ON THE STABILITY OF TWO-DIMENSIONAL PARALLEL FLOWS

PART II.—STABILITY IN AN INVISCID FLUID*

BY

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7. General considerations. The criteria of Rayleigh and Tollmien. At the end of Part I, we have shown that the study of the stability problem in an inviscid fluid gives valuable information provided it is kept in mind that we are actually dealing with the limiting case where the Reynolds number becomes indefinitely large. The study of the stability of two-dimensional parallel flows in an inviscid fluid is usually regarded as being quite complete, through the work of Rayleigh and Tollmien. Their results show that instability depends very much upon the occurrence of a point of inflection in the velocity profile. However, it seems that physical interpretations of such general results are not well known. Such an interpretation will be given in §§9, 10 of this part. There are also several points in the mathematical theory which require further development and clarification. These will be brought out for further consideration in §§7, 8.

We now proceed to make a critical survey of some aspects of the stability problem in an inviscid fluid. First, let us summarize the conclusions obtained by Rayleigh and Tollmien. These can be conveniently described as the necessary and the sufficient conditions for the existence of a disturbance, self-excited, neutral, or damped.

1) *Necessary conditions for the existence of a disturbance.*

a) If the flow possesses a self-excited or neutral mode of disturbance with finite wave length, the velocity profile has a flex at some point $y=y_s$, where $y_1<y_s<y_2$. Furthermore, in the case of a *neutral* disturbance, the phase velocity must be $c=w(y_s)$.

b) If the flow possesses a *damped* mode of disturbance, no immediate conclusion can be drawn.

2) *Sufficient conditions for the existence of a disturbance.* So far, the sufficient conditions are known only for symmetrical and for boundary-layer velocity distributions. The results may be stated as follows.

a) There is always the neutral disturbance given by $c=0$, $\alpha=0$, $\phi(y)=w(y)$.

b) If $w''(y_s)=0$, for $y_1<y_s<y_2$, there is a neutral disturbance with $c=w(y_s)$; furthermore, if $w'''(y_s)\neq 0$, self-excited disturbances also exist.

Discussion. The condition $w'''(y_s)\neq 0$ involved in (2) (b) will be shown to be actually unnecessary, by an improved method of proof to be discussed in the next section. The statement in (1) (*b*) regarding damped disturbances differs from the original conclusion of Rayleigh and Tollmien. Indeed, in the work of Lord Rayleigh, the solution is taken to be valid all along the real axis. Hence, in accordance with the discussion of §5, Part I, such considerations do not include damped disturbances. However, Rayleigh and Tollmien did not distinguish between an amplified disturbance and a damped disturbance, because they regarded them as complex conjugates. As pointed out in §5, this is not permissible. In fact, if we accept the original conclusions

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of Rayleigh and Tollmien, a profile without a flex could not execute any kind of disturbance. This can hardly be reconciled with our intuition regarding the state of affairs in a real fluid at infinitely large Reynolds numbers.[1] According to the present interpretation, only damped solutions can exist. Such a conclusion is also borne out by the investigations for a viscous fluid.[2] It is to be noted that the neutral and the self-excited disturbances, existing under the condition $w''(y_s)=0$, are free from the effect of viscosity inside the fluid, because the neutral solution is also regular at $y=y_s$ where $w=c$. Hence, we may conclude that *disturbances essentially free from the effect of viscosity inside a fluid can exist only for velocity distributions with a flex.*

The results of Rayleigh and of Tollmien discussed above tend to give the impression that the occurrence of a flex in the profile is the decisive factor in the determination of instability not only in the case of an inviscid fluid, but also in the case of a viscous fluid.[3] However, the investigation in Part III will show that this is by no means the case. When instability first occurs, as one increases the Reynolds number, viscous forces still play a dominant role, and the main characteristics of the behavior of the fluid with respect to a disturbance do not depend upon the occurrence of a flex in the velocity curve. Indeed, it is physically improbable that a slight change of the pressure gradient in the case of a boundary layer—which may cause a change from a velocity curve without a flex to one with a flex—should cause a radical change in the essential characteristics of stability. As we shall see later, the instability of a boundary layer depends more on the outside free stream than on the occurrence of a point of inflection. It might be argued that the free stream is analogous to a point of inflection in that a vanishing curvature is involved; but even if this is admitted, we must still note that the essential features in this case are not obtained from an investigation neglecting the effect of viscosity. Indeed, from inviscid investigations, it is concluded that a boundary layer with zero or favorable pressure gradient is stable, except for the very trivial type of disturbance with infinite wave-length and zero phase velocity. The present investigation shows that *all* boundary-layer profiles can be unstable, and exhibits results in agreement with the physical suggestion just discussed.

It thus seems that *any conclusion obtained from inviscid investigations must not be taken over directly to the case of the real fluid, where the stability phenomenon is largely controlled by the effect of viscosity and not decided primarily by the occurrence of a flex in the velocity curve.*

Indeed, even when we are mainly interested in the behavior in the limiting case of infinite Reynolds numbers, the existence of a flex is not as significant as it may appear to be at first sight. The existence of neutral or amplified disturbances has so far been proved only for *symmetrical* and *boundary-layer* types of velocity profiles. This may not be true for other types of velocity profiles, e.g., when the walls are in relative motion. The following example will bring out this point. Let us consider the velocity distribution $w(y)=A+B\sin y$, $y_1<y<y_2$, which has a flex at $y=0$ if $y_1<0<y_2$. According to the above necessary conditions, the only possible neutral

[1] This is the objection of Friedrichs, loc. cit. (Ref. [5]) p. 209. (The references are listed at the end of Part I.) It must also be noted that the non-linear terms are not negligible in the case of an *ideal* fluid. We shall consistently restrict the magnitude of our disturbances so that the effect of viscosity is always more important than the effect of non-linearity.

[2] See figures in Ref. [27].

[3] See Taylor's discussion on p. 308 of Ref. [70].

disturbance is the one with $c=A$. Then the equation of disturbance (6.21) reduces to $\phi''+(1-\alpha^2)\phi=0$. It has the solution

$$\phi(y) = C \sin \left\{\sqrt{1-\alpha^2}\,(y - y_1)\right\},$$

which vanishes at $y=y_1$. If $\phi(y_2)$ is also required to vanish, we must have

$$\sqrt{1-\alpha^2}\,(y_2 - y_1) = n\pi, \qquad (n = \text{integer}),$$

and hence

$$\alpha^2 = 1 - [n\pi/(y_2 - y_1)]^2.$$

Thus, if $y_2-y_1<\pi$, there is no possible neutral disturbance; if $y_2-y_1=\pi$, there is the one with $\alpha=0$; if $\pi<y_2-y_1<2\pi$, there is one with $\alpha\neq 0$; in general, if $m\pi<y_2-y_1<(m+1)\pi$, there are m neutral disturbances with $\alpha\neq 0$. In the last case, there are also m points of inflection in the velocity profile.

