THE PROBLEM OF SAINT VENANT FOR A CYLINDER WITH FREE SIDES*

BY

J. L. SYNGE

The Ohio State University

1. Introduction. Consider a cylinder composed of homogeneous isotropic elastic material (Fig. 1). The cross section is arbitrary; it may be simply or multiply connected. Body force (such as gravity) is assumed to be absent, and the sides are free. To the ends we apply any loadings which satisfy the conditions of statical equilibrium. As a result the cylinder undergoes a small deformation. We ask: what is the stress and what is the displacement throughout the cylinder?

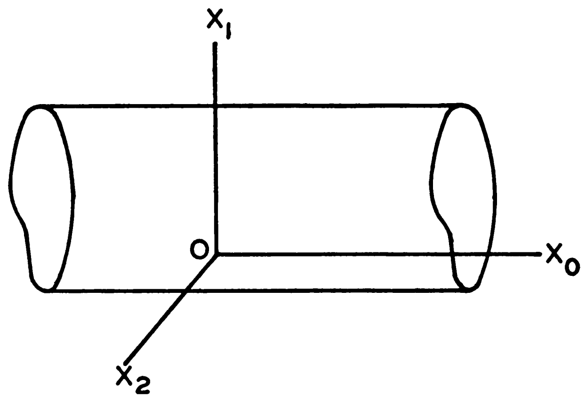

FIG. 1.

Here we have a well-formulated problem in the theory of elasticity, and one of the most important from a practical standpoint. It may be called the *problem of Saint Venant*. It includes as special cases the problems of tension, bending by couples, torsion and flexure.

Saint Venant[1] ingeniously side-stepped the major difficulty of the problem by substituting a simplified problem. Of the six components of stress (the unknowns of the original problem) he set three equal to zero, and relaxed the boundary conditions on the ends of the cylinder. In his solution the terminal loadings are not arbitrarily distributed; only the *total* load can be arbitrarily assigned. In his solution of the tension problem, for example, the load must be uniformly distributed over the cross section.

In tension or bending by couples, Saint Venant's solution is mathematically trivial. For torsion or flexure, the mathematical high spot is a problem of Dirichlet or Neumann (determination of a function harmonic in the cross section, the values of the function, or of its normal derivative, being assigned on the boundary). Mathematicians have busied themselves through the years with general and particular methods of solution. The torsion-flexure problem has been regarded as one of the central problems in the theory of elasticity.

Mathematicians should, however, be reminded that when they speak of the "torsion-flexure problem," they have in mind Saint Venant's type of solution, in which the terminal loads are not arbitrarily distributed. Relatively little attention has been

* Received May 13, 1944.

[1] B. de Saint Venant, Journal de Math., **1**, 89–189 (1856).

paid to more general types of solution. It is good, I think, to draw attention to the work that has been done, and to indicate possible methods of attack.

Section 2 gives the general mathematical statement of the problem of Saint Venant, and of the relaxed problems associated with it. Section 3 contains an old friend in new clothes—the Saint Venant solution in tensor notation. I believe that is worth while to give such a presentation, avoiding as far as possible the customary special choices of axes and particular integrals, in order to reveal the true mathematical structure of the solution. In Section 4 we pass beyond the Saint Venant solution, and set up the basic eigen-value problem associated with the exponential type of solution. In Section 5 we link this formulation with the solution given by Dougall for a cylinder of circular section. Section 6 contains some questions.

