SOLUTION OF LINEAR AND SLIGHTLY NONLINEAR DIFFERENTIAL EQUATIONS*

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Considering the practical importance of linear differential equations of the second order, or the equivalent systems of the first order equations, it is surprising that treatises give little attention to effective and sufficiently general methods for their solution. The treatises seem to be concerned primarily with power series expansions, Picard's method of successive approximations, numerical methods based on difference equations—methods which in theory are applicable to almost any differential equation and which are practically useless in the case of wave equations. On the positive side, in treatises on mathematical physics one finds a very effective asymptotic approximation which in this country is known as the Wentzel-Kramers-Brillouin approximation and in England as Jeffries' approximation and, of course, the Rayleigh-Schrödinger perturbation method. The former has its obvious limitations and the latter is suitable only for a special class of boundary value problems.

Our purpose is to call attention to another perturbation method which we developed several years ago in connection with the antenna problem. As time went on the virtues of the method became increasingly apparent. Searching for previous references to this method, we came across one by Bôcher[1] to a paper by Liouville.[2] In Liouville's paper we have found the Jeffries-Wentzel-Kramers-Brillouin approximation and a thorough discussion of the usual boundary value problem and associated orthogonal series but very little that has any direct bearing on the present paper.

The method is based on the idea that solutions of linear differential equations may be regarded as distorted or "perturbed" sinusoidal or exponential functions—the same idea which is back of the asymptotic approximation, of the Rayleigh-Schrödinger method, and of the Sturmian theory. It is hardly surprising that this method gives better results than Picard's method which regards the solutions as perturbed straight lines; but the difference is so remarkable that it deserves a special display in a separate note. In this paper, we restrict ourselves to an outline of the procedure and a statement of specific formulas reduced to a point where only simple integrations are needed in any special case. The exposition is based on the second order equation; the extension to higher order linear equations is simple enough. When it comes to nonlinear equations, excepting those which are only slightly nonlinear,† the virtues of the method are not quite clear at present. There is no question that the results

* Received July 6, 1945.

[1] Maxime Bôcher, *An introduction to the study of integral equations*, Cambridge University Press, Cambridge, 1914.

[2] Joseph Liouville, *Mémoires sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable*, J. de Math., **2**, 16–35, 418–436 (1837).

† The meaning of "slightly" depends on the goodness of results expected from the process. Beyond that we shall not attempt to define it.

should be better when compared to those obtained by Picard's method; but the more complicated technique for numerical calculations may offset the advantages. This is something to be explored.

Suppose that our problem is to find the solutions of

$$\frac{dV}{dx} = -Z(x)I, \qquad \frac{dI}{dx} = -Y(x)V, \tag{1}$$

subject to the initial conditions

$$V = V(a), \quad I = I(a), \quad \text{if} \quad x = a. \tag{2}$$

Picard simply integrates (1) and obtains a pair of integral equations

$$V(x) = V(a) - \int_a^x Z(\xi)I(\xi)d\xi, \qquad I(x) = I(a) - \int_a^x Y(\xi)V(\xi)d\xi. \tag{3}$$

Thus the stage is set for successive approximations and the solution is obtained in the form of the infinite series

$$V(x) = V_0(x) + V_1(x) + V_2(x) + \cdots, \qquad I(x) = I_0(x) + I_1(x) + I_2(x) + \cdots, \tag{4}$$

where

$$\begin{aligned} V_0(x) &= V(a), \qquad V_n(x) = -\int_a^x Z(\xi)I_{n-1}(\xi)d\xi, \\ I_0(x) &= I(a), \qquad I_n(x) = -\int_a^x Y(\xi)V_{n-1}(\xi)d\xi. \end{aligned} \tag{5}$$

