400124000315	.00010000 .00 .00 .000000	013 233	.00012 01225 00832 00832 000078 .00001 .00001 .00002 .00518 .00381 .00381 .00266 .00206 .00206 .00008 .00154 .00124 .00003	.00012012170082700078 .00002 .00001 .0000200516003800020500205002050000800123001230000335 -2.3228375802	.00012012090082300077 .000020051400378002050020500008001230012300003
3, \$\delta\} 3, \$\delta\} 3, \$\epsilon\} 3, \$\delta\} 3, \$\delta\} 3, \$\delta\} 4, \$\delta\} 4, \$\delta\} 4, \$\delta\} 4, \$\epsilon\} 5, \$\delta\} 5, \$\delta\} 6, \$\delta\} 6, \$\delta\} 6, \$\delta\} 7, \$\delta\} 7	012580085100082 .00003 .00001 .0000200526003860002200268002070000800155001240000305050519322075.067079.4623875141531941928100311	012500084600081 .00002 .00001 .0000200524003850002200207002080012500124000031 -9.35137 -2.56220 -4.48740 .3549124724	012410084200080 .00002 .00001 .000020038300022002070020700207000080154001240000315	0100000000000000	233	01225 00832 00078 00002 00001 00002 00518 00381 00022 00266 00206 00008 3 3	01217 00827 00078 .00002 .00001 .00002 00380 00205 00205 00205 00008 00123 00123 00123 00123 00123 00123 00123	012090082300077 .00072 .00001 .00002005140037800022002640020500008001230012300003
3. e   3. 7   3.	0085100082 .00003 .00001 .0000200526003860002200268002070000800124000030505015400035.067079.46238 .7514153194 .19281 .1928100311	0084600081000020000100002005240038500022002680020700008001550012400003135137 -2.56220 -4.48740 .3549124724	00842 00080 .00001 .00001 .00001 00522 00383 00022 00207 00207 00207 00124 00124 00124 00124 004482 - 1.72447 - 2.84360 .22402 15352	0000 .00 .00 .00 .0000000	837	.00832 .00078 .00002 .00001 .00002 .00518 .00381 .00381 .00206 .00206 .00206 .00124 .00124 .00124 .0003 3	00827 00078 00072 00002 00516 00380 00022 00205 00205 00123 00123 00123 35	00823 00077 .00002 .00001 00514 00378 00022 00264 00205 00008 00123 00123 00003
3. e   3. 7   3.	0085100082 .00003 .00001 .0000200526003860002200268002070000800124000030505015400035.067079.46238 .7514153194 .19281 .1928100311	0084600081000020000200524003850002200268002070000800155001240000311351372.562204.487403549124724	00842 00080 .00001 .00001 .00001 00522 00383 00022 00207 00207 00207 00124 00124 00124 00124 004482 - 1.72447 - 2.84360 .22402 15352	0000 .00 .00 .00 .0000000	837	.00832 .00078 .00002 .00001 .00002 .00518 .00381 .00381 .00206 .00206 .00206 .00124 .00124 .00124 .0003 3	00827 00078 00072 00002 00516 00380 00022 00205 00205 00123 00123 00123 35	00077 .00002 .00001 .00002 00514 00378 00022 00264 00205 00008 00123 00123 00003
(5, \delta\) (4, \delta\) (4, \delta\) (5, \delta\) (5, \delta\) (5, \delta\) (5, \delta\) (6, \delta\) (6, \delta\) (6, \delta\) (7, \delta\) (8, \delta\) (9, \		00081 .00002 .00001 .00002 00524 00385 00022 00268 00207 00108 00155 00124 00003 1	00080 .00002 .00001 .0000200522003830002200207002070000800124001240000315	00 .00 .00 .00 .00 .00 .00 .00 .00 .00	002 001 002 520	.00002 .00001 .00001 .00002 .00518 .00381 .000022 .00206 .00206 .00008 .00154 .00124 .00003 3	.00002 .00001 .00002 00516 00380 00022 00205 00205 00123 00123 00123 35	.00002 .00001 .00002 00514 00378 00022 00264 00205 00103 00123 00103 4
(5, \delta\) (4, \delta\) (4, \delta\) (5, \delta\) (5, \delta\) (5, \delta\) (5, \delta\) (6, \delta\) (6, \delta\) (6, \delta\) (7, \delta\) (8, \delta\) (9, \		.00002 .00001 .00002 00524 00385 00022 00268 00207 00008 00155 00124 00003 1	.00002 .00001 .00001 .00002 00323 00323 00207 00207 00008 00124 00124 0003 15	0000000000000000	002 001 002 520	.00001 .00002 .00518 .00381 .00022 .00266 .00206 .00008 .00154 .00124 .00003 3	.00001 .00002 00516 00380 00022 00205 00205 00008 00123 00123 00003 35	.00001 .00002 00378 00022 00264 00205 00008 00123 00123 00003
3. e <sup>2</sup>   4. δ   4. ϵ   4. ϵ   5. δ   5. ϵ   5. π   6. δ   6. π    θ    (0, π	0001 .00002 00526 00386 00022 00207 00008 00155 00124 00003 05 19.32207 5.06707 9.46238 .75141 53194 .19281 00311	.00001 .00002 00524 00385 00022 00268 00207 00008 00155 00124 00003 1	.00001 .00002 00522 00383 00022 00267 00207 00124 00124 00124 00003 15	.00 .00 .00 .00 .00 .00 .00 .00 .00 .00	001 002 520 382 022 266 206 008 154 003 25 -3.42538 -1.05015	.00001 .00002 .00518 .00381 .00022 .00266 .00206 .00008 .00154 .00124 .00003 3	.00001 .00002 00516 00380 00022 00205 00205 00008 00123 00123 00003 35	.00001 .00002 00378 00022 00264 00205 00008 00123 00123 00003
