ON THE VIBRATIONS OF THE ROTATING RING*

BY

G. F. CARRIER
Harvard University

1. Introduction. An interesting addition to the group of problems dealing with thin elastic rings is the analysis of the vibration of a circular ring which is rotating with constant speed about its geometric axis. In this paper, the small bending vibrations of the unconstrained ring are analyzed and the frequencies at which such vibrations can occur are determined. For various problems of the partially constrained ring, it is shown that the "free vibrations" differ essentially in character from those of the free ring, exhibiting a group of natural modes characterized by linear combinations of trigonometric functions. The forced vibrations of both the free and supported rings are also treated.

2. The dynamic equations. The three equations needed to specify completely the plane motion of an element of a ring, such as the one shown in Fig. 1, are derived from a consideration of the forces and moments acting on the element and the components of acceleration of the element. The summation of forces along $o'a'$, the summation of moments about o', and the summation of moments about a', lead to this required set of equations, which is

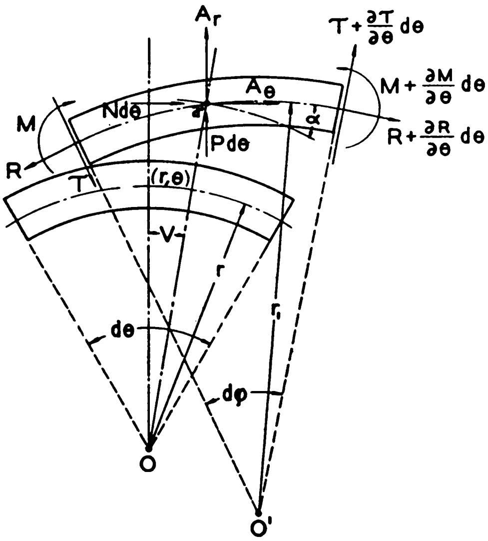

FIG. 1. Element of ring in initial and distorted positions. Center of rotation is o.

$$- Rd\phi + \left[\frac{\partial \tau}{\partial \theta} + \left(P - \frac{k}{r} A_r\right)\cos(\alpha - v) - \left(N - \frac{k}{r} A_\theta\right)\sin(\alpha - v)\right] d\theta = 0, \qquad (1)$$

$$\frac{\partial R}{\partial \theta} - \frac{1}{r_1}\frac{\partial M}{\partial \theta} + \left(N - \frac{k}{r} A_\theta\right)\cos(\alpha - v) + \left(P - \frac{k}{r} A_r\right)\sin(\alpha - v) = 0, \qquad (2)$$

$$r\tau = \frac{\partial M}{\partial \theta} - rk\frac{a}{c}\frac{\partial^2(\alpha - v)}{\partial t^2} = 0. \qquad (3)$$

Here, the notations are as follows: θ is a polar coordinate of a point in the undeformed ring, referred to axes rotating with the ring; r and r_1 are respectively the radii of curvature of the undistorted and distorted rings; b, h and I are respectively the width, thickness and cross-sectional moment of inertia of the ring; R, τ and M are respec-

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tively the tensile force, shearing force and bending moment, as shown; N/r and P/r are the components of the external forces in the directions of the tangent and normal in the undistorted state, as shown; A_r and A_θ are the components of acceleration in the directions of the tangent and normal in the undistorted state, as shown; E is the elastic modulus; ρ is the density; $k=\rho bhr^2$; $c=Ebh$; $a=EI/r^2$.

In Eqs. (1)–(3) the corrections, arising from the Poisson strains, for the moment of inertia expressions, etc., have been omitted as usual. The formulas needed to supplement the above equations are

$$M = \frac{EI}{r}\left(u_0 + u + \frac{\partial^2 u}{\partial\theta^2}\right), \quad (4a) \qquad \frac{1}{r_1} = \frac{1}{r}\left(1 - u_0 - u - \frac{\partial^2 u}{\partial\theta^2}\right), \quad (4b)$$

$$R = Ebhe_\theta = c\left(u_0 + u + \frac{\partial v}{\partial\theta}\right), \quad (4c) \qquad \alpha = \frac{\partial u}{\partial\theta}, \quad (4d)$$

$$A_r = r\left[\frac{\partial^2 u}{\partial t^2} - \omega^2(1 + u_0 + u) - 2\omega\frac{\partial v}{\partial t}\right], \quad (4e)$$

$$A_\theta = r\left[\frac{\partial^2 v}{\partial t^2} + 2\omega\frac{\partial u}{\partial t} - \omega^2 v\right], \quad (4f)$$

$$d\phi = \frac{r}{r_1}(1 + e_\theta)d\theta = \left(1 + \frac{\partial v}{\partial\theta} - \frac{\partial^2 u}{\partial\theta^2}\right)d\theta, \quad (4g)$$

