## -NOTES-

## THE PRESSURE DISTRIBUTION ON A BODY IN SHEAR FLOW*

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Problems involving shear flow have been studied recently by Tsien ${ }^{2}$ and Kuo. ${ }^{3}$ The purpose of the present note is to point out that the pressure distribution on an infinite cylindrical body immersed in a two-dimensional shear flow can be obtained by means of integral equations, at least for a sufficiently smooth contour. A direct attack on the boundary value problem for the stream function is avoided. The method used is essentially that employed by Prager ${ }^{4}$ in the case of potential flow.

If the undisturbed shear flow is given by the velocity field

$$
\begin{equation*}
v_{x}=U(1+k y), \quad v_{y}=0 \tag{1}
\end{equation*}
$$

where $U$ and $k$ are constants, then it has constant vorticity equal to $-k U$. The continuity equation implies the existence of a stream function $\psi(x, y)$ which satisfies the Poisson equation

$$
\nabla^{2} \psi=k U
$$

with the boundary conditions

$$
\frac{\partial \psi}{\partial x}=-v_{y}=0, \quad \frac{\partial \psi}{\partial y}=v_{x}=U(1+k y)
$$

at $\infty$, and $\psi=c$, a constant, on the contour $C$ of the cross section of the cylindrical body immersed in the usual position in the flow. Let us set $\psi=\psi_{0}+\psi_{1}$, where

$$
\psi_{0}=U\left(y+\frac{k}{2} y^{2}\right)
$$

is the undisturbed stream function and $\psi_{1}$ is the disturbance stream function. Then $\nabla^{2} \psi_{0}=k U$, and $\psi_{1}$ is harmonic in the region $E$ exterior to $C$, with the boundary condition

$$
\psi_{1 e}=c-\psi_{0 e}
$$

on $C$, where the parameter $s$ may be the arc length on $C$ measured from any convenient starting point.

[^0]By a well-known theorem of potential theory, we have

$$
\begin{equation*}
\psi_{1}(P)-\psi_{1}(\infty)=\frac{1}{2 \pi} \int_{c}\left(\psi_{1} \frac{\partial}{\partial n} \log \frac{1}{r}-\frac{\partial \psi_{1}}{\partial n} \log \frac{1}{r}\right) d s \tag{2}
\end{equation*}
$$

where $n$ is the exterior normal, $P$ is a point in $E$, and $r$ is the distance between $P$ and a variable point whose range will be clear from the context. We now apply Green's theorem

$$
\iint_{I}\left(u \nabla^{2} v-v \nabla^{2} u\right) d A=-\int_{c}\left(u \frac{\partial v}{\partial n^{\prime}}-v \frac{\partial u}{\partial n^{\prime}}\right) d s
$$

where $n^{\prime}$ is the interior normal and $I$ is the region interior to $C$, to the functions $u=-\psi_{0}$ and $v=\log (1 / r)$, obtaining.

$$
k U \iint_{I} \log \frac{1}{r} d A=\int_{C} \psi_{0} \frac{\partial}{\partial n^{\prime}} \log \frac{1}{r} d s-\int_{C} \frac{\partial \psi_{0}}{\partial n^{\prime}} \log \frac{1}{r} d s
$$

Using the fact that $\partial / \partial n^{\prime}=-\partial / \partial n$ and combining this with (2), we obtain
$\psi_{1}(P)-\psi_{1}(\infty)=\frac{1}{2 \pi} \int_{C} \psi \frac{\partial}{\partial n} \log \frac{1}{r} d s-\frac{1}{2 \pi} \int_{c} \frac{\partial \psi}{\partial n} \log \frac{1}{r} d s+\frac{k U}{2 \pi} \iint_{I} \log \frac{1}{r} d A$.
The first integral in (3) vanishes because $\psi=c$ on $C$, and because

$$
\int_{c} \frac{\partial}{\partial n} \log \frac{1}{r} d s
$$

is the angle subtended by $C$ at $P$, which is zero since $P$ is outside $C$. In the second integral of (3) we may write $-\partial \psi / \partial n=v(s)$ where $v(s)$ is the (tangential) velocity along $C$. Hence we have

$$
\begin{equation*}
\psi_{1}(P)-\psi_{1}(\infty)=\frac{1}{2 \pi} \int_{c} v(s) \log \frac{1}{r} d s+\frac{k U}{2 \pi} \iint_{I} \log \frac{1}{r} d A . \tag{4}
\end{equation*}
$$

Let us introduce

$$
V=\int_{c} v(s) \log \frac{1}{r} d s .
$$

Then there exist interior and exterior limits $\partial V_{i} / \partial n$ and $\partial V_{e} / \partial n$ such that

$$
\frac{1}{2}\left(\frac{\partial V_{i}}{\partial n}-\frac{\partial V_{e}}{\partial n}\right)=\pi v(s), \quad \frac{1}{2}\left(\frac{\partial V_{i}}{\partial n}+\frac{\partial V_{c}}{\partial n}\right)=\int_{c} v(t) \frac{\partial}{\partial n} \log \frac{1}{r} d t,
$$

so that

$$
\begin{equation*}
\frac{\partial V_{e}}{\partial n}=-\pi v(s)+\int_{c} v(t) \frac{\partial}{\partial n} \log \frac{1}{r} d t . \tag{5}
\end{equation*}
$$