3. e <sup>2</sup>   4. δ   4. α   4. α   5. δ   5. α   5. α   6. α   6. η    0. δ	.000020052600386000220026800207000080015500124000030519.322075.067079.46238 .7514153194 .1928100311	.000020052400385000220026800207000080015500124000031 -9.35137 -2.56220 -4.48740 .3549124724	.000020052200383000220026700207000080012400124000315 -6.04482 -1.72447 -2.84360 .2240215352	0000000000000000	520	00518 00381 00022 00266 00206 00008 00154 00124 00003 3 -2.77963 88013	00516 00380 00022 00265 00205 00123 00123 00003 35	00514 00378 00022 00264 00205 00008 00153 00123 00003
4. e} 4. 7 5. 6} 5. 6} 5. e} 6. 8 6. e} 6. n}  0. 6 7 0. 7 0. 7 0. 7 0. 7 0. 7 0. 7 0. 7 0.	00386000220026800207000080015500124000030519.32207 -5.06707 -9.46238 .7514153194 .1928100311	00385 00022 00268 00207 00008 00155 00124 00003 1 1 935137 2.56220 4.48740 .35491 24724	00383 00022 00207 00207 00008 00154 00124 00003 15	00 00 00 00 00 00 00 00	382 - 022 - 266 - 206 - 206 - 008 - 154 - 024 - 003 25 - 3.42538 - 1.05015	.00381 .00022 .00266 .00206 .00008 .00154 .00124 .00003 3 -2.77963 88013	00380 00022 00025 00205 00008 00123 00123 00003 35	00264 00205 00008 00153 00123 00003 4
4. e} 4. 7 5. 6} 5. 6} 5. e} 6. 8 6. e} 6. n}  0. 6 7 0. 7 0. 7 0. 7 0. 7 0. 7 0. 7 0. 7 0.	00386000220026800207000080015500124000030519.32207 -5.06707 -9.46238 .7514153194 .1928100311	00385 00022 00268 00207 00008 00155 00124 00003 1 1 935137 2.56220 4.48740 .35491 24724	00383 00022 00207 00207 00008 00154 00124 00003 15	00 00 00 00 00 00 00 00	382 - 022 - 266 - 206 - 206 - 008 - 154 - 024 - 003 25 - 3.42538 - 1.05015	.00381 .00022 .00266 .00206 .00008 .00154 .00124 .00003 3 -2.77963 88013	00380 00022 00025 00205 00008 00123 00123 00003 35	00264 00205 00008 00153 00123 00003 4
(4. η) (5. δ) (5. ε) (5. η) (6. δ) (6. ε) (6. η) (7. ε) (9. ε) (9. δ) (9. ε) (9. δε) (	0002200268002070000800155001240000305 -19.32207 -5.06707 -9.46238 .7514153194 .1928100311	000220026800207000080015500124000031 -9.35137 -2.56220 -4.48740 .3549124724	0002200267002070000800154001240000315 -6.04482 -1.72447 -2.84360 .2240215352	00 00 00 00 00 00 00 2	022 266 206 008 024 003   	.00022 .00266 .00206 .00008 .00154 .00124 .00003 3 -2.77963 88013	00265 00205 00008 00153 00123 00003 35	00264 00205 00008 00153 00123 00003 4
5, 5} 5, e} 5, e} 5, n} 6, s} 6, a} 6, a} 7 8 8 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9 9	00268 00207 00008 00155 00124 00003 05 -19.32207 -5.06707 -9.46238 .75141 53194 .19281 00311	0026800207000080015500124000031 -9.35137 -2.56220 -4.48740 .3549124724	00267 00207 00008 00154 00124 00003 15 -6.04482 -1.72447 -2.84360 .22402 15352	00 00 00 00 00 00 2 -4.40273 -1.30380 -2.03119 .15940	266 206 208 154 00325 -3.42538 -1.05015	.00266 .00206 .00008 .00154 .00124 .00003 3 -2.77963 88013	00265 00205 00008 00153 00123 00003 35	00264 00205 00008 00153 00123 00003 4
5, ε   5, η   6, δ   6, ε   6, π   7   7   7   7   7   7   7   7   7	00207 00008 00155 00124 00003 05 -19.32207 -5.06707 -9.46238 .75141 53194 .19281 00311	00207 00008 00155 00124 00003 1 -9.35137 -2.56220 -4.48740 .35491 24724	00207 00008 00124 00124 00003 15 -6.04482 -1.72447 -2.84360 .22402 15352	00 00 00 00 2 -4.40273 -1.30380 -2.03119 .15940	206 008 154 024 00325 -3.42538 -1.05015	.00206 .00008 .00154 .00124 .00003 3 -2.77963 88013	00205 00008 00153 00123 00003 35	00205 00008 00153 00123 00003 4
5. η}  6, δ}  6, σ}  6, σ}  6, σ}  6, η}  6, σ}  7, σ}  9, σ}  9, σδ  10, σδ	0000800155001240000305 -19.32207 -5.06707 -9.46238 .7514153194 .1928100311	000080015500124000031 -9.35137 -2.56220 -4.48740 .3549124724	0000800154001240000315 -6.04482 -1.72447 -2.84360 .2240215352	00 00 00 00 2 -4.40273 -1.30380 -2.03119 .15940	008 154 024 00325 -3.42538 -1.05015	.00008 .00154 .00124 .00003 3 -2.77963 88013	00008 00153 00123 00003 35 -2.32283 75802	000080015300123000034 -1.98376659
$ \begin{array}{c c} (6, \delta) \\ (6, \epsilon) \\ (6, \eta) \end{array} $ $ \begin{array}{c c} \theta_0 & = \\ (0, \delta) \\ (0, \epsilon) \\ (0, \delta^2) \\ (0, \delta^2) \\ (0, \delta \epsilon) \\ (0, \delta \eta) \\ (0, \delta^3) \\ ($	00155 00124 00003 05 -19.32207 -5.06707 -9.46238 .751141 53194 .19281 00311	00155 00124 00003 1 -9.35137 -2.56220 -4.48740 .35491 24724	00154 00124 00003 15 -6.04482 -1.72447 -2.84360 .22402 15352	00 00 00 2 -4.40273 -1.30380 -2.03119 .15940	154	.00154 .00124 .00003 3 -2.77963 88013	00153 00123 00003 35 -2.32283 75802	00153 00123 00003 4 -1.9837 6659
6, -ε   6, η	00124 00003 05 -19.32207 -5.06707 -9.46238 .75141 53194 .19281 00311	00124 00003 1 -9.35137 -2.56220 -4.48740 .35491 24724	00124 00003 15 -6.04482 -1.72447 -2.84360 .22402 15352	00 00 2 -4.40273 -1.30380 -2.03119 .15940	024 	3 -2.77963 88013	00003 35 -2.32283 75802	00123 00003 4 -1.9837 6659
