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GRAPHICAL ANALYSES OF NONLINEAR CIRCUITS*

BY

ALBERT PREISMAN

Capitol Radio Engineering Institute

1. Introduction. As a general rule, a problem in physics is not considered solved unless the solution can be expressed in analytical form. The same usually holds true in the case of engineering problems, although there the art often progresses faster than the theory under the impact of economic forces, and the engineer is often forced to seek a solution by means of an experimental setup, or possibly by means of some numerical or graphical process.

The disadvantage of a numerical or graphical method is its lack of generality, its tendency towards inaccuracy, particularly owing to cumulative errors, and its inability to exhibit optimum values for the parameters involved, particularly if these have to be in numerical rather than in symbolic form. On the other hand, these methods often yield answers to problems that the analytical method cannot handle, and furthermore are often very effective as teaching aids. This is particularly true of the graphical methods.

It is the purpose of this article to illustrate the application of graphical constructions to problems involving nonlinear circuits, particularly those containing vacuum tubes. It is the writer's hope that some mathematician will be sufficiently attracted to this method to attempt to establish it on a more general basis, possibly something akin to the collection of theorems of ordinary Euclidean or of Projective Geometry.

2. Definition of graphical method. Before proceeding with a description of the method it will be desirable to define it. By graphical constructions are meant those geometrical *manipulations* by which a solution to a problem is obtained. It may be necessary to slide a curve representing a relationship between two variables along the axis of the independent variable, and to find (geometrically) where it intersects another curve representing a second relationship between the two variables. The manipulations may be more involved than those of simple translation along the axis, and it is to be stressed that the restriction of ruler and compass constructions is not invoked in these manipulations.

It is apparent that the method is not that usually understood by the average engineer, namely, the plotting of a complicated generalized analytical expression to permit values to be taken off the graph in order to obviate the need for computing the value of the expression every time the problem arises.

3. Simple series nonlinear circuit. As an elementary example of a graphical construction, let us consider the circuit shown in Fig. 1, that of a diode (two-element

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vacuum tube) in series with a resistance R and a source of d.c. potential E. It is desired to find the current flow in this circuit.

It is necessary to know the voltage-current $e-i$ relationship for the diode, and for the resistor R. We assume for simplicity that the latter is a linear resistance. Then the $e-i$ relationship is that shown in Fig. 2. The curve is a straight line making an angle θ with the voltage axis, such that

$$\cot \theta = R, \tag{1}$$

the resistance of the device. This slope is constant and hence R has a fixed value: so many volts per ampere, or ohms.

On the other hand, the diode has the characteristic shown in Fig. 3. Here, for

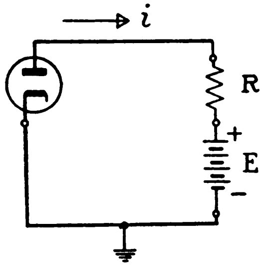


FIG. 1.

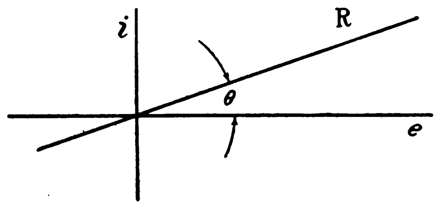


FIG. 2.

negative values of voltage (plate negative to cathode) no current can flow; while for positive values of voltage current flows in such manner as to generate the curve shown. The ideal diode would have the following equation for *positive* plate voltages

$$i = Ae^{3/2}, \tag{2}$$

but actual diodes depart to some extent from the above equation owing to such factors as initial velocity of emission of electrons from the cathode, the effect of the supporting members for the cathode and plate, etc.

The diode is a nonlinear device; first because of the break in the curve at the origin and second because even for positive plate voltages the $e-i$ relationship is usually not a straight line. One can define the resistance as

1) the reciprocal slope of the secant line to any point of the curve (this is the so-called d.c. resistance) or

2) the reciprocal slope of the tangent line to any point of the curve (this is usually called the a.c., incremental, or variational resistance of the device).

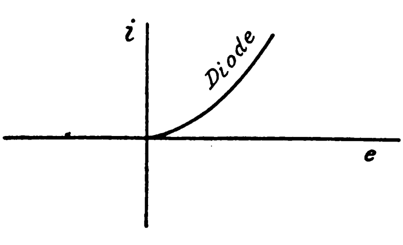


FIG. 3.

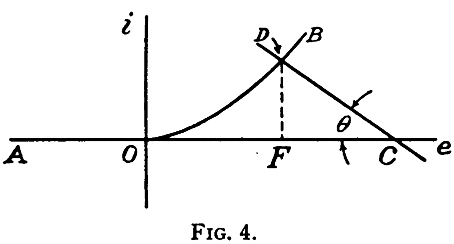


FIG. 4.