where ru_0 is the radial displacement from the rest position to the rotating equilibrium position, ru is the radial displacement from the rotating equilibrium position, rv is the tangential displacement relative to the rotating axes,[1] e_θ is the tangential strain ($e_\theta=u_0+u+\partial v/\partial\theta$), and ω is the constant angular velocity of the ring. Equations (4a) and (4b) are the well known expressions for the bending moment and curvature, respectively, of a bent ring;[2] (4c) is a one-dimensional form of Hooke's law; (4d) is the rotational displacement of the element, and is found by inspection of Fig. 1; (4e) and (4f) are the expressions for the radial and tangential components of acceleration, when u and v are referred to a rotating coordinate system;[3] and g is obtained from Fig. 1 and Eq. (4b).

The value of u_0 is obtained by writing $u=v=0$ in Eq. (1). After substitutions from Eqs. (4), it becomes $Ebhu_0=k\omega^2(1+u_0)$, or

$$u_0 = (1 + u_0)k\omega^2/c. \quad (5)$$

Throughout this analysis, we shall consider only those vibrations for which u and v are small compared to unity. We are therefore justified in disregarding terms in u^2, uv, etc., as compared to u or v. In the limit, that is, as the amplitude of u and v tend to zero, the equations obtained in this manner would be exact. However, the equations so obtained would still be encumbered by terms of the type u_0^2u, u_0^2v, $\cdots$, in addition to those found below in Eqs. (6) and (7). We also neglect these terms since

[1] This rather unconventional notation is used to provide u_0, u and v with dimensionless properties and thus produce somewhat less cumbersome equations.

[2] S. P. Timoshenko, *Strength of materials*, D. Van Nostrand Co., New York, 1930, p. 459.

[3] These are easily deduced from the vector forms given in L. Page, *Introduction to theoretical physics*, D. Van Nostrand Co., New York, 1941, p. 103.

they can produce no qualitative changes in the results and since they will drop out anyway when the procedure leading to Eq. (9a) is introduced. Finally, since the amplitudes of P and N must obviously vanish when u and v tend to zero, terms in Pu, Pv, $\cdots$, must also be omitted.

Following this procedure, using Eq. (3) to eliminate τ, and substituting from Eqs. (4) when necessary, we obtain from Eqs. (1) and (2)

$$P - c\left(u + \frac{\partial v}{\partial \theta}\right) - a\left(\frac{\partial^4 u}{\partial \theta^4} + \frac{\partial^2 u}{\partial \theta^2}\right) + k\omega^2\left(\frac{\partial^2 u}{\partial \theta^2} + u - \frac{\partial v}{\partial \theta}\right)$$
$$= k\left[\frac{\partial^2 u}{\partial t^2} - 2\omega\frac{\partial v}{\partial t} - \frac{a}{c}\,\frac{\partial^4 u}{\partial \theta^2 \partial t^2} + \frac{a}{c}\,\frac{\partial^3 v}{\partial \theta \partial t^2}\right], \tag{6}$$

$$N + c\frac{\partial}{\partial \theta}\left(u + \frac{\partial v}{\partial \theta}\right) - a\left(\frac{\partial^3 u}{\partial \theta^3} + \frac{\partial u}{\partial \theta}\right) + k\omega^2\frac{\partial u}{\partial \theta} = k\left[\frac{\partial^2 v}{\partial t^2} + 2\omega\frac{\partial u}{\partial t}\right]. \tag{7}$$

We may easily arrive at a single equation in u only by performing on Eq. (6) the operation L where,

$$L = c\frac{\partial^2}{\partial \theta^2} + k\frac{\partial^2}{\partial t^2}, \tag{8}$$

solving Eq. (7) for $L(v)$, and substituting the expression found by the latter step into that found by the former.[4] We utilize the abbreviations,

$$s = t\sqrt{a/k}, \qquad \mu = \omega\sqrt{k/a}, \qquad \epsilon = a/c = h^2/12r^2,$$