2. The problem of Saint Venant and the relaxed problems. Latin suffixes will have the range 0, 1, 2, and Greek suffixes the range 1, 2, with the usual summation convention for repeated suffixes. Let x_i be rectangular Cartesian coordinates, with the axis of x_0 in the direction of the generators of the cylinder. The position of the origin and the directions of the axes of x_1 and x_2 remain arbitrary. Let u_i be the displacement and E_{ij} $(=E_{ji})$ the reduced stress, i.e., the stress divided by Young's modulus. We have the basic stress-strain relations

$$\tfrac{1}{2}(u_{j,i} + u_{i,j}) = (1 + \sigma)E_{ij} - \sigma\delta_{ij}E_{kk}, \tag{2.1}$$

or equivalently

$$2(1 + \sigma)(1 - 2\sigma)E_{ij} = 2\sigma\delta_{ij}u_{k,k} + (1 - 2\sigma)(u_{j,i} + u_{i,j}). \tag{2.2}$$

Here σ is Poisson's ratio (a constant which may, in theory, take any value in the range $-1<\sigma<\frac{1}{2}$), and the comma denotes partial differentiation $(f_{,i}=\partial f/\partial x_i)$; δ_{ij} is the Kronecker delta.

In any problem in elasticity, we have a choice between two methods: we can work with the displacement u_i or with the stress E_{ij}. When the boundary conditions are given in terms of stress (as they are in the problem of Saint Venant), the relative advantages may be set down as follows:

Displacement method: Simple p.d.e. and complicated b.c.

Stress method: Complicated p.d.e. and simple b.c.

Here, and throughout, "p.d.e." means "partial differential equations," and "b.c." means "boundary conditions."

The two rival formulations of the problem of Saint Venant are set out below in (2.3) and (2.4). In each case, (a) contains the p.d.e., (b) contains the b.c. on the free sides (n_α are the direction cosines of the normal), and (c) contains the b.c. on the ends. T_0, T_α are the components of the assigned stress. The symbol Δ_3 is the 3-dimensional Laplacian differential operator $(\Delta_3=\partial^2/\partial x_j\partial x_j)$.

Saint Venant problem in terms of displacement:

$$(1 - 2\sigma)\Delta_3 u_i + u_{j,ji} = 0; \tag{2.3a}$$

$$(u_{\beta,0} + u_{0,\beta})n_\beta = 0, \quad 2\sigma u_{k,k}n_\alpha + (1 - 2\sigma)(u_{\beta,\alpha} + u_{\alpha,\beta})n_\beta = 0; \tag{2.3b}$$

$$\sigma u_{k,k} + (1 - 2\sigma)u_{0,0} = (1 + \sigma)(1 - 2\sigma)T_0, \quad u_{0,\alpha} + u_{\alpha,0}= 2(1 + \sigma)T_\alpha. \tag{2.3c}$$

Saint Venant problem in terms of stress:

$$E_{ij,j} = 0, \qquad (1+\sigma)\Delta_3 E_{ij} + E_{kk,ij} = 0, \tag{2.4a}$$

$$E_{0\beta}n_\beta = 0, \qquad E_{\alpha\beta}n_\beta = 0, \tag{2.4b}$$

$$E_{00} = T_0, \qquad E_{0\alpha} = T_\alpha. \tag{2.4c}$$

We note that in (2.3) there are 3 unknowns, 3 p.d.e., and 3 b.c. In (2.4) there are 6 unknowns, 9 p.d.e., and 3 b.c.; but between the 9 p.d.e. there exist 3 differential identities.

Let us cross out the (c) equations in (2.3) and (2.4), that is, *drop* the conditions on the ends of the cylinder. We have then alternative statements of what may be called the *relaxed problem of Saint Venant.* The unknowns are of course underdetermined in the relaxed problem, but the system is now linear and homogeneous, so that solutions may be superimposed. We may hope that, by superimposing solutions of the relaxed problem, we may succeed in satisfying the b.c. (c), either accurately or approximately.

In spite of the underdetermination, the relaxed problem is too complicated to yield to mere guessing, except for one very simple solution: $E_{00}=\text{const.}$, $E_{0\alpha}=0$, $E_{\alpha\beta}=0$. Indeed, it is usually more difficult to deal with an indeterminate problem than with a determinate one, and so we impose auxiliary conditions to replace the b.c. (c).

