flow, respectively. Hence $\Gamma_0=\int_C v_{0t}dt$ where $\mathbf{v}_0$ is the undisturbed velocity field given by (1) and the subscript t indicates the tangential component. By Stokes' theorem, $\Gamma_0=\iint_I(\text{curl }\mathbf{v}_0)_z dA=-UkA$ where A is the area of I. Hence in our example, $\Gamma_0=-Uk\pi a^2$. If we substitute this for Γ in (9), assuming, as Tsien does,[6] that $\Gamma_1=0$, then our result (9) reduces to (10).

[6] The author is indebted to Dr. Tsien for pointing this out. He had at first mistakenly supposed that Tsien's result was based on the assumption $\Gamma=0$.

ON PLASTIC BODIES WITH ROTATIONAL SYMMETRY*

By C. H. W. SEDGEWICK (*University of Connecticut*)

Introduction. The rotational symmetry problem in plasticity was discussed by H. Hencky[1] in 1923. In the present paper some new results are obtained. Furthermore, the presentation is different from that used by Hencky.

In the following discussion, r and z in the cylindrical coordinate system (r, θ, z) will be replaced by $\alpha(r, z)$ and $\beta(r, z)$ in such a way that α, β, θ form a curvilinear, orthogonal system. The line element ds will be written in the form

$$ds^2 = A^2d\alpha^2 + B^2d\beta^2 + r^2d\theta^2,$$

where A and B are functions of α and β. Furthermore, if the angle between the curve $\beta=$const. and the direction of increasing r is denoted by γ, we will have

$$\frac{\partial r}{\partial \alpha} = A\cos\gamma, \qquad \frac{\partial r}{\partial \beta} = -B\sin\gamma, \tag{1}$$

$$\frac{\partial z}{\partial \alpha} = A\sin\gamma, \qquad \frac{\partial z}{\partial \beta} = B\cos\gamma. \tag{2}$$

From these, we get

$$\frac{\partial A}{\partial \beta} = -B\frac{\partial \gamma}{\partial \alpha}, \tag{3}$$

$$\frac{\partial B}{\partial \alpha} = A\frac{\partial \gamma}{\partial \beta}. \tag{4}$$

The stress components will be designated by $\sigma_{\alpha\alpha}, \sigma_{\beta\beta}, \sigma_{\theta\theta}, \sigma_{\alpha\beta}, \sigma_{\alpha\theta}, \sigma_{\beta\theta}$. In the problem under discussion, $\sigma_{\alpha\theta}=\sigma_{\beta\theta}=0$.

1. Lines of principal stress. Along the lines of principal stress, $\sigma_{\alpha\beta}=0$. In this case the equations of equilibrium[2] reduce to

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[1] H. Hencky, *Über einige statisch bestimmte Fälle des Gleichgewichts in plastischen Körpern*, Zeitschr. für angew. Math. u. Mech. **3**, 241 (1923).

[2] A. E. H. Love, *The mathematical theory of elasticity*, 4th edition, Cambridge University Press, 1934, p. 90.

$$\frac{1}{ABr}\left[\frac{\partial}{\partial\alpha}(Br\sigma_{\alpha\alpha})\right]-\frac{\sigma_{\beta\beta}}{A}\frac{\partial}{\partial\alpha}(\ln B)-\frac{\sigma_{\theta\theta}}{A}\frac{\partial}{\partial\alpha}(\ln r)=0,$$

$$\frac{1}{ABr}\left[\frac{\partial}{\partial\beta}(Ar\sigma_{\beta\beta})\right]-\frac{\sigma_{\theta\theta}}{B}\frac{\partial}{\partial\beta}(\ln r)-\frac{\sigma_{\alpha\alpha}}{B}\frac{\partial}{\partial\beta}(\ln A)=0.$$

On simplifying, these become

$$\frac{\partial\sigma_{\alpha\alpha}}{\partial\alpha}+(\sigma_{\alpha\alpha}-\sigma_{\beta\beta})\frac{\partial}{\partial\alpha}(\ln B)+(\sigma_{\alpha\alpha}-\sigma_{\theta\theta})\frac{\partial}{\partial\alpha}(\ln r)=0, \tag{5}$$

$$\frac{\partial\sigma_{\beta\beta}}{\partial\beta}-(\sigma_{\alpha\alpha}-\sigma_{\beta\beta})\frac{\partial}{\partial\beta}(\ln A)+(\sigma_{\beta\beta}-\sigma_{\theta\theta})\frac{\partial}{\partial\beta}(\ln r)=0. \tag{6}$$

