A METHOD OF SOLUTION OF FIELD PROBLEMS BY MEANS OF OVERLAPPING REGIONS*

BY

H. PORITSKY AND M. H. BLEWETT
General Electric Company

1. Introduction. In problems involving the determination of fields, it often happens that the region R for which the field is to be determined is difficult to handle directly, but can be broken up into several overlapping regions $R_1, R_2, \cdots$ for each of which the field can be determined by standard methods. We suppose that the breaking up is carried out in such a manner that every point of the region R falls into *at least one* of the regions $R_1, R_2, \cdots$. In the following, Schwartz' "alternating procedure" is applied to the solution of field problems for such regions R.

To illustrate, let us consider the determination of a function u harmonic over the region R shown in Fig. 1, bounded by two circular arcs ABC and CDA with centers at O and O'. For simplicity we assume that the radii of the two circles are equal and the boundary values of u are symmetric about the straight line through A and C. It will be noticed that by completing the circular arcs by means of the dotted curves AEC and CFA one obtains the circular regions over which the determination of a harmonic function in terms of boundary values is well known. Here R is the region bounded by the solid circular arcs ABC and CDA, while the regions R_1 and R_2 are the circular regions bounded by the complete circles with centers at O and O'. The problem then is to utilize the relatively easy solution of the Dirichlet problem for the circular regions R_1 and R_2 in an efficient manner toward the solution of the Dirichlet problem over R.

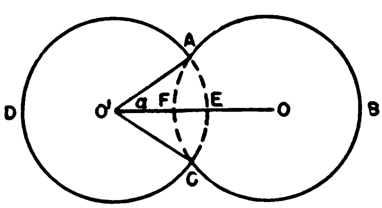

Fig. 1.

This is done by assuming the values of u over the arc AFC; the solution of the Dirichlet problem for the circle R_1 with center O, based on these assumed values and on the known boundary values over ABC, is then utilized to furnish the values of u over AEC. The procedure is then repeated by solving the Dirichlet problem for the circle R_2 with center at O', and the values over AFC are recalculated. By alternating between R and R' in this way, continual improvement of the values of u over both arcs AFC and AEC results; in the limit this leads to a solution of the Dirichlet problem for the region of Fig. 1.

In the following we shall illustrate the procedure, not for the Laplace equation

$$\nabla^2 u = 0, \tag{1.1}$$

but for the equation

$$(\nabla^2 + k^2)u = 0 \tag{1.2}$$

which is encountered in wave motion under the assumption of sinusoidal oscillations, for the region shown in Fig. 2. Other cases of interest in connection with (1.2) which

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can also be treated by the present method are given by the "open end correction of an organ pipe," wave passage through a change of width of a channel, T-sections, etc.

2. Wave propagation around a corner. We consider a solution of the differential equation (1.2) for the region shown in Fig. 2; this solution is to satisfy the boundary conditions

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad DOG,\ EBF, \tag{2.1}$$

$$u = A_1 e^{ikx} + B_1 e^{-ikx} \quad \text{for large positive } x, \tag{2.2}$$

$$u = A_3 e^{iky} + B_3 e^{-iky} \quad \text{for large positive } y, \tag{2.3}$$

where A_1, B_1, A_3 and B_3 are proper constants. Equations (2.2) and (2.3) can be described physically by the statement that u behaves as a plane wave at infinity.

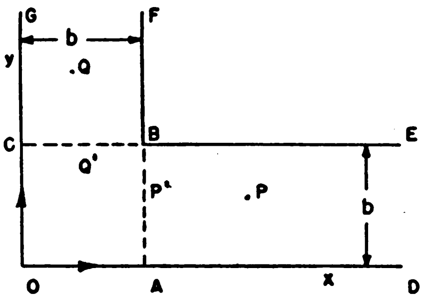

FIG. 2.

The above problem is encountered in the propagation of a transverse electromagnetic wave around a corner or through a channel the section of which is shown in Fig. 2. Here the channel has an infinite depth in the z-direction; the field components are assumed to be independent of z, and the only non-vanishing magnetic field component is H_z. At the boundaries, which are assumed to be metallic and perfectly conducting, the electric field is normal; this leads to the vanishing of the normal derivative of H_z, i.e., $\partial H_z/\partial n = 0$. Formulation of the field in terms of H_z leads to the above problem.