This procedure is so simple that it would be easy to overlook the fact that in substance we are regarding the solutions of (1) as perturbations of the solutions of

$$\frac{dV}{dx} = 0, \qquad \frac{dI}{dx} = 0, \tag{6}$$

and that we are dealing with a special application of a much more general perturbation method. Let*

$$Z(x) = Z_0(x) + \hat{Z}(x), \qquad Y(x) = Y_0(x) + \hat{Y}(x), \tag{7}$$

and suppose that the solutions of

$$\frac{dV_0}{dx} = -Z_0(x)I_0, \qquad \frac{dI_0}{dx} = -Y_0(x)V_0, \tag{8}$$

subject to the initial conditions (2), are known. Then the solutions of (1) are identical with those of the following integral equations

$$\begin{aligned} V(x) &= V_0(x) - \int_a^x \hat{Z}(\xi)I(\xi)V_1(x,\xi)d\xi - \int_a^x \hat{Y}(\xi)V(\xi)V_2(x,\xi)d\xi, \\ I(x) &= I_0(x) - \int_a^x \hat{Z}(\xi)I(\xi)I_1(x,\xi)d\xi - \int_a^x \hat{Y}(\xi)V(\xi)I_2(x,\xi)d\xi, \end{aligned} \tag{9}$$

* In substance the theorem implied by equations (7), (8) and (9) is hardly new; but we have been unable to find its statement in just that form.

where $V_1(x, \xi)$, $I_1(x, \xi)$; $V_2(x, \xi)$, $I_2(x, \xi)$ satisfy (8) and are subject to the following conditions

$$V_1(\xi, \xi) = 1, \qquad I_1(\xi, \xi) = 0; \qquad V_2(\xi, \xi) = 0, \qquad I_2(\xi, \xi) = 1. \tag{10}$$

Essentially the procedure is to regard $-\hat{Z}(x)I(x)$ and $-\hat{Y}(x)V(x)$ as known functions and to write the general solution of the corresponding nonhomogeneous linear equation. The verification of the identity of the solutions of (1) and (9) is perfectly straightforward. If $I_{01}(x)$, $I_{02}(x)$, are two linearly independent solutions of (8); then, as the reader can readily verify, $I'_{01}I_{02}-I'_{02}I_{01}$, differs from Y_0 only by a constant factor. Bearing this in mind, we have

$$\begin{aligned} V_1(x, \xi) &= \frac{I'_{01}(x)I_{02}(\xi) - I'_{02}(x)I_{01}(\xi)}{I'_{01}(x)I_{02}(x) - I'_{02}(x)I_{01}(x)}, \\ I_1(x, \xi) &= -Y_0(x)\frac{I_{01}(x)I_{02}(\xi) - I_{02}(x)I_{01}(\xi)}{I'_{01}(x)I_{02}(x) - I'_{02}(x)I_{01}(x)}. \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned} V_2(x, \xi) &= -Z_0(x)\frac{V_{01}(x)V_{02}(\xi) - V_{02}(x)V_{01}(\xi)}{V'_{01}(x)V_{02}(x) - V'_{02}(x)V_{01}(x)} \\ I_2(x, \xi) &= \frac{V'_{01}(x)V_{02}(\xi) - V'_{02}(x)V_{01}(\xi)}{V'_{01}(x)V_{02}(x) - V'_{02}(x)V_{01}(x)}. \end{aligned} \tag{12}$$

Substituting $V_0(x)$, $I_0(x)$ in the integrands of (9), we obtain $V_1(x)$, $I_1(x)$; continuing the process we obtain solutions in the form (4).

In Picard's method $Z_0(x) = Y_0(x) = 0$, which is the simplest possible choice. Naturally, the method will work well when $Z(x)$ and $Y(x)$ are small; otherwise it is far better to regard $Z_0(x)$ and $Y_0(x)$ merely as constants. If we are concerned with a finite interval, these constants may be chosen as some mean values* of $Z(x)$ and $Y(x)$—the average values, for example; then for $a=0$ (9) become

$$\begin{aligned} V(x) &= V_0(x) - \int_0^x \hat{Z}(\xi)I(\xi)\cosh\Gamma_0(x-\xi)d\xi + K_0\int_0^x \hat{Y}(\xi)V(\xi)\sinh\Gamma_0(x-\xi)d\xi, \\ I(x) &= I_0(x) + \frac{1}{K_0}\int_0^x \hat{Z}(\xi)I(\xi)\sinh\Gamma_0(x-\xi)d\xi - \int_0^x \hat{Y}(\xi)V(\xi)\cosh\Gamma_0(x-\xi)d\xi \end{aligned} \tag{13}$$

where

$$\begin{aligned} V_0(x) &= V_0\cosh\Gamma_0 x - K_0 I_0\sinh\Gamma_0 x, \qquad \Gamma_0 = \sqrt{Z_0 Y_0}, \qquad K_0 = \sqrt{Z_0/Y_0}, \\ I_0(x) &= -\frac{V_0}{K_0}\sinh\Gamma_0 x + I_0\cosh\Gamma_0 x, \qquad V_0 = V_0(0), \qquad I_0 = I_0(0). \end{aligned} \tag{14}$$