From (4) and (5), we find that the normal derivative of $\psi_{1}$ at the exterior edge of $C$ is given by

$$
\begin{equation*}
\frac{\partial \psi_{1 e}}{\partial n_{s}}=-\frac{1}{2} v(s)+\frac{1}{2 \pi} \int_{C} v(t) \frac{\partial}{\partial n_{s}} \log \frac{1}{r} d t+\frac{k U}{2 \pi} \frac{\partial}{\partial n_{s}} \iint_{I} \log \frac{1}{r} d A, \tag{6}
\end{equation*}
$$

where the subscript $s$ indicates the point of $C$ at which the quantity is to be evaluated. But

$$
\begin{aligned}
v(s) & =-\frac{\partial \psi}{\partial n_{s}}=-\frac{\partial \psi_{0}}{\partial n_{s}}-\frac{\partial \psi_{1 e}}{\partial n_{s}} \\
& =-\frac{\partial \psi_{0}}{\partial n_{s}}+\frac{1}{2} v(s)-\frac{1}{2 \pi} \int_{C} v(t) \frac{\partial}{\partial n_{s}} \log \frac{1}{r} d t-\frac{k U}{2 \pi} \frac{\partial}{\partial n_{s}} \iint_{I} \log \frac{1}{r} d A .
\end{aligned}
$$

Therefore, the velocity distribution along $C, v(s)$, satisfies the integral equation

$$
\begin{equation*}
v(s)+\frac{1}{\pi} \int_{C} v(t) \frac{\partial}{\partial n_{s}} \log \frac{1}{r} d t=-2 \frac{\partial \psi_{0}}{\partial n_{s}}-\frac{k U}{\pi} \frac{\partial}{\partial n_{s}} \iint_{I} \log \frac{1}{r} d A, \tag{7}
\end{equation*}
$$

or, since the last integral may be differentiated under the integral sign,

$$
\begin{equation*}
v(s)+\frac{1}{\pi} \int_{C} v(t) \frac{\cos \left(r_{s}, n_{s}\right)}{r_{s t}} d t=-2 \frac{\partial \psi_{0}}{\partial n_{s}}-\frac{k U}{\pi} \iint_{I} \frac{\cos \left(r, n_{s}\right)}{r} d A, \tag{8}
\end{equation*}
$$

where $s$ and $t$ are points of $C,\left(r_{s t}, n_{s}\right)$ is the angle between the direction $s t$ and the exterior normal at $s, r$ is the distance from $s$ to a variable point $p$ of $d A$, and $\left(r, n_{2}\right)$ is the angle between the direction $s p$ and the exterior normal at $s$. This result reduces to Prager's equation (6a), loc. cit., ${ }^{5}$ for the special case of uniform flow, that is, when $k=0$.

The integral equation (7) or (8) for the velocity distribution on the contour $C$ may be solved in general by approximative methods. Knowledge of the velocity distribution on $C$ is equivalent to knowledge of the pressure distribution on $C$.

Example. Suppose $C$.is a circle of radius $a$ with center at 0 . In this case, the integral equation can be solved explicitly. We have, $\cos \left(r_{s t}, n_{s}\right) / r_{s t}=-1 / 2 a$. It is not difficult to show that

$$
\frac{\partial}{\partial n} \iint_{I} \log \frac{1}{r} d A=-\pi a .
$$

Finally, $\partial \psi_{0} / \partial n_{\mathrm{s}}=U \sin \theta+U k a \sin ^{2} \theta$ at the point with polar coordinates $(a, \theta)$. Hence (7) or (8) becomes

$$
\begin{equation*}
v(s)=\frac{\Gamma}{2 a \pi}-2 U \sin \theta-2 U k a \sin ^{2} \theta+U k a, \tag{9}
\end{equation*}
$$

where $\Gamma=\int_{c} v(t) d t$ is the circulation.
For the same example, Tsien (loc. cit., equation 18) finds the stream function

$$
\psi=U\left[\left(r-\frac{a^{2}}{r}\right) \sin \theta+\frac{k}{2}\left(r^{2} \sin ^{2} \theta+\frac{a^{4}}{2 r^{2}} \cos 2 \theta\right)\right] .
$$

Hence,

$$
\begin{equation*}
v(s)=-\frac{\partial \psi}{\partial r}=-2 U \sin \theta-2 U k a \sin ^{2} \theta+\frac{1}{2} U k a . \tag{10}
\end{equation*}
$$