(6, -ε ) (6, π) (7, δ)   (9, ε ) (0, π) (0, δε ) (0, δη )	00124 00003 05 -19.32207 -5.06707 -9.46238 .75141 53194 .19281 00311	00124 00003 1 -9.35137 -2.56220 -4.48740 .35491 24724	00124 00003 15 -6.04482 -1.72447 -2.84360 .22402 15352	00 00 2 -4.40273 -1.30380 -2.03119 .15940	024 	3 -2.77963 88013	00003 35 -2.32283 75802	00123 00003 4 -1.9837 6659
(6, η)  θ <sub>0</sub> =   {0, δ}   0, q   0, η   0, δ <sup>2</sup>   0, δ <sub>0</sub>   0, δη   0, δ <sup>3</sup>   0, δ <sup>3</sup>	00003 05 -19.32207 -5.06707 -9.46238 .75141 53194 .19281 00311	00003 1 -9.35137 -2.56220 -4.48740 .35491 24724	00003 15 -6.04482 -1.72447 -2.84360 .22402 15352	00 2 -4.40273 -1.30380 -2.03119 .15940	25 -3.42538 -1.05015	3 -2.77963 88013	00003 35 -2.32283 75802	00003 4 -1.9837 6659
$\theta_0 = \{0, \delta\}$ $\{0, \epsilon\}$ $\{0, \epsilon\}$ $\{0, \delta^2\}$ $\{0, \delta\epsilon\}$ $\{0, \delta\eta\}$ $\{0, \delta^2\}$ $\{0, \delta^2\}$ $\{0, \delta^2\}$ $\{0, \delta^2\}$ $\{0, \delta^2\}$	05 -19.32207 -5.06707 -9.46238 .75141 53194 .19281 00311	1 -9.35137 -2.56220 -4.48740 .35491 24724	15 -6.04482 -1.72447 -2.84360 .22402 15352	2 -4.40273 -1.30380 -2.03119 .15940	25 -3.42538 -1.05015	3 -2.77963 88013	35 -2.32283 75802	-1.9837 6659
$   \begin{cases}     0, \delta \\     0, \epsilon \\     0, \eta \\     0, \delta^2 \\     0, \delta \epsilon \\     0, \delta \epsilon \\     0, \delta \eta \\     0, \delta^3 \\     0, \delta^3 \\     0, \delta^3 \\     0, \delta^3 \\     0, \delta^2 \\     0, \delta^3 \\     $	-19.32207 -5.06707 -9.46238 .75141 53194 .19281 00311	-9.35137 -2.56220 -4.48740 -35491 24724	-6.04482 -1.72447 -2.84360 .22402 15352	-4.40273 -1.30380 -2.03119 .15940	-3.42538 -1.05015	-2.77963 88013	-2.32283 75802	-1.9837 6659
(0, e) (0, η) (0, δ <sup>2</sup> ) (0, δε) (0, δη) (0, δ <sup>3</sup> )	-5.06707 -9.46238 .75141 53194 .19281 00311	-2.56220 -4.48740 .35491 24724	-1.72447 -2.84360 .22402 15352	-1.30380 -2.03119 .15940	-1.05015	88013	·75802	6659
(0, e) (0, η) (0, δ <sup>2</sup> ) (0, δε) (0, δη) (0, δ <sup>3</sup> )	-5.06707 -9.46238 .75141 53194 .19281 00311	-2.56220 -4.48740 .35491 24724	-1.72447 -2.84360 .22402 15352	-1.30380 -2.03119 .15940	-1.05015	88013	·75802	6659
(0, η) (0, δ <sup>2</sup> ) (0, δε) (0, δη) (0, δ <sup>3</sup> )	-9.46238 .75141 53194 .19281 00311	-4.48740 .35491 24724	-2.84360 .22402 15352	-2.03119 .15940	_1 55045	.00013	1 01222	.0001
(0, δe) (0, δη) (0, δ²)	53194 19281 00311	.35491 24724	. 22402 15352	.15940		-1.23488		
{0, δe} {0, δη} {0, δ <sup>2</sup> }	53194 .19281 00311	24724	<b>-</b> . 15352	.13940	.12120	.09616	.07860	.0656
$ \begin{cases} 0, \delta \eta \\ 0, \delta^2 \end{cases} $	. 19281 00311	24/24	<del></del> . 13332	10745	08037	06274	05042	0414
(0, 84)	00311		OF470	10745 .04017	.03036	.02396	.01947	.0161
{n =2}		.09051	.05679 — .00092	00065	00048	00038	00031	0002
0, εη ) 0, εη )		00146 .26550			.09061	.07188	.05874	.0490
0. 2	.56222 .01609	.20330	. 16755 . 00500	.11919 .00363	.00281	.00226	.00188	.0016
10.70						.00228	.00188	.0005
10.00	.00724	.00338	.00211	.00149	.00112			.0003
(U, 0'e)	.00488 00048	.00227	.00141	.00098	.00074 00008	.00059 00006	.00047 00006	0000
(0, δ <sup>2</sup> η) (0, δe <sup>2</sup> )	00048	00023	00014	00010 00073	00055	00044	00035	0002
(0. de)	00357 .00023	00167 .00010	00103 .00006	.00004	.00003	.00003	.00003	.0000
{1, 8}	<b>27445</b>	<b>26046</b>	24772	23607	22538	21553	20643	1980
[1, e]	12880	12318	11805	11332	10897	10494	10120	0977
1, η) 1, δ <sup>2</sup> 1, δε	05614	05282	04981	04707	04457	04227	04016	0382
$\{1, \delta^2\}$	.00299	.00281	.00264	.00249	.00235	.00223	.00211	.0020
{1, δe}	00076	<b>00069</b>	00064	00059	00054	00050	00047	0004
$\{1, \delta\eta\}$ $\{1, \delta^3\}$	.00046	.00043	.00040	.00038	.00036	.00034	.00032	.0003
{1, & }	00001	00001	00001	<b>-</b> .00001				
{1, e2}	.00213	.00200	.00188	.00178	.00168	.00159	.00151	.0014
$ \begin{cases} 1, e\eta \\ 1, \eta^2 \end{cases} $	.00009	.00009	.00008	.00008	.00007	.00007	.00007	.0000
$\{1, \eta^2\}$	.00001	.00001	.00001	.00001	.00001	.00001	.00001	
[2, 8]	03929	03873	03819	03766	03714	03663	03614	0356
2, e   2, η   2, δ <sup>2</sup>    2, δε   2, δη	02357	02328	02301	02275	02249 00387	02223	<b>02198</b>	0217
12. "	00417	00409	00401	00394	00387	00380	00373	0036
12. 821	.00016	.00016	.00016	.00015	.00015	.00015	.00015	.0001