and the equation resulting from the foregoing procedure takes the form,

$$\Biggl\{\Biggl[\left(\frac{\partial^2}{\partial \theta^2} - 1\right)\frac{\partial^2}{\partial s^2} + 4\mu\frac{\partial^2}{\partial \theta \partial s} + \frac{\partial^2}{\partial \theta^2}\left(\frac{\partial^2}{\partial \theta^2} + 1\right)^2 - \mu^2\frac{\partial^2}{\partial \theta^2}\left(\frac{\partial^2}{\partial \theta^2} + 3\right)\Biggr]$$
$$- \epsilon\Biggl[\frac{\partial^4}{\partial s^4} + \left\{2\frac{\partial^2}{\partial \theta^2}\left(\frac{\partial^2}{\partial \theta^2} + 1\right) - \mu^2\left(\frac{\partial^2}{\partial \theta^2} - 3\right)\right\}\frac{\partial^2}{\partial s^2}$$
$$+ 2\mu\left(\frac{\partial^2}{\partial \theta^2} + 1 - 2\mu^2\right)\frac{\partial^2}{\partial \theta \partial s} + (\mu^4 - \mu^2)\frac{\partial^2}{\partial \theta^2} - \mu^2\frac{\partial^4}{\partial \theta^4}\Biggr]$$
$$+ \epsilon^2\left[\frac{\partial^6}{\partial \theta^2 \partial s^4} + 2\mu\frac{\partial^4}{\partial \theta \partial s^3} - \left(\mu^2\frac{\partial^2}{\partial \theta^2} - \frac{\partial^4}{\partial \theta^4} - \frac{\partial^2}{\partial \theta^2}\right)\frac{\partial^2}{\partial s^2}\right]\Biggr\}u$$
$$= \frac{\partial^2}{\partial \theta^2}\left(\frac{P}{a}\right) + \frac{\partial}{\partial \theta}\left(\frac{N}{a}\right) - \epsilon\left[\frac{\partial^2}{\partial s^2}\left(\frac{P}{a}\right) + 2\mu\frac{\partial}{\partial s}\left(\frac{N}{a}\right)\right] + \epsilon^2\frac{\partial^3}{\partial \theta \partial s^2}\left(\frac{N}{a}\right). \tag{9}$$

This equation governs the motion of the ring, provided the driving functions N and P do not imply that the deformations be large. The validity of this equation may be partially checked by considering a physically trivial problem. We consider the freely spinning ring (no supports) and suppose N to P to vanish. Under these conditions, the motion of the ring which is initially not deformed from its equilibrium shape is given by $u = (\alpha + \beta s)\cos(\theta + \mu s)$, $v = -(\alpha + \beta s)\sin(\theta + \mu s)$. That is, the ring moves as

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[4] A similar procedure will provide an analogous equation in v.

a rigid body with (dimensionless) angular velocity μ and translational velocity β. Eq. (9) must and does allow this solution for all α and β.

In each of those problems to be considered, Eq. (9), in its present form, leads to a solution which is simple in form but which requires the solution of unnecessarily long algebraic equations. This computational work may be eliminated at the expense of small errors in accuracy when we consider only rings for which h/r is small. In this case, terms in Eq. (9) of order ϵ and ϵ^2 may be neglected compared to those of order one. This leads to the following equation, which is exact for the limiting case where u, v, ϵ tend to zero:

$$L_1(u) = \left[\mu^2 \frac{\partial^2}{\partial\theta^2}\left(\frac{\partial^2}{\partial\theta^2} + 3\right) - \frac{\partial^2}{\partial\theta^2}\left(\frac{\partial^2}{\partial\theta^2} + 1\right)^2 - 4\mu\frac{\partial^2}{\partial\theta\partial s} - \left(\frac{\partial^2}{\partial\theta^2} - 1\right)\frac{\partial^2}{\partial s^2}\right]u$$
$$= -\frac{\partial^2}{\partial\theta^2}\left(\frac{P}{a}\right) - \frac{\partial}{\partial\theta}\left(\frac{N}{a}\right). \tag{9a}$$

The analogous equation in v is

$$L_1(v) = \frac{N}{a} + \frac{\partial}{\partial\theta}\left(\frac{P}{a}\right). \tag{9b}$$

In each of the problems to follow, the expressions obtained from Eqs. (9a) and (9b) for the natural frequencies and amplitudes are valid to within errors of order h^2/r^2 for those frequencies of order $(a/k)^{1/2}$. These vibrations may be termed "bending vibrations" since they are essentially inextensional forms of motion. When the frequencies are of order $(c/k)^{1/2}$, accurate results may be obtained by direct use of Eq. (9). In this paper, we shall use only Eq. (9a) or (9b) since all of the characteristics of the effects of rotation on the behavior of the ring will appear in the solutions so obtained. The one exception to this statement is found in connection with the dilatory vibrations. This type of motion can not be predicted by Eq. (9a) or (9b) because these equations are those for essentially inextensional motions.[5] We shall, then, when investigating this mode, refer to Eq. (9). For this mode, u and v are independent of θ as must be those parts of P and N which excite such a motion. Hence Eq. (9) reduces for this case to

$$L_0(u) = \left[\frac{\partial^4}{\partial s^4} + \left(3\mu^2 + \frac{c}{a}\right)\frac{\partial^2}{\partial s^2}\right]u = \frac{\partial^2}{\partial s^2}\left(\frac{P}{a}\right) + 2\mu\frac{\partial}{\partial s}\left(\frac{N}{a}\right), \tag{9c}$$

or, in terms of v,

$$L_0(v) = \left(\frac{\partial^2}{\partial s^2} + \frac{c}{a} - \mu^2\right)\frac{N}{a} - 2\mu\frac{\partial}{\partial s}\frac{P}{a}. \tag{9d}$$