Such concepts have limited utility however, since the resistance in either case is no longer a constant, but a function of the applied voltage or current through the device.

The graphical method to be described takes the fundamental $e-i$ relationship, or terminal characteristic as it has been called by Kirschstein, and operates directly with it. Furthermore, the curve does not have to be analytic, nor even expressible in the form of an equation; it can simply be a plot of experimental data, although this involves interpolation between experimentally determined points.

The process of finding the current through the two in series for the impressed voltage E is essentially that of solving the two equations for the terminal characteristics simultaneously under the condition that the sum of the voltage drops across the two elements must equal the impressed voltage E. Thus, if the relationship for the one element is $i=f_1(e)$, then that for the other is $i=f_2(E-e)$, and it is desired to find a common value of i that satisfies both relationships.

Since one or both of the above equations may be of degree higher than unity, the analytical solution cannot be effected by the method of determinants, but rather by the method of substitution, and finally results in the necessity for solving an equation of degree higher than unity.

This, however, assumes that terminal characteristics can be represented by power series. The graphical method requires no such condition; it operates on the graphical plots directly. Thus, suppose the terminal characteristic of the diode is represented by AOB, Fig. 4. Let OC represent the magnitude of the impressed voltage E. Through C draw DC at an angle θ, as shown, such that $\cot \theta = R$. Then the intersection of CD and AOB in D represents the required solution, in that DF is the common current in this series circuit; OF is the voltage drop in the diode; FC is the voltage drop in the resistor R; and clearly OF+FC equals the impressed voltage E. If E varies with time, DC can be shifted back and forth along the voltage axis at positions corresponding to the instantaneous values of E, and the intersections will furnish the corresponding instantaneous values of the current.

The above solution represents a well-known method for solving two equations simultaneously when the equations are of degree higher than the first or even of transcendental nature. It will be of interest, however, to see how this method is applied to a more complicated circuit.

4. Triode tube and resistance in series. The next example will be that of a three-element or triode tube in series with a resistance and a source of d.c. voltage E_{bb}. The electrical connections are shown in Fig. 5. The additional complication is that in the triode the plate current is a function of two variables; the grid voltage and the plate voltage. The terminal characteristic must therefore be represented by a three-dimensional plot involving the plate current i_p, the grid voltage e_g (which is the sum of the instantaneous value of the alternating signal voltage e_s and the constant, d.c. bias voltage E_c), and the plate voltage e_p.

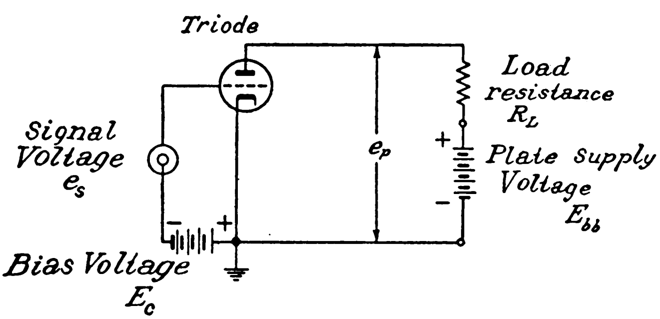


FIG. 5.

The resulting plot is a curved surface in space. It can be represented in two dimensions by a family of curves which represent discrete projections of this surface upon any one of the three coordinate planes. For the problem at hand the most useful set of projections is that upon the $e_p - i_p$ coordinate plane, in the form of a family of $e_p - i_p$ curves with e_g as the parameter. This is shown in Fig. 6 (solid lines). Curves for which e_g is positive have been omitted for simplicity.

Assume further for simplicity that R_L, the load resistance, is linear. The current through it is a function of but one voltage, that which must be applied across its terminals to produce the above current flow. To represent its terminal characteristic in three dimensions, it is plotted as a plane whose intersection with the $e_p - e_g$ coordinate plane is a straight line parallel to the e_g axis. In this way the current in it is independent of the e_g coordinate, and is a (linear) function of but one voltage, that corresponding to the plate voltage e_p of the tube. All points of this plane representing R_L project over to the $i_p - e_p$ coordinate plane as a straight line that is also the intersection of the above R_L plane with the $i_p - e_p$ plane. The straight line makes an angle θ with the e_p axis such that cot $\theta = R_L$, i.e., the R_L plane is inclined at the angle θ to the $e_g - e_p$ coordinate plane.

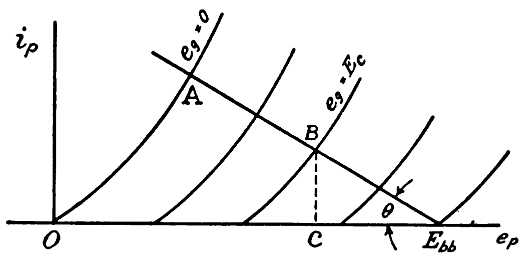


FIG. 6.