On account of the vanishing of the normal derivative over the y-axis, reflection across it is possible, thus extending the region of Fig. 2 into the region shown in Fig. 3. This reflection is carried out in accordance with the relation

$$u(-x, y) = u(x, y). \tag{2.4}$$

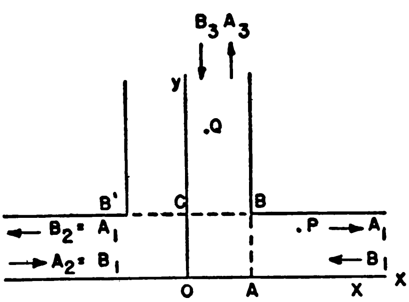

FIG. 3.

In view of this reflection the behavior of u at $x = -\infty$ is given by the expression

$$u = B_1 e^{ikx} + A_1 e^{-ikx}. \tag{2.5}$$

As a result of this reflection the semi-infinite strip $DOCE$ of Fig. 2 can be replaced by the 2-way infinite strip of Fig. 3

$$-\infty < x < \infty, \qquad 0 < y < b.$$

Similar reflection can be carried out across the lower boundary $y = 0$ of Fig. 3; this allows us to replace the semi-infinite vertical channel by a vertical channel extending to infinity both up and down. A proper behavior for u at $y = -\infty$ can be obtained from (2.3).

The general procedure which was outlined in §1 is applied to the present case. First, we consider the strip $0<y<b$ of Fig. 3, and assume values for $\partial u/\partial n$ over the dotted part $B'CB$ of its upper boundary. Since $\partial u/\partial n$ vanishes over the rest of its boundary and the behavior of u at ∞ is described by (2.2) and (2.5), it is possible to determine u at any point interior to this strip. This determination is carried out by means of a Green's function G. The derivation of G will be described presently; for the present it will suffice to say that the value of u at an interior point P of the strip is given by the relation

$$u_p = u(x_0, y_0) = 2B_1 \cos kx_0 + \frac{1}{2\pi}\int_{-b}^{b} \left(\frac{\partial u}{\partial y}\right)_{y=b} G dx. \tag{2.6}$$

G has a pole at $P=(x_0, y_0)$, and (2.6) requires that G be evaluated on the dotted line $B'CB$. After u is determined in this way, differentiation of (2.6) with respect to x allows one to compute $\partial u/\partial x$, and in particular to determine this derivative over AB. Turning now to the infinite vertical strip $0<x<b$, we repeat the same procedure and determine the function u at any point interior to this strip; in particular, we evaluate u and $\partial u/\partial y$ over CB. The process is then repeated.

The definition of the Green's function for the differential equation (1.2) and the boundary condition (2.1) for a finite region R is specified by the following:

a) G satisfies (1.2) everywhere in R except at the pole P;
b) $\partial G/\partial n$ vanishes along the boundary of R;
c) at the pole P, G becomes infinite like $-\ln r'$, where r' is the distance from P. (2.7)

We apply Green's theorem in the form

$$\int\cdot [u(\nabla^2 + k^2)v - v(\nabla^2 + k^2)u]dA = \int \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds \tag{2.8}$$

to the region R, exclude the neighborhood of the point by means of a small circle of radius ϵ and let ϵ approach zero. This yields the equation

$$u_p = \frac{1}{2\pi}\int \frac{\partial u}{\partial n} G ds, \tag{2.9}$$

where the integration is carried out over the boundary of R. In the present case, for the infinite strip $0<y<b$ special additional considerations are required. It will be assumed that in addition to the requirements (2.7) the Green's function G behaves at infinity like a divergent plane wave. Solutions of (1.2) which depend on x only are

$$e^{\pm ikx}. \tag{2.10}$$

We consider the wave equation

$$\frac{\partial^2 U}{\partial t^2} = c^2\nabla^2 U, \tag{2.11}$$

and look for solutions of the form $ue^{\pm i\omega t}$. If we set $k=\omega/c$, we find that u satisfies Eq. (1.2), and that e^{ikx} represents a plane wave traveling in the direction of positive x while e^{-ikx} represents a similar wave traveling in the direction of negative x. It will be assumed that at $x = \pm\infty$ the Green's function G behaves like a divergent plane wave of the same amplitude at $x = +\infty$ as at $x = -\infty$.

It will be assumed that the dimension b satisfies the inequality

$$b < \pi/k. \tag{2.12}$$

Physically this assumption means that the width b of the strip is less than half the wave length $\lambda/2 = \pi/k$ of a plane wave at the frequency in question. The effect of this assumption and the features which arise when it is not satisfied will appear presently.