It can thus be seen that the general shape of the velocity profile plays a very important role even in the limit of infinite Reynolds numbers. Indeed, it will become clear from Part III that the eigen solution with eigen values $\alpha=c=0$ is not as trivial as it might appear at first sight, for it actually represents a limiting case with $R\to\infty$. This solution exists for symmetrical and boundary-layer profiles, but its existence is not immediately evident for other types of profiles.

In spite of all these points against the *decisive* nature of the flex, it must be admitted that its occurrence certainly makes the motion comparatively unstable. This can be expected from the original results of Rayleigh and Tollmien, and can be seen more clearly from the interpretation of the mechanism of inertia forces to be given in §§9, 10. However, these results must not be taken to indicate any decisive nature of a flex. The essential features of instability can only be obtained through consideration of the effect of viscosity.

We shall now conclude this section by making some critical discussions of Heisenberg's classification of velocity profiles and the use of broken linear profiles for the study of stability problems.

Heisenberg's classification of velocity distributions. Heisenberg attempted the case of flow between solid walls in relative motion with the condition that $\text{Re}(w-c)$ vanishes only once for $y_1<y<y_2$ (loc. cit., p. 592). Regarding α^2 as small, he approximated the condition (6.18) by $K_1(c)=0$ [cf. (6.26), (6.24)]. He then classified the profile into four classes: (*i*) those for which $K_1(c)=0$ has a complex root; (*ii*) those for which $K_1(c)=0$ has a real root; (*iii*) those for which the real part of $K_1(c)$ vanishes for a certain real value of c; (*iv*) those for which none of the above three cases is true. Heisenberg concluded that the first class is unstable, the second generally unstable, the rest stable.

In discussing the validity of these conclusions, the following point must be borne in mind. If we can show that a certain type of disturbance exists for $\alpha^2=0$ and $\alpha R\to\infty$, it may also be expected to exist for sufficiently large values of αR and sufficiently small values of α^2. However, the non-existence of a certain type of disturbance for $\alpha^2=0$ and $\alpha R\to\infty$ does not exclude the possibility of its existence for finite values of α^2 and αR. It appears therefore that we can only expect to conclude the *instability* of a velocity distribution by discussing the roots of $K_1(c)=0$. Thus, apart from some flaws in Heisenberg's mathematical deductions, only the first two classes can have any decisive significance.

If $K_1(c)$ has a root with a *positive* imaginary part, the motion is unstable. If $K_1(c)$ has a *real* root, Heisenberg shows that the motion would be unstable when the effect of viscosity is considered. This will be studied more fully in a generalized form in §11. However, if $K_1(c)$ has a root with a *negative* imaginary part, we cannot conclude the instability of the flow by taking the complex conjugate of $K_1(c)=0$ (as Heinsenberg did). For if

$$\int_C dy(w - c)^{-2} = 0,$$

then (cf. Fig. 5)

$$\int_{C'} dy(w - c)^{-2} = - 2\pi i R_0,$$

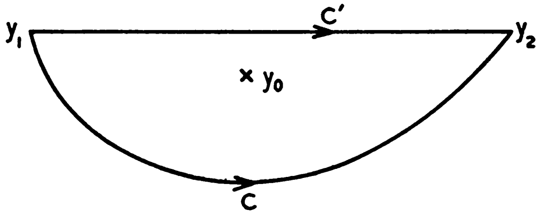

FIG. 5. Path around the critical point in the case $c_i<0$.

where R_0 is the residue of $(w-c)^{-2}$ at y_0. In fact $R_0=-w_0''/w_0'^3$. Now $K_1(\bar{c})$ is the complex conjugate of $\int_{C'}dy(w-c)^{-2}$. Hence,

$$K_1(\bar{c}) = 2\pi i\bar{R}_0 = - 2\pi i\bar{w}_0''/\bar{w}_0'^3,$$

which does not vanish unless $w_0''=0$. Hence, the equation $K_1(c)=0$ tells us nothing about the existence of the root $\bar{c}$ or any other root with a positive imaginary part.

Thus, Heisenberg's attempt appears to be not as successful as Tollmien's later work [75], which at least brings out the characteristic properties of symmetrical and boundary-layer distributions. A complete classification of velocity distributions, however, is not yet existent.

Approximation using broken linear profiles. Some investigations of Lord Rayleigh were carried out by approximating the velocity profile with straight-line segments. With this approximation, the solutions of (6.21) can be expressed in terms of elementary functions. Lord Rayleigh also tried to verify his conclusions by considering the roots of $K_1(c)=0$, using the same approximation for the velocity. However, the results of his investigations are doubtful, because the number of roots obtained for $K_1(c)$ is equal to the number of corners chosen in the approximation. This was demonstrated by Heisenberg to be inherent in the method of approximation. The general idea is as follows. As discussed above, the stability condition (6.18) may be approximated by $K_1(c)=0$ in certain cases. Although Rayleigh's approximation may be made very close so far as the velocity distribution is concerned, the approximation to $(w-c)^{-2}$ is always bad in the neighborhood of the corners. Consequently, the integral $K_1(c)$ is not properly approximated. In fact, a *continuous* broken profile $w(y)$ does not allow itself to be continued analytically to the complex y-plane without introducing *discontinuities* (cuts). It thus appears that all results deduced from the consideration of broken profiles must be regarded with reserve. The same criticism applies to Tietjen's work with the viscous fluid. His analysis failed to give a minimum Reynolds number below which all small disturbances are damped out.

8. Rigorous proof and extension of Tollmien's result for the existence of unstable modes of oscillation. In this section, we want to give a rigorous proof of the existence of amplified solutions of (6.21) satisfying the second and the third boundary conditions of (6.22) when the velocity profile $w(y)$ has a flex at $y=y_s$, i.e.,

$$w''(y_s) = 0. \tag{8.1}$$

The idea of the proof is essentially the same as that used by Tollmien, but the method is improved. It has the further advantage of enabling us to extend the results to cover cases where $w'''(y_s)=0$,—a condition which had to be excluded by Tollmien.

According to previous results, the neutral disturbance must have a phase velocity c equal to

$$c_s = w_s = w(y_s). \tag{8.2}$$

Let the corresponding value of α be denoted by α_s. The essential idea of the proof is (1) to show that there exist eigen-values of c and $\alpha>0$ in the neighborhood of the values of c_s and α_s such that the imaginary part of c does not vanish, and then (2) to show that the imaginary part is actually positive. The first statement can be expected and can be readily established, if we can show that the left-hand sides of (6.18)–(6.20) are analytic functions $f(\alpha, c)$ of the two variables α and c in the neighborhoods of α_s and c_s. For if this is true, we can always solve $f(\alpha, c)=0$ for c as an analytic function of α, (there may be more than one branch), by the implicit function theorem. Hence, there is at least one value of c corresponding to every *real* value of α in the neighborhood of $\alpha=\alpha_s$. Furthermore, by (8.2), this value of c, being unequal to c_s, cannot be real, and the first part of our result is established.