Assuming the Tresca yield condition, we have $\sigma_{\alpha\alpha}-\sigma_{\beta\beta}=2k$, where k is constant. Furthermore, in the so called "fully plastic state," $\sigma_{\theta\theta}$ must be equal to either $\sigma_{\alpha\alpha}$ or $\sigma_{\beta\beta}$. Let us assume first that $\sigma_{\theta\theta}=\sigma_{\alpha\alpha}$. Writing $\sigma_{\alpha\alpha}+\sigma_{\beta\beta}+\sigma_{\theta\theta}=3\sigma$, we have $\sigma_{\alpha\alpha}=\sigma_{\theta\theta}=\sigma+2k/3$, $\sigma_{\beta\beta}=\sigma-4k/3$. From (5) and (6), we then get

$$\frac{\partial\sigma}{\partial\alpha}+2k\frac{\partial}{\partial\alpha}(\ln B)=0, \tag{7}$$

$$\frac{\partial\sigma}{\partial\beta}-2k\frac{\partial}{\partial\beta}[\ln(Ar)]=0. \tag{8}$$

Elimination of σ furnishes

$$\frac{\partial^2}{\partial\alpha\partial\beta}[\ln(ABr)]=0. \tag{9}$$

If above we had assumed that $\sigma_{\theta\theta}=\sigma_{\beta\beta}$, we would have obtained

$$\frac{\partial\sigma}{\partial\alpha}+2k\frac{\partial}{\partial\alpha}[\ln(Br)]=0, \tag{7*}$$

$$\frac{\partial\sigma}{\partial\beta}-2k\frac{\partial}{\partial\beta}(\ln A)=0. \tag{8*}$$

These also lead to Eq. (9), the solution of which is

$$ABr=e^{f(\alpha)}e^{g(\beta)}. \tag{10}$$

Let us define α' and β' by[3]

$$d\alpha'=e^{f(\alpha)}d\alpha,\qquad d\beta'=e^{g(\beta)}d\beta. \tag{11}$$

This transformation merely relabels the families of surfaces $\alpha=$const. and $\beta=$const.

Now, the volume bounded by the surfaces α, $\alpha+d\alpha$, β, $\beta+d\beta$, θ, $\theta+d\theta$ is equal to $ABrd\alpha d\beta d\theta=A'B'rd\alpha'd\beta'd\theta$.

Substituting for $d\alpha'$ and $d\beta'$ from (11) and making use of (10), we get

$$A'B'r=1. \tag{12}$$

Thus, the volume contained between the co-ordinate surfaces α_1', α_2'; β_1', β_2'; θ_1, θ_2 is given by

$$\int_{\alpha_1'}^{\alpha_2'}\int_{\beta_1'}^{\beta_2'}\int_{\theta_1}^{\theta_2}d\alpha'd\beta'd\theta=(\alpha_2'-\alpha_1')(\beta_2'-\beta_1')(\theta_2-\theta_1).$$

[3] W. Prager, *Theory of plasticity*, mimeographed lecture notes, Brown University, R. I., 1942.

It follows that if the differences $\alpha_2' - \alpha_1'$, $\beta_2' - \beta_1'$, $\theta_2 - \theta_1$ are kept constant for successive co-ordinate surfaces, the resulting volumes will be equal. This result is analogous to that obtained by Boussinesq[4] in the plane problem.

Dropping the primes for the sake of simplicity we may construct a solution by setting $\gamma = g(\beta)$.

From (3), A is then seen to be a function of α alone. We set $A = \phi'(\alpha)$, and obtain from (4) $B = \phi g' + h(\beta)$. The first equation (1) leads to

$$r = \phi \cos g + l(\beta), \qquad \frac{\partial r}{\partial \beta} = -g'\phi(\sin g) + l'.$$

But, according to the second equation (1),

$$\frac{\partial r}{\partial \beta} = -(\phi g' + h) \sin g.$$

Hence $h = -l'/\sin g$. The condition (12) now takes the form

$$\phi'\left(\phi g' - \frac{l'}{\sin g}\right)(\phi \cos g + l) = 1.$$

This can be satisfied by setting $l = 0$, $g' \cos g = c$, $\phi^2\phi' = 1/c$, where c is a constant. Discarding constants of integration we thus obtain

$$\sin g = c\beta, \qquad \phi^3 = \frac{3\alpha}{c},$$

$$\gamma = \sin^{-1}(c\beta), \qquad A = 3^{-2/3}c^{-1/3}\alpha^{-2/3},$$

$$B = 3^{1/3}c^{2/3}\alpha^{1/3}[1 - c^2\beta^2]^{-1/2}, \qquad r = 3^{1/3}c^{-1/3}\alpha^{1/3}[1 - c^2\beta^2]^{1/2}.$$

Equations (2) now give $z = 3^{1/3}c^{2/3}\alpha^{1/3}\beta$. Hence

$$z/r = c\beta[1 - c^2\beta^2]^{-1/2}, \qquad r^2 + z^2 = 3^{2/3}c^{-2/3}\alpha^{2/3}.$$

The curves $\alpha = \text{const.}$ and $\beta = \text{const.}$ are thus seen to be concentric circles around $r = z = 0$, and radial straight lines, respectively.