In practice it is found that these equations represent a great improvement on Picard's method and yet the integrations which have to be performed are not more difficult. If $Z(x)$ and $Y(x)$ are constants, Picard's method leads to power series—not

* Assuming that Z and Y do not change signs; if they do, it is best (although by no means necessary) to subdivide the interval.

a satisfactory form for wave functions. John R. Carson[3] employed Picard's method for approximate solution when $Z(x)$ and $Y(x)$ are slowly varying functions and succeeded in summing the series and obtaining the first order correction terms in a usable form; but any attempt to get the higher order terms by this method would seem to be out of the question. Theoretically, we should select $Z_0(x)$ and $Y_0(x)$ as near as possible to $Z(x)$ and $Y(x)$, subject to our ability to solve (8); but the integrations will be difficult to perform.* Thus we come back to (13) as the best compromise and it works very well.

In the more explicit form the first order correction terms are

$$\begin{aligned} V_1(x) &= V_0[B(x)\cosh\Gamma_0 x - A(x)\sinh\Gamma_0 x + C(x)\sinh\Gamma_0 x] \\ &\quad - K_0 I_0[A(x)\cosh\Gamma_0 x - B(x)\sinh\Gamma_0 x + C(x)\cosh\Gamma_0 x], \\ I_1(x) &= -\frac{V_0}{K_0}[B(x)\sinh\Gamma_0 x - A(x)\cosh\Gamma_0 x + C(x)\cosh\Gamma_0 x] \\ &\quad + I_0[A(x)\sinh\Gamma_0 x - B(x)\cosh\Gamma_0 x + C(x)\sinh\Gamma_0 x], \end{aligned} \tag{15}$$

where

$$\begin{aligned} A(x) &= \frac{1}{2}\int_0^x \left[\frac{\hat{Z}}{K_0} - K_0\hat{Y}\right]\cosh 2\Gamma_0\xi d\xi, \\ B(x) &= \frac{1}{2}\int_0^x \left[\frac{\hat{Z}}{K_0} - K_0\hat{Y}\right]\sinh 2\Gamma_0\xi d\xi, \\ C(x) &= \frac{1}{2}\int_0^x \left[\frac{\hat{Z}}{K_0} + K_0\hat{Y}\right] d\xi. \end{aligned} \tag{16}$$

In some instances it is preferable to express the results in terms of progressive waves; then $V(x) = V_0(x) + V_1(x)$ and $I(x) = I_0(x) + I_1(x)$ become

$$\begin{aligned} V^+(x) &= K_0 I_0^+[e^{-\Gamma_0 x} - C(x)e^{-\Gamma_0 x} - E(x)e^{\Gamma_0 x}], \\ I^+(x) &= I_0^+[e^{-\Gamma_0 x} - C(x)e^{-\Gamma_0 x} + E(x)e^{\Gamma_0 x}]; \\ V^-(x) &= -K_0 I_0^-[e^{\Gamma_0 x} + C(x)e^{\Gamma_0 x} + D(x)e^{-\Gamma_0 x}], \\ I^-(x) &= I_0^-[e^{\Gamma_0 x} + C(x)e^{\Gamma_0 x} - D(x)e^{-\Gamma_0 x}]; \end{aligned} \tag{17}$$

where

$$\begin{aligned} D(x) &= A(x) + B(x) = \frac{1}{2}\int_0^x \left[\frac{\hat{Z}}{K_0} - K_0\hat{Y}\right] e^{2\Gamma_0\xi} d\xi, \\ E(x) &= A(x) - B(x) = \frac{1}{2}\int_0^x \left[\frac{\hat{Z}}{K_0} - K_0\hat{Y}\right] e^{-2\Gamma_0\xi} d\xi. \end{aligned} \tag{18}$$

Equations (14) and (15) express the solutions in terms of V and I at the beginning of a finite interval $(0, l)$; one also often wants the corresponding expressions in terms of the final values. These are

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[3] John R. Carson, *Propagation of periodic currents over nonuniform lines*, Electrician, **86**, 272–273 (1921).