12. če	.00002	.00002	.00002	.00002	.00002	.00002	.00002	.0000
2. 8n	.00002	.00002	.00001	.00001	.00001	.00001	.00001	.0000
{2, e <sup>2</sup> }	.00011	.00011	.00011	.00011	.00011	.00010	.00010	.0001
						- 01162	01155	0114
{3, 8}	01201 00818	01193 00814	01185 00809	01178	01170 00801	01163 00796	01155 00792	0078 0078
[3, ε] [3, η]	00818 00077		00809 00075	00805 00075	00801 00074	00796 00073	00792	0007
13. 71	00077 .00002	00076 00002	00073 .00002	00073 .00002	.00074	.00073	.00002	.0000
3, 62	.00002	.00002	.00002	.00002	.00002	.00002	.00002	.0000
(3, δe) (3, e <sup>2</sup> )	.00001	.00001	.00001	.00001	.00001	.00001	.00001	.0000
{4, 8}	00512	00510	00509	00507	00505 00372	00503	00501	0050
{4, e} {4, η}	00377	00376	— .00375	00374	00372	00371 00021	00370 00021	003 000
{4, η}	00022	00022	00021	00021	00021	00021	00021	000
15 2)			00262	00262	00261	00261	00260	002
13.01	_ 00244		00262 00204	00262 00203	00201 00203	00201 00202	00200 00202	002
18, 11	00264 00204	00263		00203	00203	00202	00008	0021
(5, δ) (5, ε)	00204	— .00204	- 00204			(MAA16		000
[5, ε] [5, η]	00264 00204 00008	00263 00204 00008	00008	00008	00008	00008		000
{5, η}	00204 00008	00204 00008	00008			- 00152		
[5, ε] [5, η] [6, δ] [6, ε]	00204	— .00204	00204 00008 00152 00123	00008 00152 00122	00008 00152 00122	00008 00152 00122	00151 00122	000 001 001

TABLE I. (Continued)

θ. =	45	5	55	6	7	8	9	-1.0
{0, 8}	-1.72274	-1.51622	-1.34910	-1.21138	99849	84237	72362	63067
(0, €)	59385	<b>-</b> .53588	<b>48819</b>	<b>44823</b>	<b>-</b> .38497	<b>-</b> .33706	<b>-</b> . 29947	26917
$\{0, \eta\}$	<b>72520</b>	62733	<b>-</b> . 54879	<b> 44863</b>	38670	31617	— . 26349	22300
$\{0, \delta^2\}$	.05584	.04813	.04195	.03691	.02924	.02374	.01965	.01652
(0, δe)	— .03464	— .02936	02516	02176	01666	01306	01043	0084
$\{0, \delta\eta\}$	.01368	.01172	.01016	.00888	.00695	.00559	.00457	.0038
0, 82	00022	00019	00016	00014	00011	00009	00007	0000
0, €2	.04172	.03595	.03133	.02757	.02184	.01772	.01467	.0123
(0, en)	.00139	.00121	.00107	.00096	.00078	.00064	.00055	.0004
$\{0, \eta^2\}$	.00050	.00043	.00037	.00032	.00025	.00020	.00016	.0001
(0, δ²ε)	.00033	.00028	.00024	.00021	.00016	.00013	,00010	.0000
$\{0, \delta^2 \eta\}$	00003	00003	00002	00002	00001	00001	00001	0000
0, δε2	00025	00021	00018	00016	00012	00010	00008	0000
(0, δεη)	.00001	.00001	.00001	.00001		100010	.00000	.0000
(4 1)	10017	10000	47/00	46074	45045	44702	42000	4006
{1, δ}	19017	18288	17608	16971	<b>15815</b>	14793	13882	1306
{1, €}	09448	09144	08859	08591	08102	07664	07272	0691
11. 7)	03641	03473	03318	03173	02911	02682	02480	0230
$\{1, \delta^2\}$	.00191	.00182	.00173	.00166	.00151	.00139	.00128	.0011
1, δε	00040	00037	00034	00032	00028	00024	00020	0001
$\{1, \delta\eta\}$	.00029	.00027	.00026	.00025	.00022	.00020	.00018	.0001
{1, e <sup>2</sup> }	.00136	.00130	.00124	.00118	.00108	.00099	.00092	.0008
{1, εη}	.00006	.00006	.00006	.00006	.00005	.00005	.00004	.0000
{2, δ}	03519	03473	03428	03384	03300	03219	03141	0306
(2, €)	02150	02126	02103	<b>-</b> .02081	<b>-</b> .02037	<b>-</b> .01995	<b>-</b> .01955	<b>-</b> .0191
$\{2, \eta\}$	00361	00354	00349	00342	00331	<b>-</b> .00320	00310	0015
$\{2, \delta^2\}$	.00014	.00014	.00014	.00013	.00013	.00012	.00012	.0001
{2, δε}	.00002	.00002	.00002	.00002	.00002	.00002	.00002	.0000
$\{2, \delta_{\eta}\}$	.00001	.00001	.00001	.00001	,00001	.00001	.00001	.0000
{2, e²}	.00010	.00010	.00010	.00009	.00009	.00009	.00008	.0000
{3, 8}	01141	01134	01127	01120	01106	01093	01080	0106
[3, €]	00784	00779	00775	00771	00763	00755	00748	0074
(3, 7)	00072	00071	00071	00070	00068	00067	00066	0006
13. 82	.00002	.00002	.00002	.00002	.00002	.00002	.00002	.0000
(3, δε)	.00001	.00001	.00001	.00001	.00001	.00001	.00001	.0000
(3, €2)	.00001	.00001	.00001	.00001	.00001	.00001	.00001	.0000
{4, δ}	00498	00496	00494	00492	00489	00485	00482	0047
[4, e]	00368	00366	00365	00364	00362	00360	00358	0035
{4, η}	00021	00020	00020	00020	00020	00020	00020	0002
{5, δ}	00259	00259	00258	00257	00256	00255	00253	0025
[5, €]	00201	00201	00200	00200	00199	00198	00197	0023
(5, 7)	00008	00008	00008	00008	00008	00198 00007	00197 00007	0000
(6, 8)	00151	00151	00150	00150	00150	00149	00149	0014
(6, e)	00131	00131	00130	00130 00121	00130 00121	00149 00120	00149 00120	
(6, 7)	00003	00121	00003	00003	00003	00120		0012
(4.7)	0003	00003	00003	00003	00003	00003	00003	0000

Table II. Expansion of  $\mathcal{D}$  (row 1  $\times$  row 2  $\times$  row 3, see Eq. 23). Numerical tabulation of the coefficients in row 3.