To reconcile this result with (9), we must observe that we can write $\Gamma=\Gamma_{0}+\Gamma_{1}$, where $\Gamma_{0}$ and $\Gamma_{1}$ are the circulations arising from the undisturbed flow and the disturbance

[^1]flow, respectively. Hence $\Gamma_{0}=\int_{c} v_{0} d t$ where $\mathbf{v}_{0}$ is the undisturbed velocity field given by (1) and the subscript $t$ indicates the tangential component. By Stokes' theorem, $\Gamma_{0}=\iint_{I}\left(\operatorname{curl} \nabla_{0}\right)_{z} d A=-U k A$ where $A$ is the area of $I$. Hence in our example, $\Gamma_{0}=-U k \pi a^{2}$. If we substitute this for $\Gamma$ in (9), assuming, as Tsien does, ${ }^{6}$ that $\Gamma_{1}=0$, then our result (9) reduces to (10).

[^2]
# ON PLASTIC BODIES WITH ROTATIONAL SYMMETRY* 

## By C. H. W. SEDGEWICK (University of Connecticut)

Introduction. The rotational symmetry problem in plasticity was discussed by H. Hencky ${ }^{1}$ in 1923. In the present paper some new results are obtained. Furthermore, the presentation is different from that used by Hencky.

In the following discussion, $r$ and $z$ in the cylindrical coordinate system ( $r, \theta, z$ ) will be replaced by $\alpha(r, z)$ and $\beta(r, z)$ in such a way that $\alpha, \beta, \theta$ form a curvilinear, orthogonal system. The line element $d s$ will be written in the form

$$
d s^{2}=A^{2} d \alpha^{2}+B^{2} d \beta^{2}+r^{2} d \theta^{2}
$$

where $A$ and $B$ are functions of $\alpha$ and $\beta$. Furthermore, if the angle between the curve $\beta=$ const. and the direction of increasing $r$ is denoted by $\gamma$, we will have

$$
\begin{array}{ll}
\frac{\partial r}{\partial \alpha}=A \cos \gamma, & \frac{\partial r}{\partial \beta}=-B \sin \gamma, \\
\frac{\partial z}{\partial \alpha}=A \sin \gamma, & \frac{\partial z}{\partial \beta}=B \cos \gamma . \tag{2}
\end{array}
$$

From these, we get

$$
\begin{equation*}
\frac{\partial A}{\partial \beta}=-B \frac{\partial \gamma}{\partial \alpha}, \quad \text { (3) } \quad \frac{\partial B}{\partial \alpha}=A \frac{\partial \gamma}{\partial \beta} \tag{3}
\end{equation*}
$$

The stress components will be designated by $\sigma_{\alpha \alpha}, \sigma_{\beta \beta}, \sigma_{\theta \theta}, \sigma_{\alpha \beta}, \sigma_{\alpha \theta}, \sigma_{\beta \theta}$. In the problem under discussion, $\sigma_{\alpha \theta}=\sigma_{\beta \theta}=0$.

1. Lines of principal stress. Along the lines of principal stress, $\sigma_{\alpha \beta}=0$. In this case the equations of equilibrium ${ }^{2}$ reduce to
[^3]
[^0]:    * Received July 6, 1944.
    ${ }^{1}$ This note was prepared while the author was a fellow in the Program of Advanced Instruction and Research in Mechanics at Brown University (Summer 1943). The author is indebted to Prof. W. Prager for suggesting the topic and for valuable advice.
    ${ }^{2}$ H. S. Tsien, Symmetrical Joukowsky airfoils in shear flow, Quarterly of Applied Mathematics, 1, 130-148 (1943).
    ${ }^{8}$ Y. H. Kuo, On the force and moment acting on a body in shear flow, Quarterly of Applied Mathematics, 1, 273-275 (1943).
    ${ }^{4}$ W. Prager, Die Druckverteilung an Körpern in ebener Potentialströmung, Physikalische Zeitschrift, 29, 865-869 (1928).

[^1]:    ${ }^{5}$ The difference in sign is due to the fact that our ( $r_{s t}, n_{s}$ ) is the angle supplementary to that so denoted by Prager.

[^2]:    ${ }^{6}$ The author is indebted to Dr. Tsien for pointing this out. He had at first mistakenly supposed that Tsien's result was based on the assumption $F=0$.

[^3]:    * Received December 5, 1944. This paper was written during the summer of 1944 while the author was a student in the Program of Advanced Instruction and Research in Mechanics at Brown University. The author wishes to express his appreciation to Dr. W. Prager for suggesting the problem and for valuable criticisms.
    ${ }^{1}$ H. Hencky, Über einige statisch bestimmte Fälle des Gleichgewichts in plastischen Körpern, Zeitschr. für angew. Math. u. Mech. 3, 241 (1923).
    ${ }^{2}$ A. E. H. Love, The mathematical theory of elasticity, 4th edition, Cambridge University Press, 1934, p. 90 .