To prove the analyticity of $f(\alpha, c)$ seems to be a trivial problem. Nevertheless, we shall find below that it is impossible to establish it in the neighborhood of $(\alpha, c)=(0, 0)$. The chief problem in the proof is to overcome the difficulty caused by the singular point of the differential equation (6.21).

If $w-c\neq 0$, we can write (6.21) in the form

$$\phi'' - \alpha^2\phi - \frac{w''}{w-c}\phi = 0. \tag{8.3}$$

Let us now consider a simply-connected region R of the y-plane which encloses the

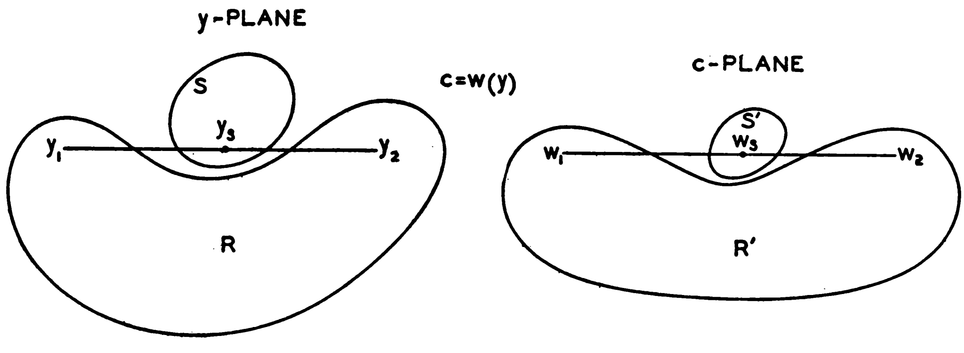

FIG. 6. The region of analyticity of the inviscid solutions.

points $y=y_1$ and $y=y_2$, but excludes the point y_s, the passage from y_1 to y_2 being taken in the *lower* half of the y-plane. We consider also a neighborhood S of y_s, mutually exclusive with the region R. Let us regard the relation

$$c = w(y) \tag{8.4}$$

as mapping the regions R and S into two regions R' and S' of the c-plane (Fig. 6). If the mapping is one-to-one, (as can be expected if $w'(y)\neq 0$ for $y_1<y<y_2$), these

regions will also be mutually exclusive. Then, if we restrict y to R and c to S', the coefficients of (8.3) are analytic functions of the independent variable y and the parameters α and c. Hence, a fundamental system of solutions of (8.3), which we denote by $\phi_1(y; \alpha, c)$ and $\phi_2(y; \alpha, c)$, are *analytic* functions of the *three* variables y, α, and c. We understand that y is restricted to the region R, c is restricted to the region S', while α may be in any finite region enclosing α_s. Thus, (for example),

$$f_2(\alpha, c) \equiv \begin{vmatrix} \phi_1(y_1; \alpha, c) & \phi_2(y_1; \alpha, c) \\ \phi_1'(y_2; \alpha, c) & \phi_2'(y_2; \alpha, c) \end{vmatrix} \tag{8.5}$$

is an analytic function of the variables α and c, as we want to prove.

We note that in the neighborhood of $(\alpha, c) = (0, 0)$, the above reasoning fails. The region R (which has to enclose the point $y = y_1$) and the region S (which has to enclose the point where $w = c = 0$) cannot be taken to be mutually exclusive. In fact, $f(\alpha, c)$ presumably has a singular point at the point $\alpha = 0$ (a logarithmic branch point). We shall discuss this case a little more closely at the end of this section.

Let us proceed to show that there actually exist values of $c = c(\alpha^2)$ with a *positive* imaginary part corresponding to positive real values of α. This is necessary because the usual argument of taking complex conjugates has been shown to be invalid. For this purpose, we consider the power series

$$c = c_s + \left(\frac{dc}{d\lambda}\right)_s (\lambda - \lambda_s) + \frac{1}{2!}\left(\frac{d^2c}{d\lambda^2}\right)_s (\lambda - \lambda_s)^2 + \cdots , \tag{8.6}$$

where $\lambda = \alpha^2$.[4] Since λ is restricted to real values, the important point to be shown is that *the first of the derivatives in* (8.6) *for which the imaginary part does not vanish is of odd order*. Then, by taking values of λ slightly greater or smaller than λ_s, we can always make $c_i > 0$. For these values of c and α^2, we can continue our solution $\phi(y)$ analytically so that it is given along the real axis between y_1 and y_2, thus obtaining an inviscid solution.

Let us now consider (8.3), writing λ for α^2. We have

$$L(\phi) \equiv \phi'' - \lambda\phi - \frac{w''}{w - c}\phi = 0. \tag{8.7}$$

Let ϕ be an eigen function with λ, c as the corresponding eigen-values. Then

$$L(\phi_\lambda) \equiv \phi_\lambda'' - \alpha^2\phi_\lambda - \frac{w''}{w - c}\phi_\lambda = \left\{1 + \frac{w''}{(w - c)^2}\frac{dc}{d\lambda}\right\}\phi, \tag{8.8}$$

where

$$\phi_\lambda = \frac{\partial\phi}{\partial\lambda} + \frac{\partial\phi}{\partial c}\frac{dc}{d\lambda}\cdot$$

We distinguish two cases: (1) the point $y = y_s$ is a simple root of $w''(y) = 0$; (2) the point $y = y_s$ is a multiple root of $w''(y) = 0$.

In the first case, $w'''(y_s) \neq 0$. In the limit $\lambda \to \lambda_s$, $c \to c_s$, Eqs. (8.7) and (8.8) become[5]

[4] Since $d\lambda/d\alpha \neq 0$ at $\alpha = \alpha_s$, the correspondence between α and λ is one-to-one in the neighborhood of $\lambda_s = \alpha_s^2$.