First, we place the pole P on the y-axis. We shall obtain G as a series in the form

$$G = \sum_{n=0}^{\infty} A_n u_n \tag{2.13}$$

where u_n are product solutions of the wave equation (1.2), i.e., u_n consist of the product of a function of x and a function of y; more explicitly,

$$\left.\begin{aligned} u_0 &= \exp\left[ik\cdot |x|\right], \\ u_n &= \cos\frac{ny}{b}\exp\left[-\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\cdot|x|\right], \quad (n > 0). \end{aligned}\right\} \tag{2.14}$$

These product solutions u_n $(n>0)$ have been chosen so that they don't become infinite at $x = \pm\infty$, while u_0 represents a divergent plane wave. If the inequality (2.12) were not satisfied, several radicals in u_n $(n>0)$ would be imaginary, infinitely large values of u_n could not be avoided, and additional stipulations regarding the behavior of G at infinity would have to be made.

The functions u_n are symmetric about the vertical line $x=0$ through the pole P, and continuous at $x=0$. However, $\partial u_n/\partial x$ is discontinuous at $x=0$, the discontinuity being

$$\Delta\left(\frac{\partial u_n}{\partial x}\right) = \left\{\begin{aligned} &-2ik && \text{for} \quad n = 0, \\ &2\sqrt{\left(\frac{n\pi}{k}\right)^2 - k^2}\cos\frac{n\pi y}{b} && \text{for} \quad n > 0. \end{aligned}\right\} \tag{2.15}$$

Thus each term u_n may be considered as representing the wave function corresponding to a sinusoidal distribution of sources* over the line $x=0$. The density σ of the sources is given by the familiar condition from potential theory

$$\text{discontinuity in normal derivative} = \Delta\left(\frac{\partial u}{\partial x}\right) = -2\pi\sigma, \tag{2.16}$$

and in the present case is given by

$$\sigma_n = \frac{1}{\pi}\sqrt{\frac{n^2\pi^2}{b^2} - k^2}\cos\frac{n\pi y}{b}. \tag{2.17}$$

* By a "source" is meant here a solution of (1.1) which depends only upon the distance r from a fixed point, is singular at $r=0$ like $-\ln r$, and behaves at infinity like a divergent cylindrical wave. The distributions of such sources satisfy continuity-discontinuity relations similar to those in the case of logarithmic potentials.

For (2.13) this yields

$$\sigma = \frac{1}{\pi}\sum_{n=0}^{\infty} A_n \sqrt{\frac{n^2\pi^2}{b^2} - k^2} \cos\frac{n\pi y}{b}\,. \tag{2.18}$$

Let us now consider the concentrated point source at the pole P, and express it as a Fourier series of cosines over the interval $x=0$, $0<y<b$, obtaining

$$\sigma = \frac{1}{b} + \frac{2}{b}\sum_{n=1}^{\infty}\cos\frac{n\pi y_0}{b}\cos\frac{n\pi y}{b} \tag{2.19}$$

where $x=0$, $y=y_0$ are the coordinates of the pole P. Solving for A_n, we obtain for the Green's function G the Fourier series

$$G = \frac{2\pi}{b}\Bigg\{-\frac{1}{2ik}\exp\left[ik\cdot|x|\right] + \sum_{n=1}^{\infty}\frac{\cos(n\pi y_0/b)}{\sqrt{(n\pi/b)^2 - k^2}}\cos\frac{n\pi y}{b}\exp\left[-\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\cdot|x|\right]\Bigg\}. \tag{2.20}$$

Due to the behavior of G at infinity it is found that after applying the Green's theorem (2.8) over the rectangular region $-l'<x<l$ and letting l and l' recede to infinity, certain additional terms R' and R'' arise from the boundaries $x=l$ and $x=l'$. Equation (2.9) is now replaced by

$$u(x_0, y_0) = \frac{1}{2\pi}\int_{-l'}^{l}\frac{\partial u(x,y)}{\partial y}\bigg|_{y=b} G\,dx + R' + R'', \tag{2.21}$$

where

$$R' = \frac{1}{2\pi}\int_0^b\left(\frac{\partial u}{\partial x}G - u\frac{\partial G}{\partial x}\right)dy\bigg|_{x=l}, \tag{2.22}$$

$$R'' = \frac{1}{2\pi}\int_0^b\left(-\frac{\partial u}{\partial x}G + u\frac{\partial G}{\partial x}\right)dy\bigg|_{x=-l'}. \tag{2.23}$$

In view of (2.4), (2.2), (2.20), (2.21) and (2.22), Eq. (2.21) can be given the form (2.6).