The best-known auxiliary condition is the so-called *hypothesis of Saint Venant:*

$$E_{\alpha\beta} = 0. \tag{2.5}$$

This leads to the Saint Venant solution, which will be discussed in Sect. 3. Another auxiliary condition, which we may call the *exponential condition,* is

$$E_{ij} = e^{kx_0}F_{ij}(x_1, x_2), \qquad (k \neq 0). \tag{2.6}$$

This is suggested by the fact that if $E_{ij}=f_{ij}$ satisfy the relaxed problem, then so also do $E_{ij}=f_{ij,0}$. The problem of determining solutions subject to (2.6) will be discussed in Sect. 4.

Dougall[2] has used the expression *permanent free modes* for solutions of the relaxed problem with (2.5) imposed, and *transitory free modes* for solutions of the relaxed problem with (2.6) imposed. (This terminology is suggested by the theory of vibrations, but of course our problem is statical.) According to Dougall, these two types of solutions are fundamental, in the sense that any solution of the relaxed problem is a linear combination of them.

3. Saint Venant's solution. Any treatise on elasticity contains a treatment of Saint Venant's solution of the relaxed problem, usually broken up into the problems of tension, bending, torsion, and flexure[3,4] for pedagogic reasons. Clebsch[5] appears to have been the first to see that Saint Venant's solution follows logically from (2.5). Marcolongo[6] has given an elegant general treatment, using displacement as fundamental. In the following treatment, stress is used and the axes $0x_1x_2$ remain completely general.

---

[2] J. Dougall, Trans. Roy. Soc. Edinburgh, **49**, 895–978 (1913). See also Proc. Fifth International Congress of Mathematicians, **2** (Cambridge 1913), 328–340.

[3] A. E. H. Love, *Mathematical theory of elasticity* (Cambridge, 1934), pp. 329–334.

[4] I. S. Sokolnikoff, *Elasticity,* Brown University, 1941, pp. 193–202.

[5] A. Clebsch, *Theorie der Elasticität fester Körper,* Leipzig, 1862.

[6] R. Marcolongo, *Teoria matematica dello equilibrio dei corpi elastici,* Milano, 1904, pp. 296–310.

We substitute the auxiliary conditions (2.5) in (2.4a, b), and obtain

$$\left.\begin{aligned} E_{00,0} + E_{0\beta,\beta} &= 0, & (1+\sigma)\Delta_3 E_{00} + E_{00,00} &= 0, \\ E_{\alpha 0,0} &= 0, & (1+\sigma)\Delta_3 E_{\alpha 0} + E_{00,\alpha 0} &= 0, \\ & & E_{00,\alpha\beta} &= 0; \end{aligned}\right\} \tag{3.1a}$$

$$E_{0\beta} n_\beta = 0. \tag{3.1b}$$

It follows at once that $E_{\alpha 0}$ are independent of x_0, and that

$$E_{00} = -x_0(A_\beta x_\beta + A) + B_\beta x_\beta + B, \tag{3.2}$$

where A_β, A, B_β, B are six constants, at present arbitrary. The remaining equations in (3.1a) are equivalent to

$$\Delta E_{\alpha 0} = (1+\sigma)^{-1} A_\alpha, \qquad E_{\beta 0,\beta} = A_\beta x_\beta + A, \tag{3.3}$$

where Δ is the plane Laplacian ($\Delta = \partial^2/\partial x_\beta \partial x_\beta$). These three equations are to be solved for the two unknowns $E_{\alpha 0}$, with the b.c. (3.1b). The problem is a plane problem, the domain being the section of the cylinder.

Given the vector $E_{\alpha 0}$, there exist invariants ϕ, ψ, such that

$$E_{\alpha 0} = \phi_{,\alpha} - \epsilon_{\alpha\gamma}\psi_{,\gamma} \qquad (\epsilon_{11} = \epsilon_{22} = 0,\ \epsilon_{12} = -\epsilon_{21} = 1). \tag{3.4}$$