[5] A subscript s denotes that the parameters λ and c are put equal to λ_s and c_s respectively. A subscript λ denotes differentiation with respect to λ.

$$L_s(\phi_s) \equiv \phi_s'' - \lambda_s\phi_s - \frac{w''}{w - c_s}\phi_s = 0, \tag{8.9}$$

$$L_s(\phi_{\lambda s}) \equiv \phi_{\lambda s}'' - \lambda_s\phi_{\lambda s} - \frac{w''}{w - c_s}\phi_{\lambda s} = \left\{1 + \frac{w''}{(w - c_s)^2}\left(\frac{dc}{d\lambda}\right)_s\right\}\phi_s. \tag{8.10}$$

From these, we deduce that

$$\phi_s L_s(\phi_{\lambda s}) - \phi_{\lambda s} L_s(\phi_s) = \frac{d}{dy}\{\phi_s\phi_{\lambda s}' - \phi_{\lambda s}\phi_s'\} = \left\{1 + \frac{w''}{(w - c_s)^2}\left(\frac{dc}{d\lambda}\right)_s\right\}\phi_s^2.$$

Now, ϕ_λ satisfies the same boundary conditions as ϕ does, because those conditions are satisfied by ϕ for each pair of values of λ and c, and ϕ is an analytic function of them. Hence, integrating $\phi_s L_s(\phi_{\lambda s}) - \phi_{\lambda s} L_s(\phi_s)$ between the limits (y_1, y_2), we have

$$\left(\frac{dc}{d\lambda}\right)_s \int_{y_1}^{y_2} \frac{w''}{(w - c_s)^2}\phi_s^2 dy + \int_{y_1}^{y_2} \phi_s^2 dy = 0,$$

or

$$\left(\frac{dc}{d\lambda}\right) = -\int_{y_1}^{y_2} \phi_s^2 dy \Big/ \int_{y_1}^{y_2} w''\phi_s^2 (w - c_s)^{-2} dy. \tag{8.11}$$

The denominator of the above expression is equal to

$$\int_{y_1-y_s}^{y_2-y_s} (w_s''' y + \tfrac{1}{2}w^{\text{iv}}y^2 + \cdots)(w_s' y + \tfrac{1}{2}w_s''' y^2 + \cdots)^{-2}(\phi_{ss}^2 + 2\phi_{ss}\phi_{ss}' y + \cdots)dy$$
$$= \frac{w_s'''}{w_s'^2}\phi_{ss}^2 \int_{y_1-y_s}^{y_2-y_s} \left\{\frac{1}{y} + A_0 + A_1 y + \cdots\right\} dy,$$

where ϕ_{ss} is the value of ϕ_s at $y = y_s$, and $A_0, A_1, \cdots$ are real. Hence, the imaginary part of the above expression is $\pi\phi_{ss}w_s'''/w_s'^2$. Since $\phi_s(y)$ is real and ϕ_{ss} does not vanish,[6] we have arrived at the required result. The above argument is a rigorous formulation of Tollmien's work.

In case $w''(y)$ has a multiple root at $y = y_s$, the proof of Tollmien does not hold, but the above method can still be carried through. The restriction must be made, however, that the point y_s is a point of inflection where $w''(y)$ actually changes its sign. Then, y_s is a root of $w''(y)$ of odd multiplicity, and the first of the derivatives $w^{\text{iv}}(y_s), w^{\text{v}}(y_s), \cdots$ which does not vanish is of *odd* order. Such a point always exists when the curvature of the velocity curve has different signs at y_1 and y_2. If we differentiate (8.7) n times with respect to λ, we have the following equation for each value of n:

$$L(\phi_{\lambda^{(n)}}) = n\phi_{\lambda^{(n-1)}} + \sum_{r=1}^{n} C_r^n \phi_{\lambda^{(n-r)}} \frac{d^r}{d\lambda^r}\left(\frac{w''}{w - c}\right). \tag{8.12}$$

Let $w''(y)$ have the root y_s up to the multiplicity $2m+1$, $m > 0$. Then, (8.10) is regular in a neighborhood of $y = y_s$, and the value of $(dc/d\lambda)_s$ as given by (8.11) is *real*. Let us consider the boundary value problem of the differential equation (8.10), requiring $\phi_{\lambda s}$ to satisfy the same boundary conditions as ϕ_s. The solution can be ob-

[6] Tollmien [74], p. 92.

tained from $\phi_\lambda(y)$ by making $\lambda \to \lambda_s$, and is moreover real along the real axis, by a direct consideration of (8.10).

Continuing the same argument with equations of the type (8.12) with $n=2, 3, \cdots, 2m$ and $c \to c_s$, $\lambda \to \lambda_s$, we find that

$$(\phi_{\lambda^{(2)}})_s,\ (\phi_{\lambda^{(3)}})_s,\ \cdots,\ (\phi_{\lambda^{(2m)}})_s;\ \left(\frac{d^2c}{d\lambda^2}\right)_s,\ \left(\frac{d^3c}{d\lambda^3}\right)_s,\ \cdots,\ \left(\frac{d^{2m}c}{d\lambda^{2m}}\right)_s$$

are all real. Finally, for $n=2m+1$, we obtain a relation of the type

$$(2m+1)!\left(\frac{dc}{d\lambda}\right)_s^{2m+1}\int_{y_1}^{y_2} w''\phi_s^2(w-c)^{-(2m+2)}dy + \left(\frac{d^{2m+1}c}{d\lambda^{2m+1}}\right)_s\int_{y_1}^{y_2}\phi_s^2 dy = \text{real}. \quad (8.13)$$

Just as in the case of the equation preceding (8.11), it can be easily seen that the above integral $\int_{y_1}^{y_2} w''\phi_s^2(w-c)^{-(2m+2)}dy$ has the imaginary part $w_s^{(2m+3)}\phi_{ss}^2/(2m+1)!(w_s')^{(2m+2)}$ while the other term on the left-hand side of (8.13) is real. Thus, $(d^{2m+1}c/d\lambda^{2m+1})_s$ has a non-vanishing imaginary part. This is the result desired.

This completes the proof of the existence of amplified solutions near the neutral solution $c=c_s$, $\alpha=\alpha_s$ when the velocity curve has a point of inflection.

The proof of the existence of amplified solutions near the neutral solution $c=0$, $\alpha=0$ cannot be so easily formulated into a rigorous form. From the solutions (4.14), it is very easy to obtain the solution ϕ_I which approaches the eigen solution $\phi=w(y)$ as $c\to 0$, $\alpha^2\to 0$, with $\alpha^2=O(c)$. The solution is

$$\phi_I = -cw_1'(w-c)\int_{y_1}^{y}(w-c)^{-2}dy$$
$$\times\left\{1+\alpha^2\int_{y_1}^{y}dy(w-c)^2\int_{y_1}^{y}dy(w-c)^{-2}+\cdots\right\}. \quad (8.14)$$

As can be easily verified from (6.21) the condition that ϕ_I be an eigen function is

$$cw_1' + \alpha^2\int_{y_1}^{y_2}(w-c)\phi_I dy = 0. \quad (8.15)$$

From this, it follows that

$$\left(\frac{dc}{d\lambda}\right)_0 = \frac{1}{w_1'}\int_{y_1}^{y_2} w^2 dy, \quad (8.16)$$

and that the imaginary part of $(d^2c/d\lambda^2)_0$ is $2\pi(dc/d\lambda)_0 w_1'/w_1'^2$, which is positive if there is one flex in the velocity profile ($w_1'>0$). However, the real part of $(d^2c/d\lambda^2)_0$ becomes logarithmically infinite, and hence the argument is not rigorous. Also, it does not seem easy to make suitable modifications and extensions in case $w''(y_1)=0$. It should be remarked that Tollmien's proof is not essentially different from the argument just given.