As explained above, in the present case not (2.6) but its x-derivative will be found useful. Differentiation of (2.6) yields

$$\frac{\partial u(x_0, y_0)}{\partial x_0} = -2kB_1\sin kx_0 + \frac{1}{2\pi}\int_{-b}^{b}\left(\frac{\partial u}{\partial y}\right)_{y=b}\frac{\partial G}{\partial x_0}dx. \tag{2.24}$$

To obtain this equation, the integral in (2.6) has been differentiated under the integral sign; this is permissable since the limits of integration are independent of x_0. Since $(\partial u/\partial y)_{y=b}$ is also independent of x_0, only G has to be differentiated. The explicit form of (2.24) is given by the relation

$$g(y_0) = \left(\frac{\partial u}{\partial x_0}\right)_{x_0=b} = -2kB_1\sin kb - \frac{1}{b}\int_{-b}^{b}f(x)\left[K_0 + \sum_{n=1}^{\infty}K_n\right]dx. \tag{2.25}$$

where

$$\left.\begin{aligned} f(x) &= \left(\frac{\partial u}{\partial y}\right)_{y=b}, \qquad K_0 = \tfrac{1}{2} \exp\left[-ik(x-b)\right], \\ K_n &= (-1)^n \cos\frac{n\pi y_0}{b} \exp\left[\sqrt{\left(\frac{n\pi}{b}\right)^2 - k^2}\,(x-b)\right]. \end{aligned}\right\} \tag{2.26}$$

A similar expression holds for $(\partial u/\partial y_0)$ along CB;

$$f(x_0) = \left(\frac{\partial u}{\partial y_0}\right)_{y_0=b} = -2kB_3 \sin kb - \frac{1}{b}\int_{-b}^{b} g(y)\left[K_0 + \sum K_n\right]dy \tag{2.27}$$

where K_0, K_n are as in (2.26) but with the coordinates interchanged.

In applying (2.25) one must assume not only $f(x)$ but also B_1. Likewise in applying (2.27), B_3 is required along with $g(y)$. Furthermore, A_1 and A_3 are essential to the complete solution. In this connection it is advisable to keep the following relations between $f(x)$, $g(y)$ and the constants A_1, B_1, A_3 and B_3 in mind:

$$A_1 = B_1 - \frac{1}{2ikb}\int_{-b}^{b} f(x)e^{-ikx}dx, \tag{2.28}$$

$$A_3 = B_3 - \frac{1}{2ikb}\int_{-b}^{b} g(y)e^{-iky}dy, \tag{2.29}$$

$$A_1e^{ikb} - B_1e^{-ikb} = \frac{1}{2ikb}\int_{-b}^{b} g(y)dy, \tag{2.30}$$

$$A_3e^{ikb} - B_3e^{-ikb} = \frac{1}{2ikb}\int_{-b}^{b} f(x)dx. \tag{2.31}$$

These relations enable one to express A_1, B_1, A_3 and B_3 in terms of $f(x)$ and $g(y)$.

The relation (2.28) is established by applying (2.6) to $u(x_0, y_0)$ for x_0 so large that G reduces to its first term in (2 20), and comparing the result with (2.2). A similar derivation over $0<x<b$ yields (2.29). As regards (2.30) it is established by expanding $\partial u/\partial x$ in the horizontal strip $0<y<b$ in a series of cosines of $n\pi y/b$ and comparing for large positive x this expansion with $\partial u/\partial x$ as derived from (2.2); a similar procedure applied over the vertical strip $0<x<b$ to $\partial u/\partial y$ leads to (2.31).

In the present example, in view of the geometric symmetry of the region of Fig. 2 about the diagonal OB, any function u over the region can be expressed as the sum of a function which is odd about this diagonal, and one which is even about it. The calculations outlined are simplified considerably for even and odd functions u, are quite similar for the two cases and will be illustrated for the odd case.

In the odd case,

$$A_3 = -A_1, \qquad B_3 = -B_1, \tag{2.32}$$

$$g(x) = -f(x), \tag{2.33}$$

and the integral relations (2.28)–(2.31) reduce to

$$g(y) = -2kB_1 \sin kb - \frac{1}{b}\int_{-b}^{b} f(x)\left[K_0 + \sum_{n=1}^{\infty} K_n\right]dx, \tag{2.34}$$

$$\left.\begin{aligned} A_1 - B_1 &= -\frac{1}{2ikb}\int_{-b}^{b} f(x)e^{-ikx}dx, \\ A_1e^{ikb} - B_1e^{-ikb} &= -\frac{1}{2ikb}\int_{-b}^{b} f(x)dx. \end{aligned}\right\} \quad (2.35).$$

After (2.25) has been applied for an initially assumed $f(x)$ curve, the resulting $g(y)$ shape, changed in sign and plotted against x, can be considered as the next approximation to $f(x)$, in view of the relation (2.33) and the symmetry of the region about $x=y$.