In the above example, it may easily be verified that, corresponding to a set of equidistant values of α, β and θ, the resulting volumes will be equal.

By substituting the value for B above in (7) and integrating, an expression for σ is obtained.

2. Lines of maximum shearing stress. Along the lines of maximum shearing stress, $\sigma_{\alpha\beta} = k$ and $\sigma_{\alpha\alpha} = \sigma_{\beta\beta} = \sigma$. $\sigma_{\theta\theta}$ will be equal to either $\sigma + k$ or $\sigma - k$. Let us assume first that $\sigma_{\theta\theta} = \sigma + k$. In this case, the equations of equilibrium (2) are

$$\frac{1}{ABr}\left[\frac{\partial}{\partial \alpha}(Br\sigma) + \frac{k}{A}\frac{\partial}{\partial \beta}(A^2 r)\right] - \frac{\sigma}{A}\frac{\partial}{\partial \alpha}(\ln B) - \frac{(\sigma + k)}{A}\frac{\partial}{\partial \alpha}(\ln r) = 0,$$

$$\frac{1}{ABr}\left[\frac{\partial}{\partial \beta}(Ar\sigma) + \frac{k}{B}\frac{\partial}{\partial \alpha}(B^2 r)\right] - \frac{1}{B}(\sigma + k)\frac{\partial}{\partial \beta}(\ln r) - \frac{\sigma}{B}\frac{\partial}{\partial \beta}(\ln A) = 0.$$

These reduce to

[4] J. Boussinesq, *Lois géométrique de la distribution des pressions, dans un solide homogène et ductile soumis à des déformations planes.* Comptes Rendus Ac. Sci. Paris **74**, 242 (1872).

$$\frac{\partial\sigma}{\partial\alpha}+\frac{k}{ABr}\left[A^2\frac{\partial r}{\partial\beta}+2Ar\frac{\partial A}{\partial\beta}\right]-k\frac{\partial}{\partial\alpha}(\ln r)=0,$$

$$\frac{\partial\sigma}{\partial\beta}+\frac{k}{ABr}\left[B^2\frac{\partial r}{\partial\alpha}+2Br\frac{\partial B}{\partial\alpha}\right]-k\frac{\partial}{\partial\beta}(\ln r)=0.$$

Making use of Eqs. (1), we obtain

$$\frac{\partial\sigma}{\partial\alpha}+k\left[-\frac{A}{r}\sin\gamma-2\frac{\partial\gamma}{\partial\alpha}\right]-k\frac{\partial}{\partial\alpha}(\ln r)=0, \tag{13}$$

$$\frac{\partial\sigma}{\partial\beta}+k\left[\frac{B}{r}\cos\gamma+2\frac{\partial\gamma}{\partial\beta}\right]-k\frac{\partial}{\partial\beta}(\ln r)=0. \tag{14}$$

Eliminating σ, we find

$$\frac{\partial}{\partial\beta}\left[\frac{-A\sin\gamma}{r}-2\frac{\partial\gamma}{\partial\alpha}\right]=\frac{\partial}{\partial\alpha}\left[\frac{B\cos\gamma}{r}+2\frac{\partial\gamma}{\partial\beta}\right].$$

Carrying out the differentiations and substituting for $\partial A/\partial\beta$, $\partial B/\partial\alpha$, $\partial r/\partial\alpha$, $\partial r/\partial\beta$ from Eqs. (1), (3) and (4), we obtain

$$\frac{\partial^2\gamma}{\partial\alpha\partial\beta}+\frac{1}{2r}\left[A\cos\gamma\frac{\partial\gamma}{\partial\beta}-B\sin\gamma\frac{\partial\gamma}{\partial\alpha}\right]-\frac{AB}{4r^2}\cos 2\gamma=0. \tag{15}$$

We may remark that as $r\to\infty$, Eq. (15) reduces to that governing the case for plane strain, i.e., $\partial^2\gamma/\partial\alpha\partial\beta=0$.