The graphical solution consists in drawing the line of intersection EA at the angle θ to the e_p axis. The intersection of EA with the tube family of curves gives the common value of current flowing through the plate circuit of the triode and R_L in series, for any given value of grid voltage e_g. For example, at a moment when the signal voltage e_s is passing through zero, the instantaneous value of the grid voltage e_g is simply that of the bias battery, E_c. The instantaneous value of the plate current is BC, where B is the intersection of AC with that curve of the plate family for which $e_g = E_c$. It is further to be noted that the instantaneous plate voltage e_p is OC, and the instantaneous value of the voltage drop across R_L is EC.

For other instantaneous values of e_g, other curves of the plate family are involved, and the process of determining the instantaneous values of plate current, plate voltage, and load voltage (across R_L) is identical to that described above. Thus, for a signal voltage impressed upon the input or grid circuit, the output signal voltage between the plate and ground can be found. Such matters as the amplification of the stage, distortion in the output, etc., can then be determined.

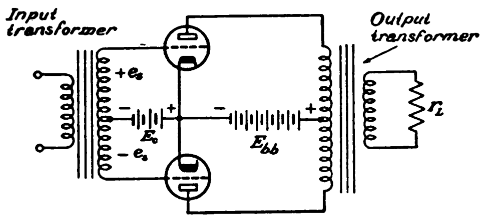
In passing, we may note here that the locus of the plate current for various values of e_g is the intersection of the tube surface and the R_L plane in space. This intersection is a curve in space, but fortunately its projection on the $e_p - i_p$ plane is a straight line, namely the intersection of the R_L plane itself with the $e_p - i_p$ plane. It is for that reason that the $e_p - i_p$ family of the tube curves is employed; the graphical construction is simply the points of intersection of a *straight line* representing R_L with the above plate family.

The above problem can become much more complicated under certain conditions. For example, if the input signal voltage is great enough, the grid can be driven positive with respect to the cathode, whereupon it draws current during the positive peak of the a.c. cycle. If the signal source has appreciable internal impedance, then a voltage drop will occur in the source during the above portion of the cycle, and the actual voltage applied to the grid will differ from the generated voltage e_s.

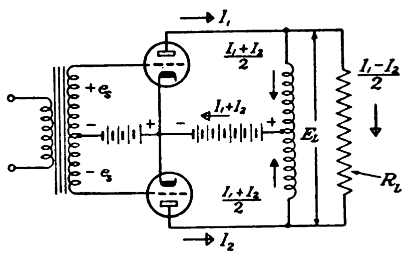
It is therefore necessary to determine the actual grid voltage before the plate current can be found. Another complication arises however, in that the grid current (and hence the actual grid voltage) is a function not only of the positive grid voltage, but of the plate voltage as well. This is because the space current divides between the two electrodes in a manner depending upon the two electrode voltages. At the same time the plate voltage is a function of R_L and the grid voltage. Thus the above simple graphical construction can become quite involved if merely the input signal is increased to a point where the grid is driven positive.

5. The balanced amplifier. Instead of investigating such details, important though they may be, it will be of interest to examine another type of circuit very important in the communication industry. Reference is made to the push-pull or balanced amplifier. The circuit is shown in Fig. 7.

In (A) is shown the actual circuit, whereas in (B) is shown an idealization or equivalent form better suited for the purpose of analysis. In the actual circuit (A), two tubes are employed, inductively coupled to each other and the output load resistance r_L by an output transformer. The signal on one grid is 180 degrees out of phase with that on the other grid, as is suggested by the symbols $+e_s$ and $-e_s$. The bias voltage E_c, on the other hand, is applied to both grids in the same polarity; and the plate supply voltage is applied to the two tubes in the same polarity too, as shown.



A



B

FIG. 7.

The actual load resistance r_L and the output transformer can be replaced by the center-tapped inductance and reflected load resistance R_L as far as the tubes are concerned. The simplified circuit is shown in (B), Fig. 7. In using this equivalent circuit, it is tacitly assumed that the actual output transformer is an ideal transformer having infinite primary and secondary open-circuit inductance, no distributed capacity, unity coefficient of coupling between windings, etc. In the equivalent circuit the center-tapped inductance is assumed to be infinite in value and to have unity coupling between the two halves of the complete winding. Ordinarily this is a reasonable assumption.

As a result, the current in one-half of the winding cannot at any moment exceed that in the other half for otherwise an infinite counter-electromotive force would be induced in the windings that would tend to prevent such an unequality from taking place. The currents in the windings can vary, however, provided they remain equal to one another at all times. Finally, two further assumptions are made, namely, that the signal voltages e_s and $-e_s$ applied to the two grids are at all times equal and opposite to one another, and that the two tubes have identical terminal characteristics. These two assumptions seem also reasonable.