From (2.35) the coefficients A_1 and B_1 may be determined in terms of $(\partial u/\partial y)_{y=b}=f(x)$. This yields

$$\left.\begin{aligned} A_1 &= -\frac{1}{4kb\sin kb}\int_{-b}^{b} f(x)\left[e^{-ik(x+b)} - 1\right]dx, \\ B_1 &= -\frac{1}{4kb\sin kb}\int_{-b}^{b} f(x)\left[e^{ikb(b-x)} - 1\right]dx. \end{aligned}\right\} \quad (2.36)$$

The procedure used consisted in assuming $f(x)$, calculating A_1 and B_1 from (2.36), then applying (2.34) to calculate $g(y)$, and using the shape of the latter with the sign

TABLE 1

y_0	x	$\sum K_n$	$bG/2\pi$
$.1b$	$.9b$	$-.4347$	$.0613+.0627i$
	$.7b$	$-.3025$	$.1625+.1841i$
	$.5b$	$-.1934$	$.2111+.2938i$
	$.3b$	$-.1151$	$.2034+.3852i$
	$.1b$	$-.0682$	$.1448+.4524i$
	$-.1b$	$-.0392$	$.0545+.4912i$
	$-.3b$	$-.0222$	$-.0527+.4991i$
	$-.5b$	$-.0126$	$-.1671+.4775i$
	$-.7b$	$-.0070$	$-.2756+.4217i$
	$-.9b$	$-.0040$	$-.3567+.3410i$
$.3b$	$.9b$	$-.4143$	$.0817+.0627i$
	$.7b$	$-.2546$	$.2104+.1841i$
	$.5b$	$-.1458$	$.2587+.2938i$
	$.3b$	$-.0806$	$.2379+.3852i$
	$.1b$	$-.0453$	$.1677+.4524i$
	$-.1b$	$-.0252$	$.0679+.4912i$
	$-.3b$	$-.140$	$-.0445+.4991i$
	$-.5b$	$-.0078$	$-.1625+.4775i$
	$-.7b$	$-.0044$	$-.2730+.4217i$
	$-.9b$	$-.0025$	$-.3632+.3410i$
$.5b$	$.9b$	$-.3348$	$.1612+.0627i$
	$.7b$	$-.1373$	$.3277+.1841i$
	$.5b$	$-.0437$	$.3608+.2938i$
	$.3b$	$-.0133$	$.3052+.3852i$
	$.1b$	$-.0039$	$.2091+.4524i$
$.5b$	$-.1b$	$-.0011$	$.0920+.4912i$
	$-.3b$	$-.0003$	$-.0308+.4991i$
	$-.5b$	0	$-.1545+.4775i$
	$-.7b$	0	$-.2686+.4217i$
	$-.9b$	0	$-.3607+.3410i$
$.7b$	$.9b$	$-.1433$	$.3527+.0627i$
	$.7b$	$.1255$	$.5905+.1841i$
	$.5b$	$.1149$	$.5194+.2938i$
	$.3b$	$.0725$	$.3910+.3852i$
	$.1b$	$.0429$	$.2559+.4524i$
	$-.1b$	$.0244$	$.1181+.4912i$
	$-.3b$	$.0138$	$-.0167+.4991i$
	$-.5b$	$.0078$	$-.1467+.4775i$
	$-.7b$	$.0044$	$-.2642+.4127i$
	$-.9b$	$.0025$	$-.3582+.3410i$
$.9b$	$.9b$	1.1279	$1.6239+.0627i$
	$.7b$	$.5691$	$1.0341+.1841i$
	$.5b$	$.2862$	$.6727+.2938i$
	$.3b$	$.1387$	$.4572+.3852i$
	$.1b$	$.0744$	$.2874+.4524i$
	$-.1b$	$.0410$	$.1341+.4912i$
	$-.3b$	$.0226$	$-.0079+.4991i$
	$-.5b$	$.0126$	$-.1419+.4775i$
	$-.7b$	$.0070$	$-.2616+.4217i$
	$-.9b$	$.0040$	$-.3567+.3410i$

changed as the starting point of the next step. To prevent the solution from becoming infinite, at each step $f(x)$ is divided by A_1, thus yielding the case $A_1 = 1$. In the following numerical work the assumption $b = 0.2\lambda$, $kb = 72°$ is made.