Substitution in (3.3) leads at once to

$$\Delta\phi = A_\beta x_\beta + A, \quad (1+\sigma)\Delta\psi = \sigma\epsilon_{\beta\gamma} A_\beta x_\gamma + C, \tag{3.5a}$$

where C is another arbitrary constant. (Note that $\epsilon_{\alpha\beta}\epsilon_{\alpha\gamma} = \delta_{\beta\gamma}$.) The b.c. is

$$\phi_{,\beta} n_\beta - \epsilon_{\beta\gamma} n_\beta \psi_{,\gamma} = 0 \quad \text{or} \quad \partial\phi/\partial n - \partial\psi/\partial s = 0, \tag{3.5b}$$

where ∂n is a element of the normal n_α, drawn out of the material and to the right of the element ∂s of the bounding curve (Fig. 2). On integrating (3.5b) around the bounding curve, and using (3.5a), we find that

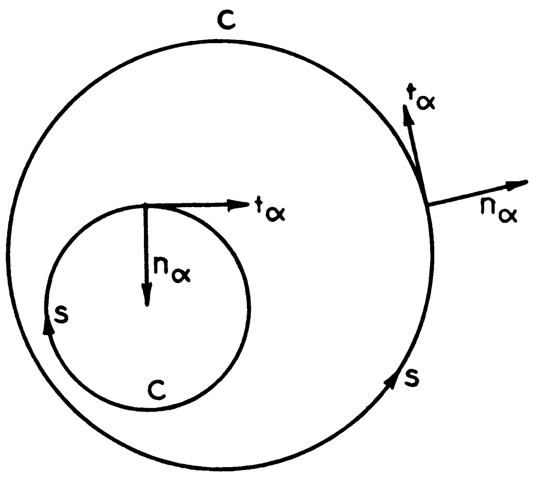

FIG. 2.

$$A = -A_\beta \bar{x}_\beta, \tag{3.6}$$

where $\bar{x}_\beta$ are the coordinates of the centroid of the section. Thus there are only six arbitrary constants, A_β, B_β, B, C.

Remembering the Riemann-Cauchy relations, we note that ϕ, ψ are indeterminate to the extent of adding to $\phi + i\psi$ an arbitrary analytic function of $x_1 + ix_2$.

Let $\phi^{(1)}$, $\psi^{(1)}$ be any single-valued particular solutions of (3.5a). Let us define Φ, Ψ by

$$\Phi = \phi - \phi^{(1)}, \qquad \Psi = \psi - \psi^{(1)}, \tag{3.7}$$

so that Φ, Ψ are harmonic. Let Ω be the harmonic conjugate of Ψ, so that

$$\Omega_{,\alpha} = \epsilon_{\alpha\gamma}\Psi_{,\gamma} = \epsilon_{\alpha\gamma}\psi_{,\gamma} - \epsilon_{\alpha\gamma}\psi^{(1)}_{,\gamma}. \tag{3.8}$$

If P is defined by

$$P = \Phi - \Omega, \tag{3.9}$$

then (3.4) reads

$$E_{\alpha 0} = P_{,\alpha} + \phi^{(1)}_{,\alpha} - \epsilon_{\alpha\gamma}\psi^{(1)}_{,\gamma}, \tag{3.10}$$

and the b.c. (3.5b) reads

$$\partial P/\partial n = -\partial\phi^{(1)}/\partial n + \partial\psi^{(1)}/\partial s. \tag{3.11}$$

Thus the relaxed problem of Saint Venant reduces to a problem of Neumann—to find a harmonic function P under the b.c. (3.11). The solution must be single-valued in order that the displacement may be single-valued.

This presentation of the problem has a fictitious simplicity; the constants A_β, C are wrapped up in the Neumann problem. We may split up the problem into problems involving only the geometry of the section and Poisson's ratio. To do this, we write

$$\begin{gathered} P = A_\beta P_\beta + (1+\sigma)^{-1}CQ, \\ \phi^{(1)} = A_\beta\phi^{(1)}_\beta, \qquad \psi^{(1)} = A_\beta\psi^{(1)}_\beta + (1+\sigma)^{-1}C\chi^{(1)}. \end{gathered} \tag{3.12}$$