Consider first that e_s equals zero (no signal is applied). The bias on each grid is E_c, and the plate voltage for either tube is E_{bb}, hence the two plate currents I_1 and I_2 are equal to one another. Since they flow in opposite directions from the ends of the winding to the center tap, they balance each other magnetically in the output inductance and produce no voltage across the ends. Consequently no current flows in the load resistance R_L.

Now suppose that a signal voltage is impressed such that the top grid is driven positive by an amount e_{s1} from its normal d.c. negative bias value of E_c, and that the bottom grid is driven more negative by an equal amount, i.e., $-e_{s1}$. The two plate currents will now vary in opposite directions, namely, I_1 will increase and I_2 will decrease. However, the sum of these two currents flows through the plate power supply, and owing to the infinite inductance of the center-tapped winding, $(I_1+I_2)/2$ flows down through the top half of the winding, and an equal amount flows up through the bottom half, to combine at the center tap to furnish the sum (I_1+I_2) flowing through the power supply.

Since $(I_1+I_2)/2$ is the average between I_1 and I_2, it is equal to neither, and from the principle of continuity of current flow, the difference

$$I_1 - \tfrac{1}{2}(I_1 + I_2) = \tfrac{1}{2}(I_1 + I_2) - I_2 = \tfrac{1}{2}(I_1 - I_2) \tag{3}$$

must flow through R_L. A quick check will indicate that Kirchhoff's current law is satisfied at each junction.

The current $(I_1-I_2)/2$ is the output current. In flowing through R_L, it sets up a voltage drop

$$E_L = \tfrac{1}{2}(I_1 - I_2)R_L. \tag{4}$$

Half of this or $E_L/2$ appears across each half of the output winding of such polarity that the instantaneous plate voltage of the top tube is $E_{bb}-(E_L/2)$ and that of the bottom tube is $E_{bb}+(E_L/2)$.

Thus the following facts have been brought to light:

1) The grid voltages change by equal but opposite increments from their common bias value E_c owing to the center tap on the input transformer secondary.

2) The plate voltages change by equal but opposite increments from their common supply value E_{bb} owing to the center tap on the output inductance. Moreover, the plate voltage increments are opposite in sign to the corresponding grid voltage increments.

3) The plate currents change in opposite directions in the same sense as the corresponding grid voltages, but not necessarily to an equal degree. If the tubes are nonlinear, as is usually the case, then the increase in plate current of either tube for a positive increment in grid voltage is not necessarily the same as the decrease in plate current for an equal negative increment in grid voltage.

From the above facts several graphical constructions are available to determine the plate current and plate voltage variations in the tubes, the output current and voltage, the power output, and the d.c. power input. The following graphical method is preferred by the author. In Fig. 8 is shown the plate family of curves for either tube. If there is no signal input, the only voltages present are the d.c. potentials E_{bb} applied to the two plates and E_c applied to the two grids. The current through either tube is then $I_b = E_{bb}B$, a direct current.

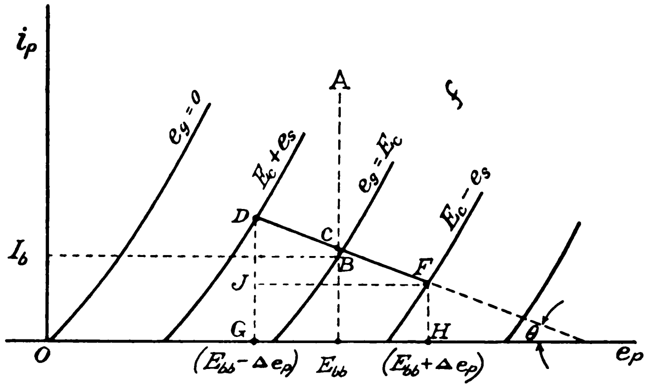


FIG. 8.

Now suppose that equal and opposite signal voltages e_s and $-e_s$ are applied to the grids in addition to E_c. Then the current in the one tube will increase from BE_{bb} to DG, and that in the other tube will drop to FH, as shown. The plate voltage of the first tube will drop from OE_{bb} to $OG = (E_{bb} - \Delta e_p)$, and that in the other tube will rise *by an equal amount* to $OH = (E_{bb} + \Delta e_p)$.

It is also clear from Fig. 8 that DJ represents the difference between the two currents or $(I_1 - I_2)$, and JF represents $2\Delta e_p$, the voltage across the output inductance and previously denoted by E_L in Fig. 7. From Eq. (4), it is evident that

$$JF/DJ = E_L/(I_1 - I_2) = R_L/2. \tag{5}$$

Thus DF makes the angle θ with the e_p axis such that

$$\cot \theta = R_L/2. \tag{6}$$

It is also evident from the geometry of the figure that DC = CF, i.e., that the ordinate through E_{bb} bisects line DF in C.