θ.	row 1 row 2 row 3 {	$(D_1)$ $(D_1)$	$\left\{ egin{array}{l} lacksquare b \ D_1 \ y_{01} \end{array}  ight\} = \left\{ egin{array}{l} D_1 D \ -y_0 \end{array}  ight.$			η D <sub>1</sub> D <sub>2</sub> 2 y <sub>012</sub> }	{D <sub>2</sub> D <sub>3</sub> -4 y <sub>0112</sub>	$ \begin{bmatrix} D_1D_4 \\ -2 y_{0122} \end{bmatrix} $
.5	1			7143 -4.000			9.14285	
. 45				2598 -3.305			8.27740	
. <u>4</u> . 35	1			9445 —2.777 8278 —2.366			7.71606 7.41094	
.3			76190 .9	0090 -2.040	13.6054	3 2.57400	7.35428	
. 25		1 5.	.33333 1.0	6667 -1.777	78 14.2222	2 2.84445	7.58519	.32508
.2	i	1 6.	25000 1.3	1579 -1.562	50 15.6250	0 3.28948	8.22369	.37380
. 15 . 1	į		.84314 1.7 .11111 2.5	3160 —1.384 6410 —1.234	08 18.4544 57 24.6913	5 4.07436 5 5.69800	9.58673 12.66222	
.05				6329 -1.108	03 44.3213	5 10.65956	22.44119	
0*		0 1.	.00000 .2	5000 0	2.0000	0 .50000	1.00000	.05556
05		1 -19.	04762 -4.9		03 -36.2811	8 -9.40624		
10 15			.09091 -2.4 .79710 -1.6				-8.06287 -4.85876	
13 2			16667 -1.1				-3.30686	
25		1 -3.	200009	4118640	00 -5.1200	0 -1.50588	-2.40941	16280
3				7519591			-1.83478	
35 4	1			5681 — .548 6818 — .510	370 -3.1354 320 -2.5510		-1.44157 -1.15955	
45 45		1 -1	532564	9938 — .475	$\frac{-2.3310}{62}$	868880	95006	07289
5			.333334	4444444	44 -1.7777	8 — .59259	79012	06238
55	i			9960416			66530	
6 7	.			6232 — .390 0395 — .346			56612 42069	
	1			6042 — .308	64 — .7716	028935	32150	02953
— . A								
8 9		1	.58480 — .2	<b>2676 — .277</b>			25126	
9 -1.0		1 -:	.58480 — .2	2676 — .277 0000 — .250			20000	
<b>`−.9</b>	{D <sub>1</sub> D <sub>4</sub> {y <sub>0122</sub>	1 -:	.58480 — .2			020000 { D <sub>2</sub> D <sub>6</sub>		
9 -1.0 σον 1 σον 2 row 3	{ D <sub>1</sub> D <sub>4</sub> y <sub>0133</sub>	$ \begin{array}{ccc} 1 & - \\ 1 & - \end{array} $ $ \begin{array}{ccc} D_2D_2 \\ 2 & y_{012} \end{array} $	584802 500002 {D <sub>2</sub> D <sub>4</sub> {4 y <sub>1112</sub>	0000250 εη D <sub>1</sub> D <sub>6</sub> 2 y <sub>0124</sub>	Di 2 yeiiii}	020000 {D <sub>2</sub> D <sub>b</sub> -4 y <sub>11124</sub>	20000  \$\frac{\delta^2}{D_1D_4}  -4 y_011212  .30732	02000 -2 y <sub>01226</sub>
9 -1.0 **row 1 row 2 row 3	{D <sub>1</sub> D <sub>4</sub> y <sub>0133</sub> 13445 - 13312	$ \begin{array}{ccc} 1 & - & \\ 1 & - & \\ & & \\ 2 & & \\ 2 & y_{013} \end{array} $ $ \begin{array}{c} D_2D_2 \\ 2 & y_{013} \end{array} $ $ \begin{array}{c} -4.57143 \\ -4.13870 \end{array} $	584802 500002 { D <sub>2</sub> D <sub>4</sub> 4 y <sub>91112</sub> 1.07563 96812	0000250  εη D <sub>1</sub> D <sub>6</sub> 2 y <sub>0124</sub> .01735 .01712	Di 2 yenni 1 .30612 1 .16583	0 − .20000 {D <sub>2</sub> D <sub>6</sub> −4 y <sub>11134</sub> .06941 .06226	20000  \$\frac{\delta^2}{D_1D_4}  -4 \text{ y}_{011222}  .30732 .27271	02000  -2 y <sub>012146</sub> -2 y <sub>012146</sub> .00071 .00070
row 1  # row 2  row 3	13445 - 13312 13458	1	584802 500002 {D <sub>2</sub> D <sub>4</sub> 4 y <sub>9113</sub> 1.07563 96812 .89721	0000 — .250	Di 2 yenn }  1.30612 1.16583 1.07168	.06941 .06226 .05752	20000  \$\frac{\delta e^2}{D_1 D_4}\$  -4 yours:  .30732 .27271 .24922	$ \begin{array}{c}02000 \\ \hline02000 \\02000 \\00071 \\ .00070 \\ .00070 \end{array} $
row 1 row 2 row 3	$ \begin{cases} D_1D_4 \\ y_{1111} \end{cases} $ $13445 \\13312 \\13458 \\13922 $	1	584802 500002 { D <sub>2</sub> D <sub>4</sub> { 4 y <sub>01123</sub> 1.07563 96812 .89721 .85676	.01735 .01772 .01772 .01772 .01779	D <sub>1</sub> 2 y <sub>0111</sub> 3 1.30612 1.16583 1.07168 1.01520	0 — .20000 { D <sub>2</sub> D <sub>6</sub> —4 уенты .06941 .06226 .05752 .05475	20000  \$\frac{\delta e^2}{D_1 D_4}\$  -4 yours:  .30732 .27271 .24922 .23498	$ \begin{array}{c}02000 \\ \hline02000 \\02000 \\00071 \\ .00070 \\ .00072 \end{array} $
row 1 # row 2 row 3  .5 .45 .4 .35 .3 .25	$ \begin{cases} D_1D_4 \\ y_{0132} \end{cases} $ $13445 \\13312 \\13458 \\13922 \\14793 $	1 2 2 D2D1 2 yms 3 -4.57143 -4.13870 -3.85803 -3.70547 -3.67714	584802 500002 {D <sub>2</sub> D <sub>4</sub> 4 y <sub>9113</sub> 1.07563 96812 .89721	0000 — .250	Di 2 yenn 1 1.30612 1.16583 1.07168	.06941 .06226 .05752	20000  \$\frac{\delta e^2}{D_1 D_4}\$  -4 yours:  .30732 .27271 .24922	$ \begin{array}{c}02000 \\ \hline02000 \\02000 \\02000 \\00070 \\ .00070 \\ .00070 \end{array} $
9 -1.0  row 1 row 2 row 3	D <sub>1</sub> D <sub>1</sub>   y <sub>9113</sub>  13445  13425  13458  13922  14793  16254  18690	1	584802 500002 { D <sub>2</sub> D <sub>4</sub> { 4 y <sub>01133</sub> 1.07563 96812 .89721 .85676 .84532 .86688 .93450	0000 — .250  гр ДіДь 2 уним  .01735 .01712 .01726 .01779 .01885 .02064 .02366	Di5000  Di 2 years }  1.30612 1.16583 1.07168 1.01520 .99382 1.01136 1.08206	D1D620000  {D1D6 - 4 yellism  .06941 .06226 .05752 .05475 .05384 .05504 .05915	20000  \$\frac{\delta e^2}{D_1 D_1} -4 \frac{\gamma \cdots \cdots \cdots}{27271} -24922 -23498 -22846 -23117 -24592	02000  D1D0   -2 y 11146   .00071 .00070 .00072 .00076 .00083 .00095