* This objection would not apply in strictly numerical handling of equations.

$$
\begin{aligned}
V_0(x) &= V(l)\cosh\Gamma_0(l-x) + K_0 I(l)\sinh\Gamma_0(l-x),\\
I_0(x) &= \frac{V(l)}{K_0}\sinh\Gamma_0(l-x) + I(l)\cosh\Gamma_0(l-x),\\
V_1(x) &= V(l)\{[B(x)-B(l)]\cosh\Gamma_0(l+x) - [A(x)-A(l)]\sinh\Gamma_0(l+x)\\
&\qquad\qquad - [C(x)-C(l)]\sinh\Gamma_0(l-x)\}\\
&\quad + K_0 I(l)\{[B(x)-B(l)]\sinh\Gamma_0(l+x) - [A(x)-A(l)]\cosh\Gamma_0(l+x)\\
&\qquad\qquad - [C(x)-C(l)]\cosh\Gamma_0(l-x)\}. \qquad (19)\\
I_1(x) &= \frac{V(l)}{K_0}\{[A(x)-A(l)]\cosh\Gamma_0(l+x) - [B(x)-B(l)]\sinh\Gamma_0(l+x)\\
&\qquad\qquad - [C(x)-C(l)]\cosh\Gamma_0(l-x)\}\\
&\quad + I(l)\{[A(x)-A(l)]\sinh\Gamma_0(l+x) - [B(x)-B(l)]\cosh\Gamma_0(l+x)\\
&\qquad\qquad - [C(x)-C(l)]\sinh\Gamma_0(l-x)\}.
\end{aligned}
$$

Suppose now that the interval is infinite and that $Z(x)$ and $Y(x)$ are slowly varying functions. In this case, there exists the Liouville-Jeffries-Wentzel-Kramers-Brillouin approximation

$$
\begin{aligned}
V(x) &= \pm A\sqrt{K(x)K(x_0)}\exp\left[\mp\int_{x_0}^{x}\Gamma(\xi)d\xi\right],\\
I(x) &= A\sqrt{K(x_0)/K(x)}\ \exp\left[\mp\int_{x_0}^{x}\Gamma(\xi)d\xi\right],
\end{aligned}
\qquad (20)
$$

where

$$
K(x) = \sqrt{Z(x)/Y(x)}, \qquad \Gamma(x) = \sqrt{Z(x)Y(x)}. \qquad (21)
$$

To the communication engineer these approximations seem natural even without formal analysis. He would reason as follows. If the "characteristic impedance" $K(x)$ is independent of x, a progressive wave moving either to the left or to the right would suffer no reflection; it is only the sudden changes in the impedance that causes reflections. Hence the voltage $V(x)$ and current $I(x)$ associated with the progressive waves will be proportional to $\exp\mp[\int_{x_0}^{x}\Gamma(x)dx]$. If $K(x)$ is a slowly varying function, we can ignore the reflections and in the first approximation consider the line as continuously "matched" and thus acting as a transformer. This means that the voltage will vary directly and the current inversely as the square root of the characteristic impedance: hence, equations (20).

There are several formal derivations;[4] but the one which appeals to us most because it corresponds closely to the physical argument is also the one which permits further improvements in the approximation. Let us consider the "transfer parameter" Θ

$$
\Theta = \int^{x}\Gamma(\xi)d\xi, \qquad \frac{d\Theta}{dx} = \Gamma(x), \qquad (22)
$$

as the new independent variable. Substituting in (1), we obtain

[4] John C. Slater and Nathaniel H. Frank, *Introduction to theoretical physics*, McGraw-Hill Book Co., Inc., New York, p. 148 (1933); John C. Slater, *Microwave transmission*, McGraw-Hill Book Co., Inc., New York, p. 73 (1942).

$$\frac{dV}{d\Theta} = -K(\Theta)I, \qquad \frac{dI}{d\Theta} = -\frac{V}{K(\Theta)}. \tag{23}$$