Although from physical considerations one would be able to make a reasonably

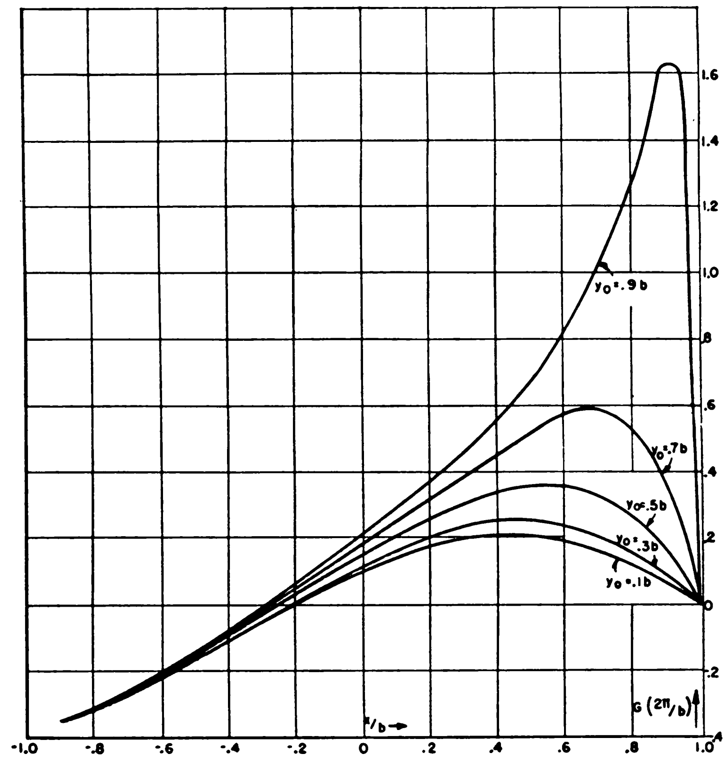

FIG. 4.

good guess for the value of $f(x)$, it was felt that in order to test the method thoroughly, the assumption

$$\text{along } BC,\ f(x) = \left(\frac{\partial u}{\partial y}\right)_{y=b} = \text{constant} = 1, \tag{2.37}$$

would be more advisable. Making this assumption, solving for A_1 and B_1 from (2.36) (with $b = 0.2\lambda$, $kb = 72°$), and dividing by A_1, we obtain

$$\left.\begin{aligned} A_1 = 1, \qquad B_1 &= \frac{bk - e^{ikb}\sin kb}{bk - e^{-ikb}\sin kb} = .0634 - .999i, \\ f(x) &= \left(\frac{\partial u}{\partial y}\right)_{y=b} = k(1.320 - 1.238i). \end{aligned}\right\} \tag{2.38}$$

The Green's function was evaluated for five positive and five negative values of x, and for five values of y_0, as shown in Table 1. The real part of this family of curves is shown in Fig. 4, the imaginary part being merely $(\pi/b)\sin k(x-b)$. These values, with

(2.38), were inserted in (2.34), the integration being made graphically with areas found by the trapezoidal rule, except near $y=b$.

As y approaches b, the value of $g(y)$ increases so rapidly that extrapolation for the curve and the resulting graphical integration is difficult in this region. From physical considerations based on the fact that in a region which is small compared to a wavelength the function u behaves like a harmonic function, it may be shown that a fairly accurate approximation is obtained by assuming $g(y)$ to vary as $(y-b)^{-1/3}$ as y approaches b. By picking two points y_1 and y_2, two constants A and B can be found such that $g(y)=A+B(b-y)^{-1/3}$ is fitted to the curve already drawn in this neighborhood for $y<.9b$; then the area is equal to

$$\int_{y_2}^{b} g(y)dy = A(b - y_2) + \frac{3B}{2}(b - y_2)^{2/3}.$$

The resulting first approximation for $\partial u/\partial x$ is shown in Table 2. By means of (2.36) the values of A_3 and B_3 (the negatives of A_1 and B_1) corresponding to these values were found to be $B_3=.1728-1.006i$, $A_3=.998-.102i$; thus $B_1/A_1=.2735-.980i$.

TABLE 2.

y	$g(y)=\partial u/\partial y=(\partial u/\partial x$ for corresponding values of $x)$
$.1b$	$-k[1.004-\ .9445i]$
$.3b$	$-k[1.054-\ .992i]$
$.5b$	$-k[1.173-1.103i]$
$.7b$	$-k[1.382-1.299i]$
$.9b$	$-k[1.382-1.299i]$
$.9b$	$-k[1.876-1.761i]$

In order to keep A_1 fixed at the value unity which we have assumed, we retain this value of the B_1/A_1 ratio and rename it B_1 as before. We must then divide the values of $\partial u/\partial y$ in Table 2 by A_1. Reinsertion now into (2.27) gives us the second approximation to $\partial u/\partial y$ shown in Table 3. The corresponding A_1 and B_1 yield the ratio $B_1/A_1=.2658-.960i$. The third approximation is then carried out in similar fashion, with the the results shown in Table 4. In this case, we have $B_1/A_1=.266-.964i$. The approximations to $\partial u/\partial y$ are shown in Fig. 5. Figure 6 shows the ratio B_1/A_1 and thus we see that this ratio is converging toward the value $.266-.962i$, with the absolute value .997.