Here $\phi^{(1)}_\beta$, $\psi^{(1)}_\beta$, $\chi^{(1)}$ are single-valued particular solutions of

$$\Delta\phi^{(1)}_\beta = x_\beta - \bar{x}_\beta, \qquad (1+\sigma)\Delta\psi^{(1)}_\beta = \sigma\epsilon_{\beta\gamma}x_\gamma, \qquad \Delta\chi^{(1)} = 1, \tag{3.13}$$

and P_β, Q are single-valued harmonic functions satisfying the b.c.

$$\partial P_\beta/\partial n = -\partial\phi^{(1)}_\beta/\partial n + \partial\psi^{(1)}_\beta/\partial s, \quad \partial Q/\partial n = \partial\chi^{(1)}/\partial s. \tag{3.14}$$

The determination of the vector P_β constitutes the *flexure problem*, and the determination of Q the *torsion problem*.

Since $\Delta(r^2x_\alpha) = 8x_\alpha$, $\Delta(r^2) = 4$, where $r^2 = x_\beta x_\beta$, particular solutions of (3.13) are given by

$$\begin{gathered} \phi^{(1)}_\beta = r^2x_\beta/8 - r^2\bar{x}_\beta/4, \\ (1+\sigma)\psi^{(1)}_\beta = \sigma r^2\epsilon_{\beta\gamma}x_\gamma/8, \qquad \chi^{(1)} = r^2/4. \end{gathered} \tag{3.15}$$

The determination of the displacement in terms of (3.2) and (3.10) in this general notation is interesting, because it reveals why the components u_α are independent of the solution of the Neumann problem (3.11). Using (2.5) in (2.1), we have

$$\begin{gathered} u_{0,0} = E_{00}, \qquad u_{\alpha,0} + u_{0,\alpha} = 2(1+\sigma)E_{\alpha 0}, \\ u_{\beta,\alpha} + u_{\alpha,\beta} = -2\sigma\delta_{\alpha\beta}E_{00}. \end{gathered} \tag{3.16}$$

By (3.2), the first two equations give at once

$$\begin{aligned} u_0 &= -\tfrac{1}{2}x_0^2(A_\beta x_\beta + A) + x_0(B_\beta x_\beta + B) + f_0, \\ u_\alpha &= \tfrac{1}{6}x_0^3A_\alpha - \tfrac{1}{2}x_0^2B_\alpha - x_0f_{0,\alpha} + 2(1+\sigma)x_0E_{\alpha 0} + f_\alpha, \end{aligned} \tag{3.17}$$

where f_0, f_α are unknown functions of x_1, x_2. When we substitute this expression for u_α in the last of (3.16), we get

$$f_{0,\alpha\beta} = (1+\sigma)(E_{\alpha 0,\beta} + E_{\beta 0,\alpha}) - \sigma\delta_{\alpha\beta}(A_\gamma x_\gamma + A), \tag{3.18}$$

$$f_{\beta,\alpha} + f_{\alpha,\beta} = -2\sigma\delta_{\alpha\beta}(B_\gamma x_\gamma + B). \tag{3.19}$$

Now, by (3.10),

$$E_{\alpha 0,\beta} - E_{\beta 0,\alpha} = -\epsilon_{\alpha\gamma}\psi^{(1)}_{,\gamma\beta} + \epsilon_{\beta\gamma}\psi^{(1)}_{,\gamma\alpha} = -\epsilon_{\alpha\beta}\Delta\psi^{(1)}, \tag{3.20}$$

and so, since $\psi^{(1)}$ satisfies (3.5a),

$$(1+\sigma)(E_{\alpha 0,\beta} - E_{\beta 0,\alpha}) = -\epsilon_{\alpha\beta}(\sigma\epsilon_{\mu\nu}A_\mu x_\nu + C). \tag{3.21}$$