The above facts suggest the following method of graphical construction. We hold a rule at the angle θ and slide it up or down until the segment between the desired $e_p - i_p$ curves (corresponding to equal and opposite grid voltage excursions from the bias value E_c) is bisected by the ordinate through E_{bb}. The intersections of the rule with the two $e_p - i_p$ curves gives the two instantaneous values of the two tube currents I_1 and I_2, corresponding to the signal voltages e_s and $-e_s$ and to the plate load resistance R_L, or rather to $R_L/2$.

Then another pair of equal and opposite grid signal voltages are chosen, and the process repeated. This is continued until as many pairs of instantaneous grid signal voltages have been used as is desired. For a symmetrical signal voltage, such as a sine wave, instantaneous values for only one-quarter of a cycle are required.

When the above graphical construction is performed, there is obtained a curve on the plate family of curves such as that shown in broken lines ABCDE in Fig. 9. This represents the locus of the current for *either* tube over a cycle of grid signal volt age. It also represents the terminal characteristic for R_L *as it appears to either tube in the presence of the other tube.*

The significance of the last statement is as follows: the two tubes may be regarded as two generators connected to a common load R_L. Owing to their nonlinear characteristics, the tubes do not share the load equally throughout the signal cycle; that tube whose apparent internal resistance is lower takes a greater share of the load, i.e., furnishes more than half of the load current $(I_1-I_2)/2$ flowing through R_L. As a result, R_L appears as a variable or nonlinear resistance to either tube even though it is actually a linear resistance, and its terminal characteristic on either tube's e_p-i_p family of curves is in itself a curved rather than a straight line.

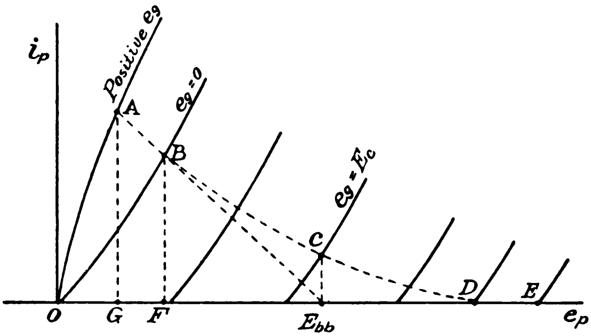


FIG. 9.

Lack of space precludes a detailed discussion of this interesting circuit. However, several important features will be presented. As indicated in Fig. 9, the two e_p-i_p curves passing through B and D, respectively, represent equal and opposite grid swings. The corresponding currents I_1 and I_2 for the two tubes are BF and zero; in short, the tube experiencing the negative grid swing has just reached plate current cutoff.

For e_p-i_p curves passing through A and E, corresponding to a still greater grid swing for either tube, I_1 is AG, and I_2 still remains zero. This means that the second tube is inoperative over this part of the cycle and acts therefore as if it were disconnected. Under these conditions R_L appears to the operative tube as $R_L/4$, which can be expected since the 2 to 1 turns ratio of the output inductance will produce this 4 to 1 impedance transformation if it is unhampered by the other tube.

Portion BA is therefore a straight line whose reciprocal slope corresponds to $R_L/4$. It is easy to show that if it were prolonged, it would pass through E_{bb}. Normally the tubes are operated so that maximum grid signal voltage drives each tube alternately to cutoff or beyond. Maximum output occurs if $R_L/4$ equals either tube's apparent internal plate resistance at the peak of the cycle. The plate resistance of either tube is given by the reciprocal slope of the e_p-i_p curve at point A. Hence a quick determination for the optimum value of R_L, or rather $R_L/4$, is to draw a line through E_{bb} at an angle equal to that of the e_p-i_p curve at point A, and calculate from the reciprocal slope of this line the value of $R_L/4$ and hence of R_L. The complete characteristic can then be determined by means of the sliding rule as described previously.

Fig. 8 also reveals an interesting point. CE_{bb} is the average between DG and FH, i.e., it represents $(I_1+I_2)/2$. This is the mid-branch current that flows through the plate supply, as indicated in Fig. 7(B). For various pairs of values of I_1 and I_2 as determined by the sliding rule, the average, or $(I_1+I_2)/2$ moves up and down along $E_{bb}A$. This is a vertical line or ordinate, and indicates that the resistance to the mid-branch current is zero. This has been tacitly assumed; the output inductance and the plate supply have been assumed to be free of resistance.