3. The unconstrained ring. Investigation of the solutions of Eqs. (9a) or (9b) of the form

$$u_n = U_n \cos n(\theta - \beta_n s) + W_n \sin n(\theta - \beta_n s) \tag{10}$$

yields, when N and P vanish identically,

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[5] Equation (9a) may also be derived from Eqs. (6) and (7) by use of the assumption that $u+\partial v/\partial\theta \ll u$; this excludes the dilatory motion.

$$\beta_n = -\frac{2\mu}{n^2+1} \pm \frac{(n^2-1)}{n^2+1}\sqrt{\mu^2+n^2+1} = -q_n \pm p_n, \tag{10a}$$

where p_n and q_n are self-defining.

If we further apply a set of initial conditions, such as

$$u_n = U_n \cos n\theta + W_n \sin n\theta \quad \text{and} \quad \frac{\partial u_n}{\partial s} = 0, \text{ at } s = 0,$$

to the two solutions defined for a given n by Eqs. (10) and (10a), namely

$$\begin{aligned} u_n = {} & U_n[a_n \cos n(\theta + q_n s - p_n s) + b_n \cos n(\theta + q_n s + p_n s)] \\ & + W_n[c_n \sin n(\theta + q_n s - p_n s) + d_n \sin n(\theta + q_n s + p_n s)], \end{aligned} \tag{11}$$

we obtain

$$\begin{aligned} u_n = {} & U_n\left[\cos n(\theta + q_n s) \cos np_n s + \frac{q_n}{p_n} \sin n(\theta + q_n s) \sin np_n s\right] \\ & + W_n\left[\sin n(\theta + q_n s) \cos np_n s - \frac{q_n}{p_n} \cos n(\theta + q_n s) \sin np_n s\right]. \end{aligned} \tag{12}$$

Each term in the foregoing bracket defines a possible free vibration of the ring which is unconstrained at all points against either radial or tangential displacement. Each of these terms may be interpreted as defining a "normal mode" of vibration, wherein a sinusoidal deformation of angular frequency np_n travels with respect to the rotating axes at an angular velocity[6] $-q_n$. The "nodal points" thus move with respect to coordinates fixed in the ring. The terms "normal mode" and "nodal point" have been used somewhat loosely here, but they adhere to the usual definitions of the terms if the motion is described relative to axes rotating with velocity $\Omega_n = \omega(n^2-1)/(n^2+1)$.

For the stationary ring, the value of the angular frequency reduces to

$$(a/k)^{1/2} np_n = n(n^2-1)[a/k(n^2+1)]^{1/2},$$

which is in agreement with previously derived results.[7]

A solution to Eq. (9a) may be obtained for arbitrary initial distributions of radial deflection U_0 and radial velocity U_0', provided these initial conditions do not imply an extensional motion. The restriction

$$\int_0^{2\pi} U_0(\theta)d\theta = \int_0^{2\pi} U_0'(\theta)d\theta = 0 \tag{13}$$

is certainly sufficient to insure this provision since it, together with the continuity requirements,

$$\int_0^{2\pi} \frac{\partial v}{\partial \theta} d\theta = \int_0^{2\pi} \frac{\partial^2 v}{\partial\theta\partial s} = 0$$

allows $u + \partial v/\partial\theta$ to vanish for all θ, at and shortly after time $t=0$.

The restriction defined by Eq. (13) together with the requirement that u be continuous, implies that U_0 and U_0' may be expanded in Fourier series in which the

[6] p_n and q_n are, of course, dimensionless quantities which define the angular velocities.

[7] J. P. Den Hartog, *Mechanical vibrations*, McGraw-Hill, New York, 1940, p. 123.

constant term vanishes and in which no terms corresponding to $n=1$ will appear, since these terms define no distortion.[8] These series may be used in conjunction with series of solutions of the type given by Eq. (11). Thus the coefficients, and hence the motion, are determined. The motion due to rigid body displacements may, of course, be superimposed on such solutions.