We now use this equation to sbustitute for $E_{\beta 0,\alpha}$ in (3.18). When we do so, the linear expression on the right must be a partial derivative with respect to x_β. Now if $F_{\alpha\beta\ldots\mu}$ is a single-valued homogeneous function of degree n in x_1, x_2, it follows from Euler's theorem that

$$\int F_{\alpha\beta\ldots\mu,\nu}dx_\nu = \frac{1}{n}F_{\alpha\beta\ldots\mu,\nu}x^\nu + K_{\alpha\beta\ldots\mu}, \tag{3.22}$$

where K is a constant. Hence we deduce from (3.18)

$$f_{0,\alpha} = 2(1+\sigma)E_{\alpha 0} + \epsilon_{\alpha\beta}x_\beta(\tfrac{1}{2}\sigma\epsilon_{\mu\nu}A_\mu x_\nu + C) - \sigma x_\alpha(\tfrac{1}{2}A_\gamma x_\gamma + A) + h_\alpha, \tag{3.23}$$

where h_α is constant. Now substitute for $E_{\alpha 0}$ from (3.10) and integrate; this gives

$$f_0 = 2(1+\sigma)P + 2(1+\sigma)\phi^{(1)} + \int[-2(1+\sigma)\epsilon_{\alpha\gamma}\psi^{(1)}_{,\gamma} + \epsilon_{\alpha\beta}x_\beta(\tfrac{1}{2}\sigma\epsilon_{\mu\nu}A_\mu x_\nu + C) - \sigma x_\alpha(\tfrac{1}{2}A_\gamma x_\gamma + A)]dx_\alpha + h_\alpha x_\alpha + h. \tag{3.24}$$

where h is constant. By virtue of the p.d.e. (3.5a) satisfied by $\psi^{(1)}$, this integral has the same value for reconcilable paths. But we have no guarantee that it has the same value for irreconcilable paths in the case of a multiply connected section. To secure a single valued displacement, we should choose for $\psi^{(1)}$ a function which satisfies the p.d.e. (3.5a) throughout the whole interior of the outer boundary. We may, for example, use the expression given in (3.12), (3.15).

The equations (3.19) determine f_α to within a plane rigid body displacement, and so

$$f_\alpha = \sigma(\tfrac{1}{2}B_\alpha r^2 - x_\alpha B_\gamma x_\gamma - Bx_\alpha) + k\epsilon_{\alpha\gamma}x_\gamma + k_\alpha, \tag{3.25}$$

where k_α, k are constants.

To find the displacement, we substitute for f_0 from (3.24) in the first of (3.17), and so obtain u_0. It involves the solution of the Neumann problem, since P appears in (3.24). To find u_α, we substitute in the second of (3.17). There are two substitutions, one for $f_{0,\alpha}-2(1+\sigma)E_{\alpha 0}$ from (3.23), and the other for f_α from (3.25). We see that P does not occur, and thus we verify the well known fact that the lateral displacement is independent of the solution of the Neumann problem.

4. The exponential type of solution. Let us now investigate the solution of the relaxed problem with the auxiliary condition (2.6). Since stress determines displacement to within a rigid body displacement (which we shall omit), we may write for the corresponding displacement

$$u_i = e^{kx_0}v_i(x_1, x_2). \tag{4.1}$$

Once more we have a choice between two methods—displacement and stress—and we shall choose displacement.

When we substitute from (4.1) in (2.3a, b), the equations reduce to

$$(\Delta + k^2)V = 0, \qquad (\Delta + k^2)v_\alpha = -V_{,\alpha}, \tag{4.2a}$$

$$2\sigma V n_\alpha + (v_{\beta,\alpha} + v_{\alpha,\beta})n_\beta = 0, \quad (1 - 2\sigma)V_{,\beta}n_\beta + k^2 v_\beta n_\beta - v_{\beta,\beta\gamma}n_\gamma = 0. \tag{4.2b}$$

Here V is an auxiliary variable, given by

$$(1 - 2\sigma)V = kv_0 + v_{\beta,\beta}; \tag{4.3}$$

it is the dilatation, to within a factor.