TABLE 3.
The second approximation.

x	$f(x)=\partial u/\partial y$
$.1b$	$k[1.119-\ .804i]$
$.3b$	$k[1.176-\ .844i]$
$.5b$	$k[1.309-\ .960i]$
$.7b$	$k[1.562-1.111i]$
$.9b$	$k[2.100-1.619i]$

TABLE 4.
The third approximation.

x	$f(x)=\partial u/\partial y$
$.1b$	$k[1.081-\ .785i]$
$.3b$	$k[1.132-\ .821i]$
$.5b$	$k[1.249-\ .911i]$
$.7b$	$k[1.515-1.095i]$
$.9b$	$k[2.12\ -1.557i]$

A similar calculation could be carried out for a function which is even about the diagonal OB. The results of this, together with those already found for the odd function, would enable us to cover all cases involving a corner with these dimensions.

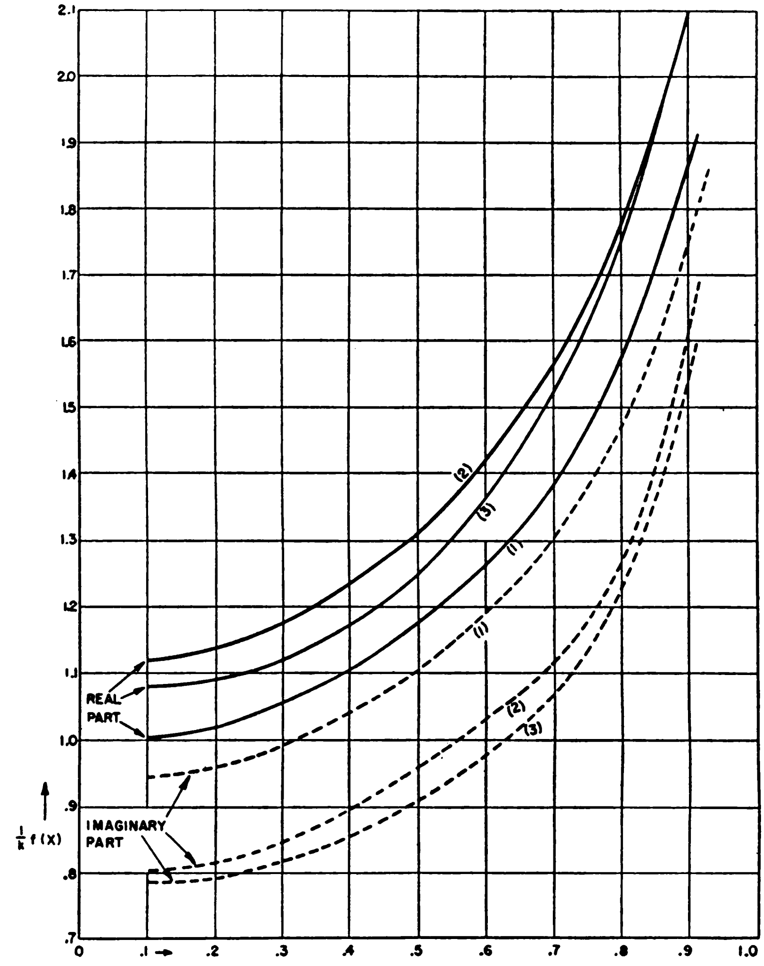
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
.9
.8
.7
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
$\frac{1}{k} f(x)$
REAL PART
IMAGINARY PART
(1)
(2)
(3)

FIG. 5.