4. Forced vibrations. As will be seen in the following section, the investigation of the possible motions of the constrained ring requires, as a preliminary step, the determination of the behavior of a ring acted upon by a force distribution $N/a=2A\cos n\theta\cos\lambda s$. The problem involving a driving function P of similar form is obviously covered by this problem. If we split N into two parts,

$$N/a = A[\cos(n\theta-\lambda s)+\cos(n\theta+\lambda s)],$$

a particular solution of the form,

$$v = b_n\cos(n\theta-\lambda s)+c_n\cos(n\theta+\lambda s) \tag{14}$$

is easily shown to exist by substituting this expression into Eq. (9b). The coefficients b_n and c_n are readily found when this is done, and are given by

$$X_n(\lambda) = b_n/A = [n^2(n^2+1)(p_n^2-q_n^2)-(n^2+1)\lambda^2-4n\mu\lambda]^{-1}, \tag{14a}$$

$$Y_n(\lambda) = c_n/A = [n^2(n^2+1)(p_n^2-q_n^2)-(n^2+1)\lambda^2+4n\mu\lambda]^{-1}, \tag{14b}$$

unless λ is one of the values given by $\lambda^2=n^2\beta_n^2$. β_n is either of the values given by Eq. (10a).

When $N/a=2B\cos n\theta\sin\lambda s$, we have

$$v = d_n\sin(n\theta-\lambda s)-e_n\sin(n\theta+\lambda s),$$

and

$$d_n/B = X_n, \qquad e_n/B = Y_n.$$

The quantities X_n and Y_n are useful later in the paper; hence the special notation.

We see now that the motion of the unconstrained ring resulting from the type of loading described is composed of two waves of different amplitudes traveling around the ring with equal but opposite velocities. We note that there is, for each n, one value of λ for which there are fixed nodal points in so far as tangential motion is concerned. This value of λ is defined by $X_n+Y_n=0$.

It follows from the linearity of our equations that the driving function N of the more general form,

$$N/a = \sum_{m,n} A_{mn}\cos n\theta\sin\lambda_m s \tag{15}$$

will correspond to a solution

$$v = \sum b_{mn}\cos(n\theta-\lambda_m s)+c_{mn}\cos(n\theta+\lambda_m s). \tag{15a}$$

Terms X_{mn} and Y_{mn} are defined as were $X_n(\lambda_m)$ and $Y_n(\lambda_m)$ in Eqs. (14a) and (14b).

The particular problem in which the exciting force is given by

$$N/a = A_0\cos\lambda s \tag{16}$$

[8] This is seen in the discussion following the introduction of Eq. (9).

has no inextensional solutions. As was mentioned previously, we must, in this case, use Eq. (9d) for the determination of v. The solution has the form $v = b_0 \cos \lambda s$ where b_0 is given by

$$X_0(\lambda) = \frac{b_0}{A_0} \frac{\lambda^{-2} + \epsilon(1 - \mu^2/\lambda^2)}{1 + 3\epsilon\mu^2 - \epsilon\lambda^2}. \tag{16a}$$

The solution arising from the loading $N/a = A_0 \sin \lambda s$ has the same coefficient. For small λ^2, $v \doteqdot A_0\lambda^{-2} \cos \lambda s$.

The natural frequency for the dilatory type of vibration is found by letting the denominator of Eq. (16a) vanish. Its value is given by

$$\lambda_0' = [(1 + 3\epsilon\mu^2)/\epsilon]^{1/2}.$$

Returning momentarily to the question of accuracy, we note that here as in all subsequent problems the exact values of X_n and Y_n differ from those obtained in this section by terms (in the denominator) of order ϵ. Our work is accurate then when $\lambda \ll \epsilon^{-1/2}$.

5. The supported ring. The first fact to observe in the investigation of the "free vibrations" of the partially constrained ring is that when N and P vanish identically, no solutions to Eq. (9a) which obey the boundary conditions can exist. Specifically, we consider the ring to be supported by a number of evenly spaced, rigid, radial supports (let there be J of them), and suppose the ring to be so fastened to these supports that radial motion is unconstrained at all points, but that $v(2\pi i/J, s)$ must vanish for all values of s and for each integer i. The first part of the appendix is devoted to the outline of a proof that Eq. (9b) has no solution under the foregoing conditions. Since the same proof holds for Eq. (9), we must conclude that the supports exert reactions which are to be accounted for in the differential equation by a function N which does not vanish identically. The problem, physically, becomes that of determining what periodic forces, applied at the supports, are capable of sustaining a motion wherein the supported points of the ring have no displacement (tangentially) at any time. (We assume that the supports must move at precisely the speed ω.) Mathematically, we must determine the eigenvalues λ_i and the corresponding solutions of the differential equation wherein we set $P = 0$ and

$$N/a = \left[1 + 2 \sum_{n=J,2J,\cdots}^{\infty} \cos n\theta\right][A \cos \lambda s + B \sin \lambda s]. \tag{17}$$

This expression defines a loading which must correspond to a motion which has period $2\pi/J$ in θ, since the force is the same at each support. When J is even, there may also be solutions periodic in π/J which don't imply extensional motion. We shall not consider these, however, since both the procedures and results are analogous in the two cases.

