$$\int \left[x_1 \frac{\partial}{\partial n} (\Delta\phi) + x_2 \frac{\partial}{\partial s} (\Delta\phi) \right] ds = 0 \tag{15}$$

and

$$\int \left[x_2 \frac{\partial}{\partial n} (\Delta\phi) - x_1 \frac{\partial}{\partial s} (\Delta\phi) \right] ds = 0. \tag{16}$$

Equations (14), (15) and (16) are Michell's conditions which are thus seen to be the natural boundary conditions of the variational problem for the stress function. The manner in which these equations are used in determining ϕ is obvious. Let $\phi_0, \phi_1, \phi_2, \phi_3$ be the biharmonic functions defined by the following boundary conditions:

1) ϕ_0 and $\partial\phi_0/\partial n$ have the prescribed boundary values on the loaded boundary curve C_1 and vanish on the other boundary curve C_2;
2) $\phi_1 = \partial\phi_1/\partial n = 0$ on C_1,
   $\phi_1 = x_1$ and $\partial\phi_1/\partial n = n_1$ on C_2;
3) $\phi_2 = \partial\phi_2/\partial n = 0$ on C_1,
   $\phi_2 = x_2$ and $\partial\phi_2/\partial n = n_2$ on C_2;
4) $\phi_3 = \partial\phi_3/\partial n = 0$ on C_1,
   $\phi_3 = 1$ and $\partial\phi_,/\partial n = 0$ on C_2.

Substituting

$$\phi = \phi_0 + a_1\phi_1 + a_2\phi_2 + b\phi_3$$

into Eqs. (14), (15) and (16), we obtain three linear equations from which a_1, a_2 and b can be determined.

THE CAPACITY OF TWIN CABLE—II*

By J. W. CRAGGS and C. J. TRANTER (*Military College of Science, Stoke-on-Trent, England*)

1. Introduction. In a recent paper[1] (subsequently referred to as "I") we have given a method for determining the capacity of two circular wires surrounded by concentric touching dielectric sheaths. The present note gives the extension of the method to the case in which the dielectric sheaths are not in contact. The problem considered is the symmetrical one of two infinite parallel circular wires each of radius R_1 surrounded by concentric sheaths of radius R_2 and dielectric constant K_1, the distance between the centers of the wires being $2L(L > R_2)$. The dielectric constant of the surrounding medium is taken as K_2.

2. The equations for solution. In line with the treatment in "I" we replace R_2 by unity, R_1/R_2 by a and L/R_2 by s; we also write $K_1/K_2 = K$. The potentials V_1, V_2 must therefore satisfy (i) the differential equations

$$\nabla^2 V_1 = 0, \qquad a \leqq r \leqq 1, \tag{1}$$

$$\nabla^2 V_2 = 0, \qquad r \geqq 1, \qquad x \geqq 0, \tag{2}$$

and (ii) the boundary conditions

$$V_1 = 1, \tag{3}$$

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[1] J. W. Craggs and C. J. Tranter, *The capacity of twin cable*, Quart. Appl. Math. **3**, 268–272 (1945).

when $r=a$,

$$V_1 = V_2, \quad (4) \qquad K\partial V_1/\partial r = \partial V_2/\partial r, \quad (5)$$

when $r=1$,

$$V_2 = 0, \tag{6}$$

when $x=0$. Here ∇^2 is Laplace's operator in two dimensions and the coordinate systems are as shown in Fig. 1.