If this is not the case, then a line must be drawn through E_{bb} whose reciprocal slope indicates one-half the value of the mid-branch resistance that is present, and the sliding rule must be bisected by this line rather than the ordinate $E_{bb}A$, as is the case in Fig. 8. From this follows several further interesting characteristics.[1]

Another point is that not only is the locus of the mid-branch current along the ordinate $E_{bb}A$ in Fig. 8, but that this current executes two alternations per cycle of the grid signal voltage. This means that the mid-branch current is at least double the frequency of the incoming signal; actually, for perfect symmetry, all the even harmonics generated by the tubes flow in parallel through the mid-branch portions of the circuit, while the odd harmonics, including of course the fundamental, flow through the output resistance R_L. Thus, if the tube characteristics are such that the second harmonic is quite prominent, but the third (and higher) harmonics are of small amplitude, then the output wave will be a fairly faithful copy of the input grid signal voltage and the stage will exhibit little distortion. Such a tube characteristic is possessed, for example, by the 6L6 and 807 beam power tubes.

As in the case of the previous constructions for the single-ended tube, various degrees of complication can arise. For example, if the grids are driven positive so that grid current flows, the signal voltage at the grids will be distorted, and this distortion must be determined separately before the above construction can be concluded. Another case is that where the mid-branch plate supply has an internal resistance that is adequately by-passed for the even harmonics, all except the d.c. component. This represents a particularly difficult problem that can be solved only by a series of approximations.

6. Reactive circuits. The previous circuits contained only resistances, linear or nonlinear. If reactances were present, such as the center-tapped output inductance, they were assumed infinite in value and so situated in the circuit as not to have any appreciable a.c. components flowing in them. However, many nonlinear circuits contain reactances of finite value that influence the behavior of the circuit directly, and hence must be taken directly into account.

Owing to lack of space, only the case of an inductance in series with a nonlinear resistance and an a.c. source will be discussed here. Consider the circuit shown in Fig. 10. Here a source of a.c. voltage e is in series with a nonlinear resistance r and inductance L. The voltage e is a known function of time, and the terminal characteristic for r and the value of L is given. It is desired to find the current flow in this circuit.

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[1] See, for example, A. Preisman, *Graphical constructions for vacuum tube circuits*, McGraw-Hill Publishing Co., New York, 1943.

We have the fundamental relation

$$e(t) = ir + L\frac{di}{dt}\cdot \tag{7}$$

Expressing Eq. (7) in terms of finite increments, we obtain

$$e_1 + \Delta e_1 = (i_1 + \Delta i_1)r + L\frac{\Delta i_1}{\Delta t}\cdot \tag{8}$$

In Eq. (8), it is assumed that at the start, the voltage e has a certain value e_1, the current i has a certain value i_1, and $t=0$. These are the initial conditions. During a

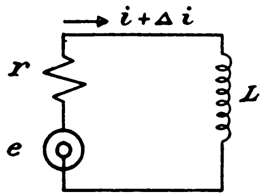

FIG. 10.

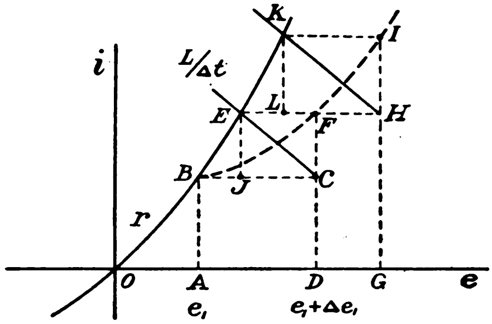

FIG. 11.

small time interval Δt, e_1 is assumed to change instantly to $e_1+\Delta e_1$ and remain at this value during the interval Δt, and similarly i_1 is assumed to change instantly by an amount Δi_1 and to remain at the value $i_1+\Delta i_1$ during the time Δt. This is of course an approximation, sufficiently close if Δt is taken sufficiently small. Under these conditions Eq. (8) holds.

The quantity $L/\Delta t$ has a finite value if Δt is finite. It can represent the cotangent of some angle θ. Then—as far as Δi_1 is concerned—the circuit consists of two resistances in series: that of r at the value i_1, and that of $L/\Delta t$. The graphical construction then takes the form shown in Fig. 11, where OA represents the initial value e_1, and AB the initial current i_1. We now suppose that the voltage changes from e_1 to $e_1+\Delta e_1$ in a small chosen time interval Δt, and let OD represent $e_1+\Delta e_1$ so that AD represents Δe_1.

The voltage across L is due to the change of current Δi_1 and not due to i_1 itself, which has already been established in L. This is indicated by the fact that OA$=e_1$ represents the drop across the nonlinear resistance r; there is no voltage drop across L for i_1 at the time $t=0$. Hence, in view of the above, a point C is located in line with B and directly over D, and through C line EC is drawn to represent $L/\Delta t$ such that

$$\cot \measuredangle ECB = L/\Delta t. \tag{9}$$

The line EC has been designated by the author as a finite operator because it resembles the Heaviside operator Lp. The intersection of this finite operator with the terminal characteristic of r in E gives the value of Δi_1, namely, EJ. Here BJ represents the additional voltage drop across r (in addition to the original voltage drop OA owing to i_1), and JC represents the voltage drop across L. In short, OA+BJ represents $(i_1+\Delta i_1)r$; JC represents $L(\Delta i_1/\Delta t)$; and OA+BC therefore represents $e_1+\Delta e_1$, and hence satisfies Eq. (8).