We have already shown [Eqs. (15) and (15a)] that for loadings of the type given by Eq. (17), solutions of the form

$$v(\theta, s) = \sum_n [b_n \cos (n\theta - \lambda s) + \cdots - e_n \sin (n\theta - \lambda s)] \tag{18}$$

exist for all λ except those for which $\lambda^2 = n^2\beta_n^2$. Using Eq. (9b) we determined all coefficients except those of index zero, and these were found with the aid of Eq. (9d). We may then, in this problem, replace Eq. (9) by the following:

$$L_0(v') = L_0'(N/a), \qquad L_1(v - v') = (N - N')/a, \tag{19}$$

where the former equation is merely Eq. (9d), v' and N' are those parts of v and N which are independent of θ, and the latter equation is Eq. (9b). The solutions are now defined by Eq. (18) and the coefficients by Eqs. (14a), (14b), and (16a).

If we now plot for a continuous range of λ,

$$\sigma(\lambda) = \sum_{n=J}^{\infty} X_n(\lambda) + Y_n(\lambda) = \sum_n (b_n + c_n)/A = \sum_n (d_n + e_n)/B,$$

we find that the resulting graph (Fig. 2) contains two singularities corresponding to each n. Furthermore, there are two values of λ which we may associate with each n for which $\sigma(\lambda)$ vanishes. One of these lies between the two singularities belonging to n; the other lies to the right of these values. We denote the smaller by λ_n the larger by λ_n^*.

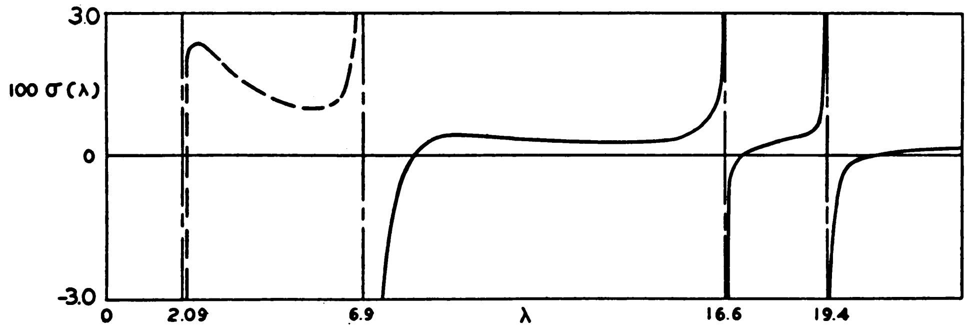

FIG. 2. Response curve for ring driven at two points. $\sigma(\lambda)$ is the amplitude of the motion of the points of application of the force. The broken section of the curve is drawn to the scale 1:5. For this curve, $\mu = 3$.

We observe now that the motion at the points $2\pi i/J$ is given by $v(0, s) = A\sigma(\lambda) \cos \lambda s + B\sigma(\lambda) \sin \lambda s$. The values λ_i and λ_i^* therefore define the frequencies for which the tangential displacements at the points of support are identically zero. They are therefore the desired eigenvalues. The motion of the ring for any λ_i or λ_i^* is given by

$$v_i(\theta, s) = \sum_n A_i X_{in} \cos (n\theta - \lambda_i s) + \cdots + B_i Y_{in} \sin (n\theta - \lambda_i s), \tag{19a}$$

where A_i and B_i are determined by the initial conditions.

The question now arises as to whether linear combinations of the v_i will always describe the motion arising from an initial set of conditions which are arbitrary except for the previously prescribed periodicity in θ. An outline of a proof that the v_i are complete in this sense is included in the appendix. Since the natural modes of the possible vibrations are described accurately by Eq. (19a) only for the smaller values of λ_i, it may seem at first that this question of completeness is superfluous. However, the completeness of such sets of solutions provides an assurance that no other possible solutions of the equations have been overlooked in the analysis. In view of the fact that the foregoing procedure started with a guess as to the probable form of the support reactions, this indication that certain initial conditions would not lead to different types of motions is helpful. We conclude, then, that all vibrations, arising from initial conditions whose Fourier series expansions are such that the low index terms predominate, can be closely approximated by sums of the form,

$$v(\theta, s) = \sum_i K_i v_i(\theta, s). \tag{20}$$

We also conclude that the supports will continue to exert exactly those reactions required to sustain this motion, or in other words, those forces required to prevent all tangential motion of the supported points of the ring.