Similar considerations can be applied to boundary-layer profiles, and similar results can be obtained confirming and extending Tollmien's original results.

9. Physical interpretation of instability in an inviscid fluid. The fact that the instability of a two-dimensional parallel laminar flow is so closely connected with the occurrence of a point of inflection in the velocity profile demands a physical interpretation. Since Eq. (6.21) is essentially the vorticity equation, we would expect $w''=0$ to indicate a maximum or minimum of the vorticity $-w'$ of the main flow. This is actually where the explanation is to be found.

Since we have neglected the effect of viscosity, a fluid element maintains its vorticity throughout the motion. From this point of view, a two-dimensional parallel flow may be regarded as the motion of a large number of vortex filaments under the action of each other. Filaments of equal vorticity are arranged in the same layer, and the whole flow is built up of a collection of such layers.

The following physical interpretation is based upon the fact that a fluid element is accelerated in such a field if it is associated with an excess or a defect of vorticity. These considerations were originally developed by von Kármán[7] for the interpretation of the failure of the simple vorticity-transfer theory of fully developed turbulence as applied to the case of parallel Couette flow. The idea is developed in greater mathematical detail here in this section and the next. It will be noticed that the consideration is essentially two-dimensional, and hence is even more suitable here than for fully developed turbulence, where the fluctuations are essentially three-dimensional. An alternative interpretation of the results of Rayleigh and Tollmien, but still based upon vorticity considerations, will also be given to demonstrate the role of the viscous forces.

Let us imagine a disturbance of the flow such that an element E_1 of fluid of the layer L_1 is interchanged with an element E_2 of a neighboring layer L_2. For definiteness, let us suppose that the layer L_2 has a higher vorticity than the layer L_1 in the undisturbed state. Since E_1 preserves its vorticity, it will appear to have a defect of vorticity when it is in L_2. Similarly, E_2 appears to have an excess of vorticity.

Let us fix our attention on one element, say E_2. It will be shown in §10 that a fluid element with an excess of vorticity is accelerated in the direction of the positive y-axis with an acceleration $\Gamma^{-1}\iint\{v'(x, y)\}^2\zeta_0'dxdy$, where $\zeta_0'(y)$ is the gradient of vorticity of the main flow, $v'(x, y)$ is the component of the disturbing velocity perpendicular to the direction of flow, and Γ is the total strength of the vortex filaments corresponding to the disturbance. Examining the signs of the various quantities in the acceleration formula, we can easily see that E_2 is accelerated toward a region of higher vorticity if the gradient of vorticity does not change sign anywhere in the fluid. Thus, E_2 is accelerated toward L_2. A similar consideration holds for the element E_1. Hence, in either case, *the fluid element is returned to the layer where it belonged* (by the acceleration due to its interaction with other vortex filaments). *The motion is therefore stable when the gradient of the vorticity does not vanish.*

When there is an extremum of vorticity, an interchange of fluid elements on opposite sides of the extremum does not give rise to an excess or a defect of vorticity. Furthermore, the gradient of vorticity vanishes there, and has opposite signs on opposite sides of that layer. It can easily be seen from the above acceleration formula that the restoring tendency mentioned above is largely impaired in such a case. Thus, exchanged fluid elements are not as strongly forced back by the action discussed above. Such an exchange constitutes a disturbance because there is an exchange of momentum. Thus, a disturbance may tend to persist and perhaps to augment. The motion is not necessarily stable.

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[7] Cf. discussions of the vorticity transfer theory of turbulence in his general lecture at the Fourth International Congress for Applied Mechanics [19]. Some developments in that direction were continued by C. B. Millikan (unpublished).

The above discussion is based on very general considerations and does not depend on the consideration of a periodic wavy disturbance as used in the mathematical analysis. We shall now support the above argument by considering a neutral wavy disturbance, with the understanding that if such a disturbance can persist (except for the exceptional case of infinite wave-length and zero phase velocity), the motion is presumably unstable. From these considerations, the importance of viscosity in the inner friction layer will also be brought out.

Let us consider an observer moving with the phase velocity of a neutral wavy disturbance. He will observe a stationary pattern of the flow (see Fig. 7).[8] Closed stream lines are inevitable unless the disturbance has no v-component of velocity in the critical layer $w=c$, for the flows on opposite sides of the critical layer are in opposite directions relative to the observer. It appears unlikely that the v-component of the disturbance should be zero throughout that layer. Indeed, it has been shown to be impossible mathematically.[9] Thus, whenever a neutral disturbance persists, it involves a steady exchange of fluid elements on opposite sides of the critical layer.

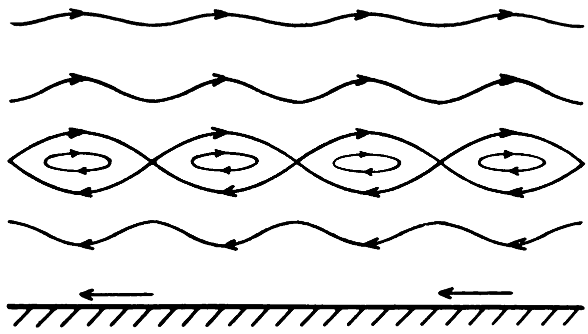

FIG. 7. Stream lines of a neutral disturbance as observed by an observer moving with the wave velocity.

If the effect of viscosity is to be negligible, fluid elements on the same stream line must have the same vorticity. If the gradient of vorticity of the main flow is zero or small near the critical point, it is easy to compensate this small difference of vorticity by the vorticity of the superposed flow, while the "scale" of disturbance [as measured in order of magnitude by $u'/(\partial u'/\partial y)$] remain the same as that of the main flow. It is thus *not impossible* to find a neutral disturbance *for which the effect of viscosity is negligible.* The motion may be unstable.

On the other hand, if the gradient of vorticity of the main flow is finite, the superposed small disturbance must also give a finite gradient of vorticity. This means that the "scale" of the disturbance must be very small in the critical layer. The diffusion of vorticity by the effect of viscosity is then inevitable. It is thus *impossible* to find a neutral disturbance *for which the effect of viscosity is negligible.* The motion is inertially stable.

10. Acceleration of vortices in a non-uniform field of vorticity. In the foregoing physical interpretation of inertial instability, we have considered the acceleration of an element of fluid in a two-dimensional parallel flow when this element of fluid does not have the same vorticity as the surrounding layer. We are now going to derive the explicit formula for the acceleration. The derivation shall be made in two different ways: (1) by kinematical considerations (using vorticity theorems); (2) by considera-

[8] This figure is due to Lord Kelvin (loc. cit.). He pointed out that the facts discussed here are "surprising," but did not attempt to explain their connection with the mechanism of hydrodynamic stability.