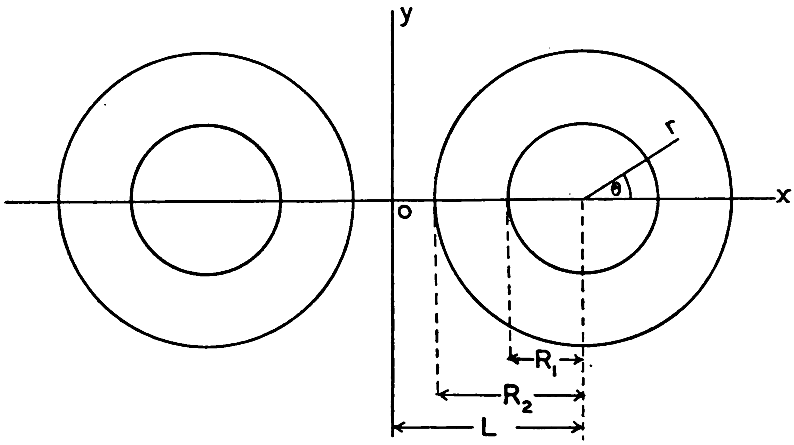

FIG. 1.

3. The analytical solution. As in "I" we write

$$V_1 = 1 + B \log \frac{r}{a} + \sum_{n=1}^{\infty} \left\{ \left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n \right\} b_n \cos n\theta. \tag{7}$$

The conformal transformation for the region $r>1$, $x>0$ can be written

$$\xi - i\eta = \log \frac{re^{i\theta} + e^{\mu}}{re^{i\theta} + e^{-\mu}}, \tag{8}$$

where

$$\mu = \log (s + \sqrt{s^2 - 1}). \tag{9}$$

The boundaries $r=1$, $x=0$ then become $\xi=\mu$, $\xi=0$ respectively.

Since V_2 is odd in ξ and even and periodic in η, we write

$$V_2 = D\xi + \sum_{m=1}^{\infty} d_m \sinh m\xi \cos m\eta. \tag{10}$$

The constants B, b_n of (7) and D, d_m of (10) are now to be determined from the boundary conditions (4) and (5).

On the boundary $r=1(\xi=\mu)$, we find from (8) and (9)

$$\cos \eta = \frac{1 + \cosh \mu \cos \theta}{\cos \theta + \cosh \mu}, \tag{11}$$

so that $0 \leqq \theta \leqq \pi$ corresponds to $0 \leqq \eta \leqq \pi$, and

$$\frac{\partial V}{\partial r} = \frac{\partial \xi}{\partial r} \cdot \frac{\partial V}{\partial \xi} = - \frac{\partial \eta}{\partial \theta} \cdot \frac{\partial V}{\partial \xi} = \frac{- \sinh \mu}{\cos \theta + \cosh \mu} \frac{\partial V}{\partial \xi} . \tag{12}$$

Thus (4) and (5) give

$$1 - B \log a + \sum_{n=1}^{\infty} \frac{1 - a^{2n}}{a^n} b_n \cos n\theta = D\mu + \sum_{m=1}^{\infty} d_m \sinh m\mu \cos m\eta, \tag{13}$$

$$KB + K\sum_{n=1}^{\infty} \frac{1 + a^{2n}}{a^n} nb_n \cos n\theta = \frac{- \sinh \mu}{\cos \theta + \cosh \mu} \left\{ D + \sum_{m=1}^{\infty} md_m \cosh m\mu \cos m\eta \right\} . \tag{14}$$

Multiplying (13) by $\cos m\eta (m=0, 1, 2, \cdots)$ and integrating with respect to η from 0 to π, we have

$$D\mu = 1 - B \log a + \sum_{n=1}^{\infty} (-1)^n e^{-n\mu} \frac{1 - a^{2n}}{a^n} b_n, \tag{15}$$

since

$$\int_0^{\pi} \cos n\theta d\eta = (-1)^n \pi e^{-n\mu},$$

and

$$d_m \sinh m\mu = \sum_{n=1}^{\infty} e^{-n\mu} \frac{1 - a^{2n}}{a^n} b_n I_m(n), \tag{16}$$

where

$$I_m(n) = \frac{2}{\pi} e^{n\mu} \int_0^{\pi} \cos n\theta \cos m\eta d\eta. \tag{17}$$