The problem of the radially constrained ring may be treated in a manner similar to the foregoing, with analogous results. When the ring is constrained at its supported points against both radial and tangential displacement, Eqs. (9a) and (9b) must both be used. Support reactions of the form

$$N/a = A\left[1 + 2\sum_n \cos n\theta\right]\cos \lambda s, \qquad P/a = B\left[1 + 2\sum_n \cos n\theta\right]\sin \lambda s,$$

and solutions of the form

$$v = \sum a_n \cos (n\theta - \lambda s) + a_n' \cos (n\theta + \lambda s),$$
$$u = \sum b_n \sin (n\theta + \lambda s) - b_n' \sin (n\theta - \lambda s),$$

are assumed to exist as before. This time we find four functions analogous to $\sigma(\lambda)$ which enter the equations for the motion of the supported points. When this motion vanishes, these equations become $A\sigma_1(\lambda)+B\sigma_2(\lambda)=0$, $A\sigma_3(\lambda)+B\sigma_4(\lambda)=0$. The critical frequencies are defined by $\sigma_1(\lambda)\sigma_4(\lambda)-\sigma_3(\lambda)\sigma_2(\lambda)=0$. Since nothing essentially different from the preceding results would be shown, the formulas for the terms in the σ_i, the explicit expressions for the v_i, etc. are omitted.

The forced vibration problem of the supported ring can now be easily treated. For example, let us consider the ring to be supported as in the first problem of this section of the paper, but to be loaded by a force distribution which may be expanded into the form $N_1/a=\sum_n D_n \cos (n\theta \pm \nu s)$. The particular solution to Eq. (9b) corresponding to the loading N_1 is found as before, and the function

$$v(0, s) = \sum_n g_n \cos \nu s = G \cos \nu s$$

representing the displacement at the supports, has an easily evaluated amplitude G. Remembering that a support reaction,

$$N_2/a = A\left[1 + 2\sum_n \cos n\theta\right]\cos \nu s$$

produces a motion at $\theta=2\pi i/J$ which is given by $v_2(0, s)=A\sigma(\nu)\cos \nu s$, we may determine from the response curve (Fig. 2) the value of A such that $A\sigma(\nu)=-G$. The motion is then given by the solution to Eqs. (19) corresponding to the loading N_1+N_2. When more than one value of ν enters the problem, the solution is changed only by the fact that the summation now takes place over two indices.

We note that when ν in the problem just discussed is equal to one of the λ_i, a resonant condition exists, as one would expect from the results of the preceding problem. Also, we note that when the loading consists of a single term, $\cos m\theta \cos \nu s$, and when ν is one of the roots of $X_m + Y_m = 0$, the solution requires no support reactions.

Perhaps the most interesting result of this analysis is the observation that, unlike most problems of this sort, the forces exerted by the rigid supports of the vibrating system must be included in the differential equation before the solution can be obtained.

6. The elastically supported ring. A rather interesting eigenvalue problem arises when we consider the ring with elastic rather than rigid supports. Let us suppose again that the ring is unconstrained radially but that the supports resist the displacement of the points of attachment by a force, $N/a = -Kv(0, s)$.

Using Eqs. (19) as before, we find that the differential equations governing the motion now have the form,

$$L_0(v') = -KL_0'[v(0, s)], \tag{21a}$$

$$L_1(v - v') = -K[v(0, s)][2\sum \cos n\theta]. \tag{21b}$$

For solutions of the form,

$$v = \sum_n [\alpha_n \cos(n\theta - \lambda s) + \cdots - \delta_n \sin(n\theta - \lambda s)],$$

Eqs. (21a) and (21b) become

$$\begin{aligned} \alpha_n &= -KFX_n(\lambda), & \gamma_n &= -KHX_n(\lambda), \\ \beta_n &= -KFY_n(\lambda), & \delta_n &= -KHY_n(\lambda), \end{aligned} \quad (n = 0, J, 2J, \cdots). \tag{21c}$$

In these equations, $F = \sum_n (\alpha_n + \beta_n)$, $H = \sum_n (\gamma_n + \delta_n)$, and the X_n and Y_n are again given by Eqs. (14a), (14b) and (16a).

When Eqs. (21c) are added by pairs and then summed over n, the following results are obtained: $F[1 + K\sigma(\lambda)] = 0$, $H[1 + K\sigma(\lambda)] = 0$. But if F and H vanish the solution is the trivial one; hence,

$$\sigma(\lambda) = -1/K. \tag{22}$$

This equation defines the eigenvalues and hence the natural modes at which the system may vibrate. We note that as K tends to infinity λ approaches that value found in the problem of the more strongly constrained ring (as it obviously should). As K tends to zero, the solution approaches that for the unconstrained ring. It is again easily shown that the set of eigen-functions obtained in this problem is complete in the previously used sense.

A final problem in forced vibrations follows easily from the foregoing. Let the ring be supported as above, but with the supports rotating at a speed $\omega + \psi\lambda \sin \lambda s$, where ω is again constant. Briefly, we replace $v(0, s)$ by $v(0, s) + \psi \cos \lambda s$, on the right sides of Eqs. (21a) and (21b). The previously used procedures lead to the familiar set of solutions with the resonant frequencies obviously defined by Eq. (22).