[9] This follows at once from Rayleigh's original results, if we apply it to the region between this layer and the solid wall (cf. Tollmien, loc. cit., 1935).

tions of the pressure gradient. In either method, we shall consider a perfect fluid in accordance with the stability problem under consideration.

1) *First derivation, by kinematical considerations.* For definiteness, let us consider a two-dimensional flow between two solid walls, which we shall take to be $y = \pm b$. Let the velocity components of the main flow be

$$U = w(y), \qquad V = 0, \tag{10.1}$$

and those of the secondary flow be

$$u' = u'(x, y), \qquad v' = v'(x, y). \tag{10.2}$$

The distribution of vorticity of the main flow is

$$\zeta_0 = \zeta_0(y) = -w'(y), \tag{10.3}$$

and that of the secondary flow is

$$\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}. \tag{10.4}$$

The latter distribution shall approximate a vortex "at" the point (ξ_0, η_0). Thus, if the signs of ζ' and ζ_0 are the same (or opposite), we have essentially a small element of fluid having an excess (or a defect) of vorticity near the point (ξ_0, η_0).

The stream function for the secondary flow is

$$\psi'(x, y) = -\frac{1}{2\pi}\iint \zeta'(\xi, \eta)G(x, y; \xi, \eta)d\xi d\eta, \tag{10.5}$$

with

$$\left.\begin{aligned} u'(x, y) &= -\frac{\partial\psi'}{\partial y} = \frac{1}{2\pi}\iint \zeta'(\xi, \eta)\frac{\partial}{\partial y}G(x, y; \xi, \eta)d\xi d\eta, \\ v'(x, y) &= \frac{\partial\psi'}{\partial x} = -\frac{1}{2\pi}\iint \zeta'(\xi, \eta)\frac{\partial}{\partial x}G(x, y; \xi, \eta)d\xi d\eta. \end{aligned}\right\} \tag{10.6}$$

In these expressions, the integrals are extended over the whole region between the planes. The function $G(x, y; \xi, \eta)$ is the Green's function of the first kind for the region under consideration. It is defined by the following conditions:

$$\left.\begin{aligned} &\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0 \text{ except at } (\xi, \eta), \\ &G(x, y; \xi, \eta) \sim -\log\left\{(x-\xi)^2 + (y-\eta)^2\right\}^{1/2} \text{ near } (\xi, \eta), \\ &G(x, y; \xi, \eta) = 0 \text{ over the solid boundaries.} \end{aligned}\right\} \tag{10.7}$$

As is well-known, it has the reciprocity property

$$G(x, y; \xi, \eta) = G(\xi, \eta; x, y). \tag{10.8}$$

For the case of a channel, it is given by the real part of

$$f(z) = -\left\{\log \operatorname{sh}\frac{\pi}{4b}(z - z_0) - \log \operatorname{ch}\frac{\pi}{4b}(z - \bar{z}_0)\right\},$$

$$(z = x + iy,\ z_0 = \xi + i\eta,\ \bar{z}_0 = \xi - i\eta). \tag{10.9}$$

Let us now consider the behavior of a particular element of fluid at (ξ, η) having an excess (or a defect) of vorticity corresponding to the secondary flow (10.5) and (10.6). It causes a distortion of the main vorticity distribution as indicated in Fig. 8. After a very small interval of time δt the vorticity at the point (x, y) is changed by the amount

$$\delta\zeta(x, y) = -v'(x, y)\delta t \zeta_0'(y), \tag{10.10}$$

because it is replaced by a fluid element from below, which retains its original vorticity. This change produces an effect at the "vortex," i.e., at the element of fluid under

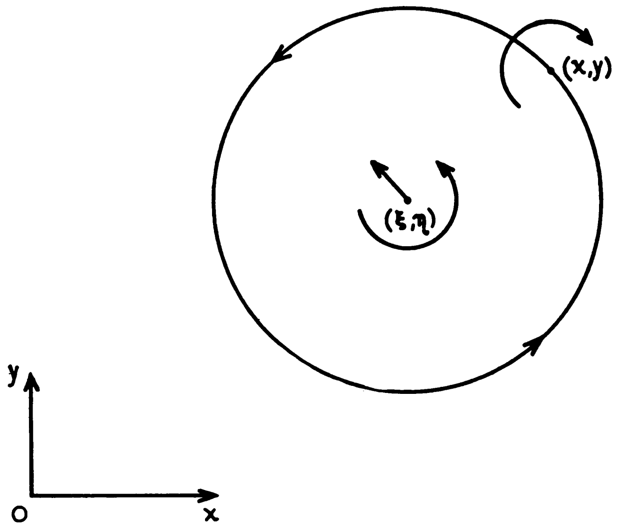

FIG. 8. Acceleration of vorticies in a non-uniform field of vorticity ($\zeta_0'(y)>0$, $\Gamma>0$).

consideration at (ξ, η). It can be easily seen that the effect is a small velocity with components

$$\left.\begin{aligned} \delta u(\xi, \eta) &= \frac{1}{2\pi}\int\int \frac{\partial}{\partial \eta} G(x, y; \xi, \eta)\delta\zeta(x, y)dxdy, \\ \delta v(\xi, \eta) &= -\frac{1}{2\pi}\int\int \frac{\partial}{\partial \xi} G(x, y; \xi, \eta)\delta\zeta(x, y)dxdy, \end{aligned}\right\} \tag{10.11}$$

the integrals being extended over the whole region between the planes. Dividing these quantities by δt and passing to the limit $\delta t \to 0$, we have the following components of acceleration at the point (ξ, η):

$$\left.\begin{aligned} a_x(\xi, \eta) &= -\frac{1}{2\pi}\int\int \frac{\partial}{\partial \eta} G(x, y; \xi, \eta)v'(x, y)\zeta_0'(y)dxdy, \\ a_y(\xi, \eta) &= \frac{1}{2\pi}\int\int \frac{\partial}{\partial \xi} G(x, y; \xi, \eta)v'(x, y)\zeta_0'(y)dxdy. \end{aligned}\right\} \tag{10.12}$$

Let us first consider the y-component of this acceleration. From the special form in which x and ξ enter into the Green's function [cf. (10.9)], we can also write

$$a_y(\xi, \eta) = -\frac{1}{2\pi}\iint v'(x, y)\zeta_0'(y)\frac{\partial}{\partial x}G(x, y; \xi, \eta)dxdy. \tag{10.13}$$

If we multiply this equation by $\zeta'(\xi, \eta)$ and integrate over the whole region, we have the final formula

$$\iint a_y(\xi, \eta)\zeta'(\xi, \eta)d\xi d\eta = \iint \{v'(x, y)\}^2\zeta_0'(y)dxdy, \tag{10.14}$$

upon using (10.6). Before discussing its signifiance, let us first notice that

$$\iint a_x(\xi, \eta)\zeta'(\xi, \eta)d\xi d\eta = 0 \tag{10.15}$$

if $\zeta'(\xi, \eta)$ is an even function of $\xi-\xi_0$, i.e., if the vorticity distribution $\zeta'(\xi, \eta)$ has a symmetry about the line $\xi=\xi_0$. For then $v'(x, y)$ is an odd function of $x-\xi_0$ and $a_x(\xi, \eta)$ is an odd function of $\xi-\xi_0$, both being the consequence of the fact that $G(x, y; \xi, \eta)$ is an even function of $x-\xi$. Hence, we have the conclusion.