	D <sub>1</sub> D <sub>1</sub>   y <sub>0111</sub>  13445  13312  13458  13922  14793  16254  18690  23019	1	584802 500002 { D.D. 4 yess 1.07563 96812 89721 85676 84532 86688 93450 1.08323	0000 — .250	D <sub>1</sub> <sup>2</sup> 2 y <sub>0111</sub> 30612 1.16583 1.07168 1.01520 .99382 1.01136 1.08206 1.24503	0 − .20000 {D <sub>2</sub> D <sub>6</sub> − 4 y <sub>01134</sub> .06941 .06226 .05752 .05475 .05384 .05504 .05915 .06834	20000  \$42 D_1D_1 -4 yourse  .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136	02000  D <sub>1</sub> D <sub>0</sub> \ -2 y <sub>01336</sub> \  .00071 .00070 .00072 .00076 .00083 .00095 .00117
-1.0  now 1  now 2  row 3  .5  .45  .4  .35  .3  .25  .2  .15  .1	\[ \begin{align*} \D_1 D_1 \\ \y \text{min} \end{align*} \] \[13445 \\13312 \\13458 \\13922 \\14793 \\16254 \\18690 \\23019 \\32011 \end{align*}	1	584802 500002 { D <sub>2</sub> D <sub>4</sub> { 4 y <sub>01131</sub> 1.07563 96812 .89721 .85676 .84532 .86688 .93450 1.08323 1.42273	еп D <sub>1</sub> D <sub>6</sub> 2 унин .01735 .01712 .01726 .01779 .01885 .02064 .02366 .02904 .04027	D1 2 years 1	D <sub>2</sub> D <sub>6</sub>  20000	20000  \$42 D_D, -4 yourss  .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136 .36480	02000  D <sub>1</sub> D <sub>6</sub> \ -2 y <sub>011146</sub> \ .00071 .00070 .00070 .00072 .00076 .00083 .00095 .00117 .00162
9 -1.0  now 1 row 2 row 3  .5 .45 .45 .3 .3 .25 .2 .15 .1 .05 0*	D <sub>1</sub> D <sub>1</sub>   y <sub>1111</sub>  13445  13312  13458  13922  14793  16254  18690  23019  32011  59550	1	584802 500002 { D.D. 4 yess 1.07563 96812 89721 85676 84532 86688 93450 1.08323	0000 — .250	Di5000  Di 2 yem;  1.30612 1.16583 1.07168 1.01520 .99382 1.01136 1.08206 1.24503 1.62336 2.84066 .12500	0 − .20000 {D <sub>2</sub> D <sub>6</sub> − 4 y <sub>01134</sub> .06941 .06226 .05752 .05475 .05384 .05504 .05915 .06834	20000  \$42 D_1D_1 -4 yourse  .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136	02000  D <sub>1</sub> D <sub>0</sub> \ -2 y <sub>01334</sub> \ .00071 .00070 .00070 .00072 .00076 .00083 .00095
-9 -1.0  now 1 now 2 row 3  .5 .45 .4 .35 .3 .25 .2 .15 .1 .05 -05	D <sub>1</sub> D <sub>1</sub>   Y = 13445   -13458   -13458   -13458   -14793   -16254   -18690   -23019   -32011   -59550   -02778   -51968	1	584802 500002 { D <sub>2</sub> D <sub>4</sub> { 4 y <sub>01133</sub> 1.07563 96812 .89721 .85676 .84532 .86688 .93450 1.08323 1.42273 2.50737 .11111 -1.9974	0000 — .250	Di 2 yenn; }  1.30612 1.16583 1.07168 1.01520 .99382 1.01136 1.24503 1.62336 2.84066 .12500 -2.21193	(D <sub>1</sub> D <sub>6</sub> ) -4 y <sub>11114</sub> -06941 -06226 -05752 -05475 -05384 -05504 -05915 -06834 -08948 -15722 -00694 -12335	20000  \$42 D1D4 - 4 yourss  .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136 .36480 .63478 .02778 .48882	02000  D <sub>1</sub> D <sub>4</sub> } -2 y <sub>11144</sub> }  .00071 .00070 .00070 .00076 .00083 .00083 .00117 .00162 .00300 .00014
9 -1.0  row 1 row 2 row 3  .5 .45 .35 .35 .25 .2 .15 .1 .05 .0*051	D <sub>1</sub> D <sub>1</sub> y <sub>111</sub> 13445 - 13312 13458 13922 14793 16254 18690 23019 32011 95550 02778 .51968 .24366	1	584802 500002 { D <sub>1</sub> D <sub>4</sub> { 4 ymms 1.07563 96812 .89721 .85676 .84532 .86688 .93450 1.08323 1.42273 2.50737 .11111 -1.97974 -88604	0000 — .250  *** *** *** *** *** *** *** *** ***	Di5000  Di 2 years }  1 .30612 1 .10583 1 .07168 1 .01520 .99382 1 .01136 1 .08206 1 .24503 1 .62336 2 .84066 .12500 -2 .2119398328	D <sub>1</sub> D <sub>6</sub>	20000  \$42 D1D, -4 yourss  .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136 .36480 .63478 .027784888221610	02000  D <sub>1</sub> D <sub>4</sub> \ -2 y <sub>11144</sub> \ .00071 .00070 .00070 .00072 .00083 .00095 .00117 .00162 .00300 .000140026000121