The point E is projected over to F directly above C and D, and FD represents then the new value of current $i_1+\Delta i_1$, at the end of the time interval Δt. Another small time interval can now be chosen, preferably equal to the previous one, so that $L/\Delta t$ remains at the same angle to the e-axis as before. We suppose that in this new time interval, e changes from $e_1+\Delta e_1$ to $e_1+\Delta e_1+\Delta e_2$. Letting OG represent the new value of voltage, we project F over to H directly above G. Through H we draw HK parallel to CE, intersecting the terminal characteristic for r in K. Then KL represents the new increment of current Δi_2, EL the additional voltage drop across r, and LH the new voltage drop across L. It is evident that Eq. (8) is once again satisfied. It is also evident that IG represents $(i_1+\Delta i_1+\Delta i_2)$, the new value of current at the end of the second time interval.

Points B, F, and I represent three points on the *overall terminal characteristic* for L and r in series for the given function $e(t)$. If $e(t)$ is a periodic voltage, the overall terminal characteristic will spiral around counter-clockwise and ultimately form a closed curve, the steady-state solution for the given circuit and given function $e(t)$. The initial open branches of this spiral represent the transient solution. If r is a linear resistance so that its terminal characteristic is a straight line instead of the curve shown in Fig. 11, the closed loop will be an ellipse inclined to both axes; if on the other hand r is nonlinear, the closed loop will be some form of distorted ellipse depending upon the nonlinearity of r. It can be shown from the graphical construction that the tangents to the closed loop at the points where it intersects the terminal characteristic for r are parallel to the e axis and hence perpendicular to the i axis.

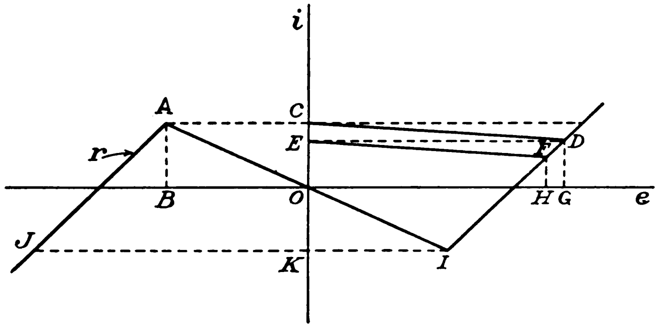

Fig. 12.

7. Relaxation oscillator. Similar methods can be developed for r in series with a condenser C, and for LCr circuits, and for parallel as well as series arrangements. Owing to lack of space these will not be treated here.[2] An interesting case is that of a nonlinear resistance having a suitable negative branch, in series with a pure inductance. For graphical purposes the simplest form for the terminal characteristic of r is possibly that of three intersecting straight lines, as shown in Fig. 12. Such a characteristic may be approximated by a tube having positive feedback, by a dynatron, etc. Usually a d.c. polarizing voltage is required, but this merely represents a translation of the axes and does not materially change the construction or results as obtained in Fig. 12, in which the impressed voltage is assumed to be zero.

[2] Cf. Preisman, loc. cit., p. 109.

Suppose that the initial conditions at $t=0$ are that $e=0$, and that $i=\text{AB}$, the peak current for the left-hand portion of r. Then C will be the starting point, where CO = AB. Through C the finite operator $L/\Delta t$ is drawn corresponding to a time interval Δt. If Δt is sufficiently small, $L/\Delta t$ will be practically a horizontal line through C. In Fig. 12, $L/\Delta t$ has been drawn with a finite tilt to clarify the construction, and is represented by CD. This finite operator curve intersects the terminal characteristic for r in D, as shown.

The current therefore decreases from CO to DG. Point D is projected over to the i axis as point E. From E, EF is drawn parallel to CD under the assumption that the second time interval is equal to the first. The current now decreases from DG to FH. Point F can now be projected over to the i axis and the process repeated. It is clear from the figure that the intersections will proceed down the right-hand branch of r to I, hop over from I to J, directly opposite I, then proceed from J up to A, hop over to D, and repeat the first set of intersections. As Δt approaches zero, the finite operator curve approaches a horizontal position, $\text{DG} \cong \text{CO} = \text{AB}$, and the points of intersection become more and more closely spaced so that they form essentially all the points of ID and JA.