If we recall that the vorticity $\zeta'(\xi, \eta)$ is spread over a small region, we may take $\Gamma=\iint\zeta'(\xi, \eta)d\xi d\eta$ as the strength of the "superposed vortex." If we divide the left-hand side of (9.14) and (9.15) by Γ, we may consider the results as giving the components of the "average acceleration." The x-component of acceleration vanishes; the sign of the y-component depends upon the sign of the superposed vortex and the sign of $\zeta_0'(y)$. This component of acceleration is the one used in the above physical considerations.

It should be mentioned that in considering the stability of a motion we deal with a vortex pair. Although this makes it difficult to obtain a compact formula for the average accelerations of the individual vortices, a kinematical consideration such as that given above (cf. Fig. 8) shows that the general tendency is not changed. Furthermore, the two vortices are soon separated, because they are situated in layers of different mean velocity.

Another point should be mentioned. If we notice the tendency for the main vorticity to be swung around the secondary vortex, there is an acceleration of every element of fluid toward the vortex. Whatever this acceleration may be, it is expected to be of minor importance, because the effect is spread out over the whole field. This point will be brought out clearly in the following derivation of (10.14), where we shall study the whole phenomenon from the point of view of pressure forces. The acceleration will be identified with the negative of the pressure gradient divided by the density of the fluid, because the effect of viscosity has been neglected. Thus, if we can calculate the pressure disturbance corresponding to a given velocity disturbance, the left-hand side of (10.14) can be calculated.

2) *Second derivation, by consideration of pressure forces correlated with vorticity fluctuations.* To calculate the pressure distribution from a given velocity distribution, we use for the pressure a differential equation of Poisson's type obtained by taking the divergence of the equations of motion. Thus, if the equations of motion are[10]

[10] The usual notation is used: x_i ($i=1, 2, 3$) are the coordinates, u_i are the components of velocity, p is the pressure, and ρ is the density of the fluid. Summation over a repeated index is understood. For a discussion of this type, see Lichtenstein's book [26].

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \Delta u_i, \qquad (i = 1, 2, 3), \tag{10.16}$$

and the equation of continuity is

$$\partial u_i/\partial x_i = 0, \tag{10.17}$$

we have

$$\Delta(p/\rho) = -\sigma, \tag{10.18}$$

where

$$\sigma = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = e_{ij}e_{ij} - \omega_{ij}\omega_{ij} = 2\left\{\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} + \frac{\partial(u_2, u_3)}{\partial(x_2, x_3)} + \frac{\partial(u_3, u_1)}{\partial(x_3, x_1)}\right\}, \tag{10.19}$$

and

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{10.20}$$

are the components of deformation and of rotation. If we integrate (10.18) under proper boundary conditions, σ being known at the *initial* instant, we obtain the *initial* distribution of pressure. The initial acceleration field is then obtained from (10.16) as the negative gradient of the pressure, if we neglect the effect of viscosity.

For a perfect fluid, the only boundary condition at a solid wall is

$$u_i n_i = 0, \tag{10.21}$$

where n_i is the outward normal of the boundary surface. If we multiply (10.16) by n_i; neglecting the effect of viscosity, we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial n} = V_0 \frac{\partial u_i}{\partial s} n_i, \tag{10.22}$$

where V_0 is the velocity along a stream line on the boundary, and ds is an element of its arc. If we write

$$u_i = V_0 l_i, \qquad \frac{\partial u_i}{\partial s} = V_0 \frac{\partial l_i}{\partial s} + l_i \frac{\partial V_0}{\partial s},$$

where l_i are the direction cosines of the velocity over the boundary surface, we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{V_0^2}{R}, \tag{10.23}$$

where R is the radius of curvature of the stream line, $R^{-1} = n_i \partial l_i/\partial s$. This relation expresses the balance of pressure and centrifugal force. With a given distribution of velocity, the right-hand side is known. We have thus a potential problem of the second kind for the pressure.

Two-dimensional flow between parallel solid walls. Returning to the problem at hand, we have the very simple boundary condition

$$\partial p/\partial y = 0 \quad \text{at} \quad y = \pm b. \tag{10.24}$$

Since the main motion is a two-dimensional parallel motion, we have

$$u_1 = w(y) + u'(x, y), \qquad u_2 = v'(x, y), \qquad u_3 = 0, \tag{10.25}$$

where $w(y)$ represents the main flow, and u' and v' give a secondary flow approximating a vortex. Equation (10.18) becomes

$$\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) = -\sigma, \qquad \sigma = 2\left\{w'(y)\frac{\partial v'}{\partial x} - \frac{\partial(u', v')}{\partial(x, y)}\right\}. \tag{10.26}$$

We note that we can write $\sigma = \sigma_1 + \sigma_2$, where

$$\sigma_1 = -2\frac{\partial(u', v')}{\partial(x, y)}, \qquad \sigma_2 = -2\zeta_0\frac{\partial v'}{\partial x}\cdot \tag{10.27}$$

σ_1 depends upon the structure of the secondary vortex itself, and σ_2 depends upon its interaction with the main flow. We shall also separate the pressure into two parts and require them to satisfy (10.24) separately. Thus,

$$\left.\begin{aligned} p = p_1 + p_2, \qquad & \sigma = \sigma_1 + \sigma_2, \\ \frac{1}{\rho}\left(\frac{\partial^2 p_1}{\partial x^2} + \frac{\partial^2 p_1}{\partial y^2}\right) = -\sigma_1, \qquad & \frac{\partial p_1}{\partial y} = 0 \quad \text{at} \quad y = \pm b, \\ \frac{1}{\rho}\left(\frac{\partial^2 p_2}{\partial x^2} + \frac{\partial^2 p_2}{\partial y^2}\right) = -\sigma_2, \qquad & \frac{\partial p_2}{\partial y} = 0 \quad \text{at} \quad y = \pm b. \end{aligned}\right\} \tag{10.28}$$