Eliminating first I and then V we have

$$V''(\Theta) - \frac{K'(\Theta)}{K(\Theta)}V'(\Theta) - V = 0, \qquad I''(\Theta) + \frac{K'(\Theta)}{K(\Theta)}I'(\Theta) - I = 0. \tag{24}$$

If $K(\Theta)$ is constant, we have simple progressive waves as anticipated; otherwise, we introduce new dependent variables in conformity with our idea of voltage and current transformation

$$V = [K(\Theta)]^{1/2}\bar{V}, \qquad I = [K(\Theta)]^{-1/2}\bar{I}. \tag{25}$$

Incidentally, this is the transformation which should remove the first derivatives from (24). Substituting, we obtain

$$\bar{V}''(\Theta) = \left[1 + \frac{3(K')^2}{4K^2} - \frac{K''}{2K}\right]\bar{V}, \qquad \bar{I}''(\Theta) = \left[1 - \frac{(K')^2}{4K^2} + \frac{K''}{2K}\right]\bar{I}. \tag{26}$$

We now have not only equations (20) but also the quantitative criterion of their goodness: $(K'/K)^2$ and $K''/2K$ should be small compared with unity.

To improve on (20), we could repeat the process beginning with (22); but the analytical work is simpler if we turn to equations (13) and apply them to an infinite interval, assuming of course that in the entire interval the bracketed quantities in equations (26) differ but little from unity. Thus, the solutions of

$$\frac{d^2y}{dx^2} = [1 + f(x)]y \tag{27}$$

are also the solutions of

$$y(x) = y_0(x) + \int_\infty^x f(\xi)y(\xi)\sinh(x - \xi)d\xi, \tag{28}$$

provided the integral is convergent. The solutions asymptotic to $e^{\mp x}$ are

$$y(x) \simeq e^{\mp x} + \int_\infty^x f(\xi)e^{\mp x}\sinh(x - \xi)d\xi, \tag{29}$$

or

$$y^+(x) \simeq e^{-x} - \tfrac{1}{2}e^{-x}\int_\infty^x f(\xi)d\xi + \tfrac{1}{2}e^x\int_\infty^x e^{-2\xi}f(\xi)d\xi,$$

$$y^-(x) \simeq e^x + \tfrac{1}{2}e^x\int_{i\infty}^x f(\xi)d\xi - \tfrac{1}{2}e^{-x}\int_{i\infty}^x e^{2\xi}f(\xi)d\xi. \tag{30}$$

From these equations we can obtain the well-known asymptotic expansions of Bessel functions as well as expansions of other types.

The case in which $\Theta = i\beta x$, where β is a constant, occurs so frequently that a repetition is justified. Equations (26) become

$$\bar{V}''(x) = -\beta^2\bar{V} + \left[\frac{3(K')^2}{4K^2} - \frac{K''}{2K}\right]\bar{V}, \qquad \bar{I}''(x) = -\beta^2\bar{I} + \left[\frac{K''}{2K} - \frac{(K')^2}{4K^2}\right]\bar{I} \tag{31}$$

and the corresponding integral equations are

$$\bar{V}(x) = \bar{V}_0(x) + \frac{1}{\beta}\int_\infty^x \left[\frac{3[K'(\xi)]^2}{4[K(\xi)]^2} - \frac{K''(\xi)}{2K(\xi)}\right]\bar{V}(\xi) \sin \beta(x-\xi)d\xi$$
$$\bar{I}(x) = \bar{I}_0(x) + \frac{1}{\beta}\int_\infty^x \left[\frac{K''(\xi)}{2K(\xi)} - \frac{[K'(\xi)]^2}{4[K(\xi)]^2}\right]\bar{I}(\xi) \sin \beta(x-\xi)d\xi. \tag{32}$$

Suppose, for example, that $K(x) = K_0 + kx$; then, asymptotically,

$$V(x) = \pm A\sqrt{K_0 + kx}\left[1 \mp \frac{3ik}{8\beta(K_0+kx)}\right]e^{\mp i\beta x},$$
$$I(x) = \frac{A}{\sqrt{K_0+kx}}\left[1 \pm \frac{ik}{8\beta(K_0+kx)}\right]e^{\mp i\beta x}. \tag{33}$$