It is easily seen that the only solution of equation (15) having two orthogonal families of straight lines occurs when $\gamma=45°$, i.e., when the two families of straight lines are inclined at an angle of 45° to the axis of symmetry. This result was obtained by Hencky.[1]

If we had assumed above that $\sigma_{\theta\theta}=\sigma-k$, our equilibrium equations would reduce to

$$\frac{\partial\sigma}{\partial\alpha}+k\left[\frac{-A}{r}\sin\gamma-2\frac{\partial\gamma}{\partial\alpha}\right]+k\frac{\partial}{\partial\alpha}(\ln r)=0, \tag{13*}$$

$$\frac{\partial\sigma}{\partial\beta}+k\left[\frac{B}{r}\cos\gamma+2\frac{\partial\gamma}{\partial\beta}\right]-k\frac{\partial}{\partial\beta}(\ln r)=0, \tag{14*}$$

which also lead to Eq. (15).

Let us assume a solution of Eq. (15) in the form $\gamma=f(\alpha)+g(\beta)$. The equation then becomes

$$2r[\{A\cos(f+g)\}g'-\{B\sin(f+g)\}f']-AB\cos 2(f+g)=0.$$

Substituting for A and B from relations (1), we get

$$\frac{\partial r}{\partial\alpha}\frac{\partial}{\partial\beta}\left[\frac{\cos 2(f+g)}{r}\right]+\frac{\partial r}{\partial\beta}\frac{\partial}{\partial\alpha}\left[\frac{\cos 2(f+g)}{r}\right]=0. \tag{16}$$

A solution of (16) is given by $r=C\cos 2(f+g)$ where C is a constant. Without loss in generality we may set $f(\alpha)+g(\beta)=\alpha-\beta$. We then have

$$r = C \cos 2(\alpha - \beta),$$

$$\frac{\partial r}{\partial \alpha} = -2C \sin 2(\alpha - \beta) = -4C \sin (\alpha - \beta) \cos (\alpha - \beta).$$

Equations (1) now furnish

$$A = -4C \sin (\alpha - \beta), \qquad B = -4C \cos (\alpha - \beta),$$

and Eqs. (2) give

$$z = -2C(\alpha + \beta) + C \sin 2(\alpha - \beta).$$

It follows that the curves $\alpha=$const. and $\beta=$const. are cycloids tangent to the lines $r=C$ and $r=-C$, respectively.

From Eqs. (14) and (15), we are able to determine σ. Substituting our values for γ, A, B, r and integrating, we find that

$$\sigma = 4k(\alpha + \beta) + k \ln [1 - \sin 2(\alpha - \beta)] + \text{const.}$$

Another solution is obtained by setting $f(\alpha)+g(\beta)=\alpha-\beta$, as before, and substituting $r=e^{\alpha+\beta}\phi$ where $\phi=\phi(\alpha-\beta)$ is a function yet to be determined. After making these substitutions and carrying out the differentiations, we get

$$[\phi^2 - \phi'^2] \cos 2(\alpha - \beta) - 2\phi\phi' \sin 2(\alpha - \beta) = 0$$

which is satisfied by

$$\phi = C[\cos (\alpha - \beta) + \sin (\alpha - \beta)].$$

We thus have

$$r = Ce^{\alpha+\beta}[\cos (\alpha - \beta) + \sin (\alpha - \beta)].$$

Using relations (1) and (2), we find that

$$z = Ce^{\alpha+\beta}[\sin (\alpha - \beta) - \cos (\alpha - \beta)].$$

The curves $\alpha=$const. and $\beta=$const. are logarithmic spirals which intersect the straight lines through the origin at an angle of $\pi/4$. This solution corresponds to the solution obtained in **1**.

It is interesting to see that these networks of cycloids or logarithmic spirals, known in the case of plane strain, are also admissible in the case of rotational symmetry.

ON THE TREATMENT OF DISCONTINUITIES IN BEAM DEFLECTION PROBLEMS*

By S. TIMOSHENKO (*Stanford University*)

In a note on the treatment of discontinuities in beam deflection problems Mr. E. Kosko[1] attributes to R. Macaulay the method whereby the number of constants of integration can be always reduced to two, independently of the number of forces. This method was, however, originated by A. Clebsch, and is discussed in his book "Theorie der Elasticität Fester Körper," 1862, page 389. In Russia it was called the Clebsch method and was widely used in textbooks on strength of materials. It was also used in German books. See, for example, A. Föppl, Festigkeitslehre, 5th ed. 1914, page 124.

* Received Jan. 14, 1945.

[1] Quarterly of Appl. Math. **2**, 271–272 (1944).