9 -1.0  now 1  now 2  row 3  .5 .45 .4 .35 .3 .25 .2 .15 .0*0515	D <sub>1</sub> D <sub>1</sub> , y <sub>111</sub> -1.13445 -1.13312 -1.13458 -1.13922 -1.14793 -1.16254 -1.18690 -2.3019 -32011 -3.9550 -0.2778 -3.1968 -2.4366 -1.5267	1 2 D <sub>2</sub> D <sub>1</sub> } 2 y <sub>min</sub> } -4.57143 -4.13870 -3.85803 -3.70547 -3.67714 -3.79260 -4.11185 -4.79337 -6.33111 -11.22059 -50000 8.95833 ,4.03144 2.42938	58480 - 2 50000 - 2 { D <sub>1</sub> D <sub>4</sub> { 4 y <sub>01131</sub> 1.07563 96812 .89721 .85676 .84532 .86688 .93450 1.08323 1.42273 2.50737 -11111 -1.97974 88604 53101	0000 — .250	D1 2 years }  1.30612 1.16583 1.07168 1.01520 .99382 1.08206 1.24503 1.62336 2.84066 .12500 -2.211939832858539	(D <sub>1</sub> D <sub>6</sub> -4 y <sub>611114</sub> -06921 .06921 .05752 .05475 .05384 .05504 .05915 .06834 .15722 .00694 -12335 -05503288	20000  \$42 D3D4 - 4 Y01111  .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136 .36480 .63478488822161012796	02000  D <sub>1</sub> D <sub>4</sub> } -2 y <sub>1134</sub> }  .00071 .00070 .00070 .00076 .00083 .00095 .00117 .00162 .00300 .00014 .0026000121 .00075
9 -1.0  row 1 row 2 row 3  .5 .45 .35 .35 .25 .2 .15 .1 .05 .0*051	D <sub>1</sub> D <sub>1</sub> y <sub>111</sub> - 13445 - 13312 - 13458 - 13922 - 14793 - 16254 - 18690 - 23019 - 32011 - 59550 - 02778 51968 24366 15267 10783 - 88140	1  2 D <sub>1</sub> D <sub>1</sub> } 2 y <sub>min</sub> }  -4.57143 -4.13870 -3.85803 -3.70547 -3.67714 -3.79260 -4.11185 -4.79337 -6.33111 -11.22059 -50000 8.95833 4.03144 2.42938 1.65343 1.20470	58480 - 2 50000 - 2 { D <sub>2</sub> D <sub>4</sub> 4 y <sub>1111</sub> 1.07563 96812 .89721 .85676 .84532 .86688 .93450 1.08323 1.42273 2.50737 .1111 -1.97974 88604 53101 35948		D1   2 yenrs   1.30612   1.16583   1.07168   1.01520   .99382   1.0136   1.24503   1.62336   2.84066   .12500   -2.21193  98328  58539  39367   .28346	D <sub>1</sub> D <sub>6</sub>	20000  \$42 D3D4 - 4 Yenses  .207271 .24922 .23498 .22846 .23117 .24592 .28136 .36480 .634784888221610127960855806129	02000  D <sub>1</sub> D <sub>4</sub> \ -2 y <sub>*1186</sub> \ .00071 .00070 .00070 .00076 .00083 .00083 .00117 .00162 .00300 .0001400260001210007500053
9 -1.0   • row 1 row 2 row 3  .5 .45 .4 .35 .3 .25 .2 .15 .0 •051152253	D.D. (901) - 13445 - 13312 - 13458 - 13922 - 14793 - 16254 - 18690 - 23019 - 32011 - 59550 - 02778 51968 24366 - 15763 - 10783 - 08140 - 064412	1	584802 500002 { D <sub>1</sub> D <sub>1</sub> { 4 y <sub>01133</sub> 1.07563 96812 .89721 .85676 .84532 .86688 .93450 1.08323 1.42273 2.50737 .11111 -1.97974 88604 53101 35943 19728	0000 — .250	1,30612 1,16583 1,07168 1,01520 99382 1,01136 1,0236 1,0236 1,24503 1,62336 2,84066 1,2500 -2,21193 -,98328 -,58539 -,28346 -,21335	D <sub>1</sub> D <sub>6</sub>  20000	20000  \$42 D1D4 - 4 y01111  .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136 .36480 .63478 .02778 .02778 .12796127960855806129 .064588	02000  D <sub>1</sub> D <sub>4</sub> } -2 y <sub>1134</sub> }  00071 .00070 .00070 .00076 .00083 .00083 .00117 .00162 .00300 .00014 .0026000121000530005300040
9 -1.0  now 1 now 2 row 2 row 3  .5 .45 .45 .35 .35 .25 .2 .15 .0*05115225353535353535353	D <sub>1</sub> D <sub>1</sub> y <sub>111</sub> 13445  - 1331213458139221479316254186902301932011595500277851968243661526710783081400641205203	1	584802 500002 {	0000 — .250  *** *** *** *** *** *** *** *** ***	Di5000  Di 2 years }  1 .30612 1 .16583 1 .07168 1 .01520 .99382 1 .01136 1 .08206 1 .24503 1 .62336 2 .84066 .12500 -2 .21193983285853939367283462133516570	D <sub>1</sub> D <sub>b</sub>	20000  \$42 D2D4 -4 y011111  .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136 .36480 .63478 .0277848882216101279608558061290458804588	02000  D <sub>1</sub> D <sub>6</sub> \ -2 y <sub>11166</sub> \ .00071 .00070 .00070 .00072 .00072 .00083 .00095 .00117 .00162 .00300 .0001400260001210007500053000400003100025
9 -1.0  now 1  now 1  now 2  row 3  .5 .45 .4 .35 .3 .25 .2 .15 .0*05152253353353353354	D <sub>1</sub> D <sub>1</sub> -13445 -13312 -13458 -13922 -14793 -16254 -18690 -23019 -32011 -59550 -02778 -24366 -10783 -08140 -06412 -05203 -04318	1	584802 500002 { D <sub>1</sub> D <sub>1</sub> { 4 y <sub>11111</sub> 1.07563 96812 .89721 .85676 .84532 .86688 .93450 1.08323 1.42273 2.50737 .1111 -1.97974 88604 35943 26048 15418 12336	0000 — .250	005000    2 years	D <sub>1</sub> D <sub>6</sub>  20000     D <sub>2</sub> D <sub>6</sub>  4 y <sub>611134</sub>  06941   .06226   .05752   .05475   .05384   .05504   .05915   .06834   .08948   .15722   .00694   .12335  05503   .03288  02218  01603   .01210   .00943  00752	20000  \$42 D3D4 - 4 yourss  .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136 .36480 .63478 .02778 .1279608558061290612906129061280	02000  D <sub>1</sub> D <sub>4</sub> \ -2 y <sub>1134</sub> \ .00071 .00070 .00070 .00076 .00083 .00095 .00117 .00162 .00300 .00014 -00260001210007500053000310002500021