Similar treatment of (14) gives for B, b_n

$$KB = - D \tag{18}$$

and

$$K \frac{1 + a^{2n}}{a^n} nb_n = 2(-1)^{n+1} e^{-n\mu} D - e^{-n\mu} \sum_{m=1}^{\infty} md_m \cosh m\mu I_m(n). \tag{19}$$

Expansion of $\cos m\eta$ in (17) in terms of $u = (1 + e^{-2\mu} + 2e^{-\mu} \cos \theta)^{-1}$ leads to

$$I_m(n) = (-1)^{m+n} \sum_{p=0}^{m} (-1)^p \, {}^nC_{m-p} \, {}^{n+p-1}C_p e^{(n-2p)\mu}. \tag{20}$$

Eliminating D, b_n from equations (15), (16), (18) and (19) we have

$$B \log a - 1 - KB\mu = 2BS + \frac{1}{2K} \sum_{m=1}^{\infty} md_m \alpha_m \cosh m\mu, \tag{21}$$

$$- d_p \sinh p\mu = B\alpha_p + \frac{1}{K} \sum_{m=1}^{\infty} md_m A_{mp} \cosh m\mu, \tag{22}$$

where

$$\left.\begin{aligned} S &= \sum_{n=1}^{\infty}\left(\frac{1-a^{2n}}{1+a^{2n}}\right)\frac{e^{-2n\mu}}{n}, \\ \alpha_p &= 2\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1-a^{2n}}{1+a^{2n}}\right)I_p(n)\frac{e^{-2n\mu}}{n}, \\ A_{mp} &= \sum_{n=1}^{\infty}\left(\frac{1-a^{2n}}{1+a^{2n}}\right)I_m(n)I_p(n)\frac{e^{-2n\mu}}{n}. \end{aligned}\right\} \tag{23}$$

Following the procedure of "*I*" we retain only a finite number p of the coefficients d_m. Writing

$$\gamma_m = \frac{K}{m}\tanh m\mu, \tag{24}$$

and eliminating $md_m \cosh m\mu$ between equations (21), (22) we find

$$\begin{vmatrix} A_{11}+\gamma_1 & A_{12} & \cdots & A_{1p} & \alpha_1 \\ A_{21} & A_{22}+\gamma_2 & \cdots & A_{2p} & \alpha_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{p1} & A_{p2} & \cdots & A_{pp}+\gamma_p & \alpha_p \\ \alpha_1 & \alpha_2 & \cdots & \alpha_p & 2\left(\frac{1}{B}-\log a+K\mu\right)+4S \end{vmatrix} = 0. \tag{25}$$

The capacity is then given by $-\frac{1}{4}K_1B$.

4. Alternative method of solution. The above treatment provides a satisfactory basis of computation when $K \geqq 1$. For completeness it is interesting to notice that, when $K<1$, more rapid convergence to the true solution is obtained by eliminating D and d_m from equations (13), (14) by treating (13) as a Fourier series in θ and (14) as one in η.

ON A. A. POPOFF'S METHOD OF INTEGRATION BY MEANS OF ORTHOGONALITY FOCI*

By HOWARD A. ROBINSON (*Research Laboratories, Armstrong Cork Company*)

In a recently published paper[1] a method is given which allows a marked reduction of the work necessary in computing the tristimulus values necessary in color specification work. The three tristimulus values are defined by the following relations:

$$X = \int E_L(\lambda)\bar{x}(\lambda)R(\lambda)d\lambda, \qquad Y = \int E_L(\lambda)\bar{y}(\lambda)R(\lambda)d\lambda, \qquad Z = \int E_L(\lambda)\bar{z}(\lambda)R(\lambda)d\lambda,$$

where $E_L(\lambda)$ are tabulated relative energy functions of a known light source L, $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$ are tabulated luminosity functions and $R(\lambda)$ are the experimentally meas-

* Received August 9, 1945

[1] Quart. Appl. Math., 3, 166–174 (1945).