The overall terminal characteristic is by definition all the points between C and K in that the overall impressed voltage has been assumed zero, so that the points must lie along the i axis, and the current range is from C to K. However, a more significant terminal characteristic in this case is the relationship between the current and the voltage across either circuit element. The voltage across the inductance, for example, is equal and opposite to that across r when taken in a circuital direction, since the algebraic sum of the two must equal the impressed voltage, which is zero.

According to this definition, the terminal characteristic is represented by such points as D, F, etc.; in this case, it is lines DI, IJ, JA, and AD, traversed in the order given. This means that for the circuit given, the terminal characteristic is very simply given by a quadrilateral involving the two positive resistance portions of the terminal characteristic for r contained between their peak values A and I.

The time required to traverse these portions depends upon the relaxation time for L in series with the incremental resistance of r for each portion, under the proper initial conditions. The time required to traverse the horizontal portions AD and IJ is infinitesimal, and is independent of the shape of the negative resistance portion AI provided it has no maxima or minima exceeding or less than A and I, respectively. The device operates continuously as an oscillator with a period of oscillation determined by the two relaxation times.

Similar conclusions can be drawn for shapes of r other than three straight lines. For example, r can have the form of a cubic parabola. This case has been treated analytically by Van der Pol.[3] However, he started with an LCr parallel circuit or double-energy condition. For such a circuit the terminal characteristic is a closed curve or loop that exceeds the above quadrilateral in size. As C approaches zero, the loop shrinks and appears to have as its limit the above quadrilateral. However, the analytical method required that some capacity be present even in this limit, relaxation case, and it has been suggested that in a practical circuit there would always be some residual stray capacitance present.

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[3] B. Van der Pol, *The nonlinear theory of electric oscillations*, Proc. I.R.E., **22**, 1051–1086 (1934).

There are other graphical methods for handling the double-energy case, notably that by Liénard[4] and another by Kirschstein.[5] Unfortunately, these constructions become indeterminate in nature as C approaches zero, so that although the relaxation condition is suggested by them, it cannot be conclusively shown to be the limit form.

The construction given here starts out with merely L and r, and requires no C for its argument. It appears to give the limit case directly and presents no indeterminate considerations. It has seemed to the author that the necessity for requiring a capacity to be present, no matter how small, was an unnecessary restriction, and that the argument advanced that any practical circuit would have some capacity, appeared to be rather irrelevant, since the notion of a circuit is in itself an idealization of what is really a field problem. In treating an electrical problem as a circuit problem one assumes that the circuit elements are ideal inductances or capacitances or resistances and develops the various theorems on this basis.

Similar results can be obtained for a capacitance in series with a nonlinear resistance having an S-shaped terminal characteristic provided that it is turned through a right angle from that shown in Fig. 12, i.e., provided that it is a single-valued function of the current rather than of the voltage. A familiar example is the neon tube relaxation oscillator employed to generate a saw-tooth voltage. It is also possible to develop a graphical construction employing the finite operator method for an LCr circuit, and in this case L or C may be permitted to approach zero, depending upon the position of the S-shaped characteristic for r, without the construction becoming indeterminate. For example, the construction reduces to the form given in connection with Fig. 12 if C is made to approach zero and r has the terminal characteristic shown in the figure.

8. Conclusions. This concludes the discussion on some graphical methods for solving nonlinear electrical circuits. Simple series circuits involving resistance elements only, are very simply solved by finding the intersections of their terminal characteristics. This can then be extended to more complicated resistances in which the current is a function of two voltages, as in the case of a triode tube.

The next circuit considered is that of the ideal balanced amplifier having perfectly matched tubes and feeding the load resistance through an ideal transformer. Here the coupling of the two tubes through this ideal transformer requires a special construction involving the sliding of a rule at a fixed angle along the tube characteristics. The wave shape of the output and of the mid-branch currents is then discussed, and it is shown that owing to the symmetry of the circuit the former can contain only odd harmonics; and the latter, even harmonics.

Finally, a simple case of a reactive circuit involving a nonlinear resistance in series with an inductance is treated. Here the concept of a finite operator curve corresponding to $L/\Delta t$ is developed and this curve is employed to solve the circuit. Similar methods are available for capacitive circuits and for double-energy circuits involving both L and C. The method is applied to a suitable negative resistance in series with an inductance, and it is shown in a direct manner that this circuit can produce relaxation oscillations.

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[4] A. Liénard, *Étude des oscillations entretenues*, Rev. Gen. Élec. **23**, 901–946 (1928). See also P. LeCorbeiller, *The non-linear theory of the maintenance of oscillations*, Journal IEE (London) **79**, 361–378 (1936).

[5] F. Kirschstein, *Über ein Verfahren zur graphischen Behandlung elektrischer Schwingungsvorgänge*, Arch. Elek. **24**, 731 (1930).