ON CERTAIN INTEGRALS IN THE THEORY OF HEAT CONDUCTIONS*

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The integrals

$$\phi = \int_0^t x^{-3/2} \exp\left(-\frac{a^2}{x} - b^2 x\right) dx,$$
$$\psi = \int_0^t x^{-1/2} \exp\left(-\frac{a^2}{x} - b^2 x\right) dx \tag{1}$$

frequently arise in the theory of heat conduction. It is the purpose of this note to express these integrals in terms of tabulated functions.

By simple transformations the above integrals may be written in the form

$$\phi = 2\int_{1/\sqrt{t}}^{\infty} \exp\left(-a^2\lambda^2 - \frac{b^2}{\lambda^2}\right) d\lambda, \qquad \psi = -\frac{1}{2b}\frac{d\phi}{db}, \tag{2}$$

Let us consider the integral

$$u = \int_c^{\infty} \exp\left(-a^2\lambda^2 - \frac{b^2}{\lambda^2}\right) d\lambda = \int_c^{\infty} F d\lambda. \tag{3}$$

By obvious transformations (3) may be written successively in the forms

$$u = \int_0^{\infty} F d\lambda - \int_0^c F d\lambda = \frac{\sqrt{\pi}}{2a} e^{-2ab} - e^{2ab}\int_0^c \exp\left[-a^2\left(\lambda + \frac{b}{a\lambda}\right)^2\right] d\lambda$$
$$= \frac{\sqrt{\pi}}{2a} e^{-2ab} - \frac{b}{a} e^{2ab}\int_{b/ac}^{\infty} \lambda^{-2} \exp\left[-a^2\left(\lambda + \frac{b}{a\lambda}\right)^2\right] d\lambda$$
$$= \frac{\sqrt{\pi}}{2a} e^{-2ab} + e^{2ab}\int_{b/ac}^{\infty} \left(1 - \frac{b}{a\lambda^2}\right) \exp\left[-a^2\left(\lambda + \frac{b}{a\lambda}\right)^2\right] d\lambda$$
$$- e^{2ab}\int_{b/ac}^{\infty} \exp\left[-a^2\left(\lambda + \frac{b}{a\lambda}\right)^2\right] d\lambda$$
$$u = \frac{\sqrt{\pi}}{a}\cosh 2ab - \frac{\sqrt{\pi}\, e^{2ab}}{2a} \operatorname{erf}\left(\frac{b}{c} + ac\right)$$
$$- e^{2ab}\int_{b/ac}^{\infty} \exp\left[-a^2\left(\lambda + \frac{b}{a\lambda}\right)^2\right] d\lambda, \tag{4}$$

where $\operatorname{erf}(x) = (2/\sqrt{\pi})\int_0^x \exp(-\xi^2)d\xi$.

In an entirely similar manner one finds

$$u = \frac{\sqrt{\pi}}{2a} e^{-2ab} \operatorname{erf}\left(\frac{b}{c} - ac\right) + e^{-2ab}\int_{b/ac}^{\infty} \exp\left[-a^2\left(\lambda - \frac{b}{a\lambda}\right)^2\right] d\lambda. \tag{5}$$

From (4) and (5) one obtains

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$$u = \int_c^\infty \exp\left(-a^2\lambda^2 - \frac{b^2}{\lambda^2}\right) d\lambda$$

$$= \frac{\sqrt{\pi}}{2a}\cosh 2ab + \frac{\sqrt{\pi}}{4a}\left[e^{-2ab}\operatorname{erf}\left(\frac{b}{c} - ac\right) - e^{2ab}\operatorname{erf}\left(\frac{b}{c} + ac\right)\right]. \tag{6}$$

Accordingly,

$$\phi = 2\int_{1/\sqrt{t}}^\infty \exp\left(-a^2\lambda^2 - \frac{b^2}{\lambda^2}\right) d\lambda$$

$$= \frac{\sqrt{\pi}}{a}\cosh 2ab + \frac{\sqrt{\pi}}{2a}\left[e^{-2ab}\operatorname{erf}\left(b\sqrt{t} - \frac{a}{\sqrt{t}}\right) - e^{2ab}\operatorname{erf}\left(b\sqrt{t} + \frac{a}{\sqrt{t}}\right)\right]. \tag{7}$$

An expression for ψ is obtained by differentiating (7) with respect to b in accordance with the second Eq. (2).

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