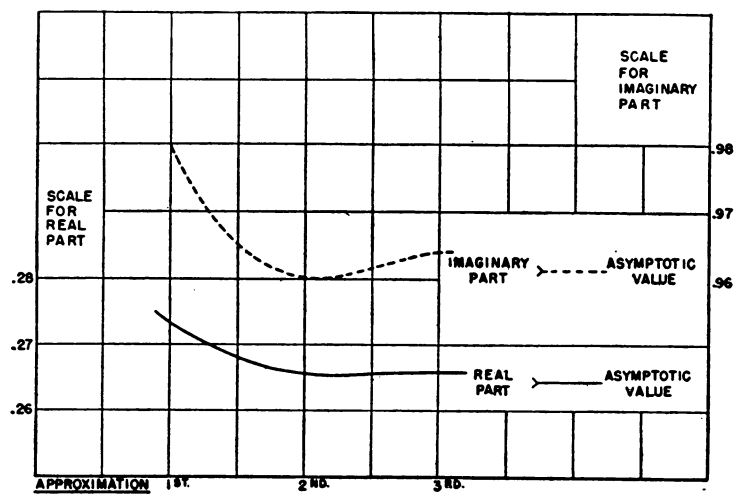
SCALE FOR IMAGINARY PART
SCALE FOR REAL PART
.98
.97
.96
.28
.27
.26
IMAGINARY PART
ASYMPTOTIC VALUE
REAL PART
ASYMPTOTIC VALUE
APPROXIMATION
1 ST.
2 ND.
3 RD.

FIG. 6.

3. An alternative method of procedure. The procedure described and illustrated in the preceding sections can also be applied in a different way. Basically, the calculation was carried out by first assuming the field over the line CB in Fig. 2, then calculating it over the line BA. It is possible to carry out the same calculation by assuming the field over CB not as a function of x or as a curve, but as a Fourier cosine series in x,

$$f(x) = \frac{\partial u}{\partial y}\bigg|_{y=b} = \sum C_n \cos \frac{n\pi x}{b}. \tag{3.1}$$

Similarly, $g(y) = \partial u/\partial x$ over AB can be converted into a similar Fourier cosine series in y,

$$g(y) = \frac{\partial u}{\partial x}\bigg|_{x=b} = \sum D_n \cos \frac{n\pi y}{b}. \tag{3.2}$$

Applying (2.34), (2.35) and (2.26) to the calculation of $g(y)$ from $f(x)$, we obtain

$$\begin{aligned} g(y_0) &= \sum D_m \cos \frac{m\pi y_0}{b} \\ &= -2kB_1 \sin kb - \frac{1}{2b}\sum_n C_n \int_{-b}^{b} \exp\,[-ik(x-b)] \cos \frac{n\pi x}{b}\,dx \\ &\quad - \frac{1}{b}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} (-1)^m C \cos \frac{n\pi y_0}{b} \int_{-b}^{b} \cos \frac{n\pi x}{b} \exp \sqrt{\left(\frac{m\pi}{b} - k^2(k-b)\right)}\,dx. \end{aligned} \tag{3.3}$$

This leads to integrals involving a cosine and an exponential in z. After these integrations are carried out, each one of the coefficients D_n of the expansion (3.2) turns out to be linearly dependent upon the coefficients C_n. Thus, instead of being given a curve $f(x)$ and computing from it the curve $g(y)$, one starts with B_1 and a series of coefficients C_n represented by the Fourier expansion (3.1) and ends up with the coefficients D_n by applying (3.4). The explicit relation between these two sets of coefficients is

$$D_0 = -2kB_1 \sin kb + \sum_{n=0}^{\infty} P_{0n}C_n, \qquad D_m = \sum P_{mn}C_n \quad \text{for} \quad m > 0, \tag{3.4}$$

where

$$P_{0n} = \frac{(-1)^n ikb(1 - e^{2ikb})}{2(n^2\pi^2 - k^2b^2)},$$

$$P_{mn} = \frac{(-1)^{n+m+1}(m^2\pi^2 - k^2b^2)^{1/2}\{1 - \exp\,[-2(m^2\pi^2 - k^2b^2)^{1/2}]\}}{(m^2 + n^2)\pi^2 - k^2b^2}, \qquad m > 0.$$

The matrix P_{mn} thus takes the place of the series of curves $\partial G/\partial x$ which were given in Table 1 and shown in Fig. 4. A similar set of equations expresses C_n in terms of D_n and B_3.

By proceeding as in §2 with the field which is odd about the 45° diagonal OB, it is clear that for the final field the coefficients D_n should be the negatives of the coefficients C_n. For the individual successive approximations, this of course is not necessarily the case. The calculation of the next improvement can be carried out by starting with D_n, changing their signs and putting them in place of C_n in (3.4).

It is possible to replace D_m by C_m in (3.4). A solution of the resulting equations would lead to a complete determination of the field problem. However, the solution of the resulting equations itself involves some method of successive approximation; hence, this procedure is not advisable, and the successive calculation of C's and D's appears to be preferable, since it agrees in spirit with the method outlined above and constitutes just a variation of it.