Appendix

In this section, we wish to show, first, that Eq. (9b) has no solutions which are periodic in both θ and s and which vanish at $\theta = k\pi/J$, $(k = 0, 1, 2, \cdots, P \equiv N \equiv 0)$.

We assume a solution periodic in θ and write it as the sum of an even and an odd function (in θ);

$$v(\theta, s) = x(\theta, s) + y(\theta, s),$$

where $x(\theta, s) = x(-\theta, s)$ and $y(\theta, s) = -y(-\theta, s)$. In order that v satisfy the specified requirements, both x and y must vanish at the prescribed points.

In an abbreviated form we write Eq. (9b) as $L_2(v) = \partial^2 v/\partial\theta\partial s$, where the operator L_2 transforms even functions into even and odd into odd. This can be seen by inspection of Eq. (9b). The equation is now separable into two parts:

$$L_2(x) = \frac{\partial^2 y}{\partial\theta\partial t}, \qquad \text{(a)} \qquad\qquad L_2(y) = \frac{\partial^2 x}{\partial\theta\partial t}\cdot \qquad \text{(b)}$$

We operate on Eq. (b) with the operator $\partial^2(\cdots)/\partial\theta\partial t$ and substitute Eq. (a) into the result, obtaining

$$L_2^2(x) = \frac{\partial^4 x}{\partial\theta^2\partial t^2}, \tag{c}$$

which has solutions of the form, $x = \sum_n a_n \cos n\theta \exp i\gamma_n s$. All even, continuous, periodic, solutions of Eq. (c) may be written in this form but all of such solutions will fail to vanish at the specified points unless the a_n vanish identically, since γ_m/γ_n is irrational and $\cos n\theta$ never vanishes at $\theta = 0$. Equation (a) then reduces to $\partial^2 y/\partial\theta\partial t = 0$. It now becomes obvious that the solutions sought do not exist.

Before showing that the functions derived as natural modes of vibration in the section on the supported ring are capable of describing all motions periodic in $2\pi/J$ which arise from arbitrary initial conditions, we introduce the following notation:

$$X_{in} + Y_{in} = \phi_{in}, \ \lambda_i[-X_{in} + Y_{in}] = \psi_{in}, \ X_{in}^* + Y_{in}^* = \phi_{in}^*, \cdots . \tag{d}$$

For any motion described by Eq. (20), the possible initial displacements and velocities may be written

$$\begin{aligned} v(\theta, 0) &= \sum_{i=0}^{\infty} \sum_{n=0,J,\cdots}^{\infty} \left[(A_i\phi_{in} + A_i^*\phi_{in}^*)\cos n\theta + (B_i\phi_{in} + B_i^*\phi_{in}^*)\sin n\theta\right], \\ \frac{\partial v}{\partial s}(\theta, 0) &= \sum_i \sum_n \left[(A_i\psi_{in} + A_i^*\psi_{in}^*)\sin n\theta + (B_i\psi_{in} + B_i^*\psi_{in}^*)\cos n\theta\right], \end{aligned} \tag{e}$$

where the $A_i, \cdots, B_i^*$ are to be determined by the initial conditions. However, any initial conditions, periodic in $2\pi/J$ can be written

$$v(\theta, 0) = \sum_n \alpha_n \cos n\theta + \beta_n \sin n\theta, \qquad \frac{\partial v}{\partial s}(\theta, 0) = \quad \gamma_n \cos n\theta + \delta_n \sin n\theta. \tag{f}$$

This leads to the relations,

$$\sum_i (A_i\phi_{in} + A_i^*\phi_{in}^*) = \alpha_n, \ \sum_i (B_i\phi_{in} + B_i^*\phi_{in}^*) = \beta_n, \cdots .$$

This set of equations may be considered as a group to be solved for the $A_i, \cdots, B_i^*$ whenever such solutions exist. Except for special cases, such a set of equations always leads to a unique set of solutions corresponding to each set of $\alpha_n, \cdots, \delta_n$. Since Eqs. (f) can express all the specified sets of initial conditions, we see that Eqs. (e) can also accomplish this purpose and hence the v_i are complete in the sense defined above.

The case where i equals zero requires a few additional words. The fact that there is no α_0 or δ_0 seems to imply that we have two too few equations for the determination of the $A_i, \cdots, B_i^*$. However, there is only one root of $\sigma(\lambda)$ corresponding to n equal to zero. Hence, the correct correspondence between the $\alpha_i, \cdots, \delta_i$ and the $A_i, \cdots, B_i^*$ exists.

The proofs outlined in this section are not claimed to be rigorous. They are presented merely to outline the reasoning by which the two hypotheses might be proven if so desired.