In this case, however, the integrals in (32) can be evaluated in terms of sine and cosine integrals. Moreover, the complete result corresponds closely to the physical picture of reflection which invariably takes place when waves are traveling in transmission lines or media with variable characteristic impedance $K(x)$. Thus

$$V(x) = A\left[\sqrt{K_0+kx}\,e^{-i\beta x} + R_V\sqrt{K_0+kx}\,e^{i\beta x}\right],$$
$$I(x) = A\left[\frac{e^{-i\beta x}}{\sqrt{K_0+kx}} + R_I\frac{e^{i\beta x}}{\sqrt{K_0+kx}}\right], \tag{33'}$$

where R_V and R_I are the first order reflection coefficients given by

$$R_V = -3R_I = -(3/4)\exp(2i\beta k^{-1}K_0)\Big[\mathrm{Ci}\,(2\beta x + 2\beta k^{-1}K_0) - i\,\mathrm{Si}\,(2\beta x + 2\beta k^{-1}K_0) + \frac{i\pi}{2}\Big].$$

The succeeding correction terms represent successive reflections. The entire series resembles an asymptotic solution of the differential equation in question but it appears to be rapidly convergent.

An another example, take the case of principal waves on a thin cylindrical antenna when

$$K(x) = 120\log(2x/a), \qquad K'(x) = \frac{120}{x}, \qquad K''(x) = -\frac{120}{x^2}\cdot \tag{34}$$

In this case we obtain

$$V(x) = A\sqrt{K(x)}\left[1 - \frac{i}{2\beta x}\left\{\frac{1}{4[\log(2x/a)]^2} + \frac{1}{2\log 2(x/a)}\right\}\right]e^{-i\beta x},$$
$$I(x) = \frac{A}{\sqrt{K(x)}}\left[1 + \frac{i}{2\beta x}\left\{\frac{-1}{4[\log(2x/a)]^2} + \frac{1}{2\log(2x/a)}\right\}\right]e^{-i\beta x}. \tag{35}$$

As the third example we shall take Rayleigh's equation for a nonlinear oscillator[5]

$$\ddot{q} + (R_1\dot{q} + R_3\dot{q}^3) + \omega^2 q = 0. \tag{36}$$

By (13) we have

$$q(t) = q_0(t) - \frac{1}{\omega}\int_0^t [R_1\dot{q}(\tau) + R_3\dot{q}^3] \sin \omega(t - \tau)d\tau, \tag{37}$$

where $q_0(t)$ is a sinusoidal function. If $q=0$ up to $t=0$, then $q_0(t) = A \sin \omega t$. Substituting in (37) and integrating, we obtain

$$q(t) = A \sin \omega t - \tfrac{1}{2}(R_1 + \tfrac{3}{4}\omega^2 R_3 A^2)t \sin \omega t - \frac{1}{32}\omega R_3 A^3(\cos \omega t - \cos 3\omega t). \tag{38}$$

For a periodic solution we must have

$$R_1 + \tfrac{3}{4}\omega^2 R_3 A^2 = 0; \tag{39}$$

then

$$q(t) = A \sin \omega t - \frac{1}{32}\omega R_3 A^3(\cos \omega t - \cos 3\omega t). \tag{40}$$

Equation (39) is precisely Rayleigh's equation for the amplitude of oscillations; equation (40) differs from his equation in that ours contains a term proportional to cos ωt. Our approximation satisfies the initial condition $q(0)=0$ while Rayleigh's does not.

Originally this work was undertaken to obtain convenient analytic approximations to a number of problems in wave theory. It has since become apparent, however, that at least for a certain class of differential equations, the method would be suitable for numerical solution. The practicability of Picard's method for this purpose has already been explored by Thornton C. Fry;[6] the present method should be quicker. The rapidity of convergence will be discussed in a separate paper.

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[5] Ph. LeCorbeiller, *The nonlinear theory of the maintenance of oscillations*, I.E.E. Journal, **79**, 361–378 (1936).

[6] Thornton C. Fry, *The use of the integraph in the practical solution of differential equations by Picard's method of successive approximations*, Proc. 2d Internat. Cong. Math. Toronto, **2**, 405–428 (1924).