We can reduce our problem to that of the *first* kind by looking for the acceleration $a_y(x, y)$ in the y-direction, $a_y = -(1/\rho)\,\partial p/\partial y$. If we differentiate (10.28) with respect to y, we have

$$\left.\begin{aligned} & a_y = \alpha_1 + \alpha_2, \\ \frac{\partial^2 \alpha_1}{\partial x^2} + \frac{\partial^2 \alpha_1}{\partial y^2} = \frac{\partial \sigma_1}{\partial y}, \qquad & \frac{\partial^2 \alpha_2}{\partial x^2} + \frac{\partial^2 \alpha_2}{\partial y^2} = \frac{\partial \sigma_2}{\partial y}, \end{aligned}\right\} \tag{10.29}$$

with $\alpha_1 = 0$, $\alpha_2 = 0$ at $y = \pm b$. The x-component of acceleration is zero, from symmetry considerations. The solutions of (10.29) are

$$\left.\begin{aligned} \alpha_1(x, y) &= -\frac{1}{2\pi}\iint G(x, y : \xi, \eta)\left(\frac{\partial \sigma_1}{\partial y}\right)_{\xi,\eta} d\xi d\eta, \\ \alpha_2(x, y) &= \frac{1}{2\pi}\iint G(x, y; \xi, \eta)\left(\frac{\partial \sigma_2}{\partial y}\right)_{\xi,\eta} d\xi d\eta, \end{aligned}\right\} \tag{10.30}$$

where the integrals are extended over the whole region between the planes. These formulae give the distribution of acceleration. Actually, it is more convenient to deal with the integrated quantities

$$I_1 = \iint \alpha_1(x, y)\zeta_0(x, y)dxdy, \tag{10.31}$$

$$I_2 = \iint \alpha_2(x, y)\zeta_0(x, y)dxdy, \tag{10.32}$$

$$J_1 = \iint \alpha_1(x, y)\zeta'(x, y)dxdy, \tag{10.33}$$

$$J_2 = \iint \alpha_2(x, y)\zeta'(x, y)dxdy. \tag{10.34}$$

The first two integrals correspond to the accelerations of the main flow by the secondary flow itself and by the interaction; the latter two quantities correspond to the accelerations of the secondary flow by itself and by the interaction. It can be verified (as will be done presently) that

$$I_1 = -\iint v'^2 \frac{d\zeta_0}{dy} dxdy, \quad (10.35) \qquad I_2 = 0, \quad (10.36)$$

$$J_1 = 0, \quad (10.37) \qquad J_2 = \iint v'^2 \frac{d\zeta_0}{dy} dxdy. \quad (10.38)$$

We note that (10.38) is essentially a reproduction of the formula (10.14), the significance of which has been discussed above. The integral I_1 is equal to the negative of J_2. This is the above-mentioned acceleration distributed among the fluid elements throughout the field. It is therefore relatively unimportant. Thus, all the statements made in the last section have been verified, if we can verify (10.35)–(10.38).

Verification of (10.35)–(10.38). To verify these equations, let us first examine the behavior of the quantities u', v', $\partial p/\partial x$, $\partial p/\partial y$ for large values of x. From the expression (10.9) for the Green's function, we see that if ζ' vanishes sufficiently rapidly as x becomes infinite, we have

$$u' = O(|x|^{-3}), \qquad v' = O(|x|^{-2}), \tag{10.39}$$

for large values of x. From the equations of motion, we then find that

$$\frac{\partial p}{\partial x} = O(u') = O(|x|^{-3}), \qquad \frac{\partial p}{\partial y} = O(v') = O(|x|^{-2}). \tag{10.40}$$

This will assure the convergence of the integrals involved and the validity of the steps taken in the following transformations.

In the first method of derivation, we have been mainly concerned with J_2. We shall therefore consider it first. Referring to (10.30) and (10.5), we see that

$$J_2 = \iint \psi'(\xi, \eta)\left(\frac{\partial \sigma_2}{\partial y}\right)_{\xi,\eta} d\xi d\eta.$$

If we now introduce the value of σ_2 as given by (10.27) and replace (ξ, η) by (x, y), we have

$$J_2 = -2\iint \psi'(x, y)\frac{\partial^2}{\partial x \partial y}(v'\zeta_0)dxdy.$$

On integrating by parts with respect to x, we obtain

$$J_2 = 2\iint v'\frac{\partial}{\partial y}(v'\zeta_0)dxdy = \iint \left\{\frac{\partial}{\partial y}(v'^2\zeta_0) + v'^2\frac{d\zeta_0}{dy}\right\}dxdy.$$

The result (10.38) or (10.14) is thereby verified.

Similar calculations can be carried out for the integrals I_1, I_2, and J_1. Thus,

$$I_1 = \iint \alpha_1 \zeta_0 dxdy = \iint w(y) \frac{\partial \alpha_1}{\partial y} dxdy$$
$$= \iint w\left(\sigma_1 + \frac{1}{\rho} \frac{\partial^2 p_1}{\partial x^2}\right) dxdy,$$

by (10.28). If we note that

$$w\sigma_1 = -2w \frac{\partial(u', v')}{\partial(x, y)} = 2 \frac{\partial}{\partial x}\left(wv' \frac{\partial u'}{\partial y}\right) - 2 \frac{\partial}{\partial y}\left(wv' \frac{\partial u'}{\partial x}\right) - \frac{dw}{dy} \frac{\partial^2 v'}{\partial y^2},$$

the above integral is easily transformed into the form (10.35). Following an exactly analogous process, we have

$$I_2 = \iint w\left(\sigma_2 + \frac{1}{\rho} \frac{\partial^2 p_2}{\partial x^2}\right) dxdy = 0,$$

when we make use of (10.27). The integral J_1 has also the significance that it is the effect of the solid boundaries upon a general flow $\psi'(x, y)$ consistent with (10.40), because it is independent of $w(y)$. Using (10.30), (10.5), and (10.27), we have

$$J_1 = \iint \alpha_1 \zeta' dxdy = \iint \psi'(\xi, \eta) \left(\frac{\partial \sigma_1}{\partial y}\right)_{\xi, \eta} d\xi d\eta$$
$$= -2 \iint u' \frac{\partial(u', v')}{\partial(x, y)} dxdy.$$

If we note that

$$u' \frac{\partial(u', v')}{\partial(x, y)} = \frac{\partial}{\partial y}\left(u'v' \frac{\partial u'}{\partial x}\right) - \frac{\partial}{\partial x}\left(u'v' \frac{\partial u'}{\partial y}\right),$$

we see that $J_1 = 0$. The results (10.35)–(10.38) are thereby verified. We have thus completed the investigations indicated at the beginning of this section.

Note added in proof. In a very early work, [Phil. Trans. Roy. Soc. London (A) **215,** 23–26 (1915)] G. I. Taylor gave a physical interpretation of Rayleigh's results on the stability of the laminar motion of an inviscid fluid, based on momentum considerations. He also indicated clearly that a motion, stable according to Rayleigh's criterion, may be unstable through the effect of viscosity.

(To be continued)