9 -1.0  now 1 row 2 row 3  .5 .45 .45 .45 .35 .3 .25 .1 .05051515253353544455	D <sub>1</sub> D <sub>1</sub> -13445 -13312 -13458 -13922 -14793 -16254 -18690 -23019 -32011 -59550 -02778 -51968 -24366 -10783 -08140 -06412 -0530 -04318 -03644 -03119	1	58480 — .2 50000 — .2 { D <sub>2</sub> D <sub>4</sub> { 4 ymms 1.07563 96812 .89721 .85676 .84532 .86688 .93450 1.08323 1.42273 2.50737 .1111 -1.97974 88604 53101 35943 26048 19728 15418 12336 10054 08317	0000 — .250	D1   2 yenrs   1.30612   1.16583   1.07168   1.01136   1.08206   1.24503   1.62336   2.84066   1.2500   -2.21193  58539  39367  28346  21335  16570  13177  10675  08779	D <sub>1</sub> D <sub>6</sub>	20000  \$4 <sup>2</sup> D <sub>3</sub> D <sub>4</sub> - 4 y <sub>01111</sub> .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136 .36480 .634784888221610127960855803544025940225904588402259	02000  D <sub>1</sub> D <sub>4</sub> \ -2 y <sub>01344</sub> \ .00071 .00070 .00070 .00076 .00083 .00095 .00117 .00162 .00300 .00014 .0026000121 .0007500053 .000400003100021000210001700017
	D.D. (901) - 13445 - 13312 - 13458 - 13922 - 14793 - 16254 - 18690 - 23019 - 32011 - 39050 - 02778 - 10783 - 08140 - 06412 - 05203 - 04318 - 03644 - 03119 - 027700	1	584802 500002 { D.D.	0000 — .250	00 — .5000    2 years   2 years   1.30612   1.16583   1.07168   1.01520   1.08206   1.24503   1.62336   2.84066   .12500   -2.21193  98328  58539  39367  28346  21335  16570  13177  10675  08779  07311	D <sub>1</sub> D <sub>6</sub>  20000	20000  \$42 D1D4 - 4 y011111  .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136 .36480 .63478 .02778 .02778 .1279612161012796085580612904588045880458803544022590184801848018480184801848	02000  D <sub>1</sub> D <sub>4</sub> \ -2 y <sub>1114</sub> \ .00071 .00070 .00070 .00072 .00076 .00083 .00083 .00014 .000260012100075 .00030 .00014 .00025000310002500031000250002100017000150001700015
9 -1.0   now 1 row 2 row 3  .5 .45 .45 .45 .35 .3 .25 .1 .050511525335354455556	D.D. \$\forall y\text{9'11}\$1344513312134581392214793162541869023019320115955002778 .51968 .24366 .15267 .10783 .08140 .06412 .05203 .04318 .03644 .03119 .02700 .02359	1	58480 — .2 50000 — .2 { D <sub>1</sub> D <sub>4</sub> { 4 ymms 1.07563 96812 .89721 .85676 .84532 .86688 .93450 1.08323 1.42273 2.50737 .11111 —1.97974 —.88604 —.53101 —.35943 —.26048 —.15418 —.15418 —.12336 —.10544 —.08317 —.06967 —.05897	0000 — .250  *** *** *** *** *** *** *** *** ***	Di	D <sub>1</sub> D <sub>6</sub>  20000	20000	02000  D <sub>1</sub> D <sub>6</sub> \ -2 y <sub>11166</sub> \ .00071 .00070 .00070 .00072 .00072 .00083 .00095 .00117 .00162 .00300 .0001400260001210007500040000310002500021000150001500015000150001500015
9 -1.0  now 1  now 2  row 3  .5 .45 .4 .35 .3 .25 .2 .15 .0*051522533544555567	D.D., ymm 13445133121345813922147931625418690230193201159550027781078381400064120520304318036440311027000235901843	1	58480 — .2 50000 — .2 { D <sub>1</sub> D <sub>4</sub> { 4 y <sub>11111</sub> 1.07563 96812 .89721 .85676 .84532 .86688 .93450 1.08323 1.42273 2.50737 .11111 -1.97974 88604 15418 15418 15418 19728 15418 1054 08317 06967 05897 04337	0000 — .250	1.30612 1.16583 1.07168 1.01520 .99382 1.01136 1.02336 2.84066 1.24503 1.62336 2.84066 .12500 -2.21193 98328 58539 39367 28346 13177 16575 06154 06154	D <sub>1</sub> D <sub>6</sub>  20000     D <sub>1</sub> D <sub>6</sub>   -4 y <sub>11114</sub>   -4 y <sub>11114</sub>   -66226   .05752   .05384   .05504   .05915   .06834   .08948   .15722   .00694   .12335  05503  01210   .00943  01210  00943  00752  00611  00752  00611  00355  00260  00260	20000  \$42 D3D4 - 4 yourss  .30732 .27271 .24922 .23498 .22846 .23117 .24592 .28136 .36480 .63478 .00778 .02778 .02778 .02778 .02778 .03544 .048821216100855806129045880354402259 .0184802259 .01848012310128200923	02000  D <sub>1</sub> D <sub>4</sub> } -2 y <sub>1134</sub> }  .00071 .00070 .00070 .00076 .00083 .00117 .00114 .0002600121000250003100021000150001300011000110001100011
9 -1.0   # row 1 row 2 row 3  .5 .45 .45 .45 .35 .3 .25 .1 .0505115253353544455556	D.D. \$\forall y\text{9'11}\$1344513312134581392214793162541869023019320115955002778 .51968 .24366 .15267 .10783 .08140 .06412 .05203 .04318 .03644 .03119 .02700 .02359	1	58480 — .2 50000 — .2 { D <sub>1</sub> D <sub>4</sub> { 4 ymms 1.07563 96812 .89721 .85676 .84532 .86688 .93450 1.08323 1.42273 2.50737 .11111 —1.97974 —.88604 —.53101 —.35943 —.26048 —.15418 —.15418 —.12336 —.10544 —.08317 —.06967 —.05897	0000 — .250  *** *** *** *** *** *** *** *** ***	Di	D <sub>1</sub> D <sub>6</sub>  20000	20000	02000  D <sub>1</sub> D <sub>4</sub> \ -2 y <sub>11144</sub> \ .00071 .00070 .00070 .00072 .00073 .00083 .00095 .00117 .00162 .00300 .00014002600012100075000400001100015000150001500015

<sup>\*</sup> Note 1. For  $\theta_0=0$  ( $y_0=\infty$ ) the coefficients of  $\mathcal{D}/y_0$  are given.