Inspection of (4.2) shows that we have before us an eigen-value problem of considerable complexity. The system will (presumably) be consistent only for certain values of k. There is no objection to complex eigen-values, with complex solutions for V, v_α, and the corresponding stress. Denoting complex conjugates by bars, we should take in such cases for the real displacement and stress

$$\tfrac{1}{2}(e^{kx_0}v_i + e^{\bar{k}x_0}\bar{v}_i), \qquad \tfrac{1}{2}(e^{kx_0}F_{ij} + e^{\bar{k}x_0}\bar{F}_{ij}). \tag{4.4}$$

If k is an eigen-value of (4.2), so also are $-k$ and $\pm\bar{k}$. In fact, the eigen-values occur in sets of two if they are real or purely imaginary, and in sets of four if complex.

By a simple and ingenious argument, Dougall[2] has shown that the system (4.2) has no purely imaginary eigen-values. A purely imaginary k implies a periodic distribution of displacement and stress. Consider the energy in a length of cylinder equal to this period. It is equal to the work done by the terminal stress in passing from the natural state to the strained state. But, from the periodicity, this is zero. Hence, the energy of a strained state is zero, which is contrary to a basic postulate of elasticity. Hence *there can be no purely imaginary eigen-value k*. It should be added that we cannot assert this if σ is arbitrary. It is necessarily true only if strain-energy is positive-definite, i.e., if $-1 < \sigma < \frac{1}{2}$.

In many problems in applied mathematics, harmonic functions play a fundamental role. In our problem (4.2) that role is taken over by plane wave functions, where by a wave function $f(x_1, x_2)$ we mean a solution of $(\Delta + k^2)f = 0$. We note that V is a wave function, but v_α is not.

We can, however, easily reduce the unknowns to wave functions by writing

$$v_\alpha = w_\alpha + w^*_\alpha, \tag{4.5}$$

where w^*_α is any particular solution of

$$(\Delta + k^2)w^*_\alpha = -V_{,\alpha}. \tag{4.6}$$

Let us not tie ourselves down to any definite particular solution. Two, however, seem to be particularly interesting:

$$w^*_\alpha = -\tfrac{1}{2}x_\alpha V, \tag{4.7}$$

$$w^*_\alpha = -W_{,\alpha}, \qquad (\Delta + k^2)W = V. \tag{4.8}$$

Our problem (4.2) may now be stated as follows:

$$(\Delta + k^2)V = 0, \qquad (\Delta + k^2)w_\alpha = 0, \tag{4.9a}$$

$$\begin{gathered} 2\sigma V n_\alpha + (w^*_{\beta,\alpha} + w^*_{\alpha,\beta})n_\beta + (w_{\beta,\alpha} + w_{\alpha,\beta})n_\beta = 0, \\ (1 - 2\sigma)V_{,\beta}n_\beta + k^2 w^*_\beta n_\beta - w^*_{\beta,\beta\gamma}n_\gamma + k^2 w_\beta n_\beta - w_{\beta,\beta\gamma}n_\gamma = 0. \end{gathered} \tag{4.9b}$$

In simplifying the p.d.e., we have complicated the b.c.

In dealing with Saint Venant's solution, we found it advisable to pass in (3.4) from a vector to two invariants. The same procedure will be pursued here. We shall make use of the following fact: *Given a vector wave function w_α, there exist wave functions ϕ, ψ such that*

$$w_\alpha = \phi_{,\alpha} - \epsilon_{\alpha\gamma}\psi_{,\gamma}, \tag{4.10}$$

and ϕ, ψ are unique.

To prove this, we have merely to put

$$\phi = -k^{-2}w_{\beta,\beta}, \qquad \psi = k^{-2}\epsilon_{\mu\nu}w_{\mu,\nu}. \tag{4.11}$$

It is easy to verify that these satisfy (4.10), by virtue of the wave character of w_α. As for uniqueness, we can add to ϕ, ψ only functions $\phi^{(1)}$, $\psi^{(1)}$ such that $\phi^{(1)}+i\psi^{(1)}$ is an analytic function of x_1+ix_2. Thus $\phi^{(1)}$, $\psi^{(1)}$ are simultaneously wave functions and harmonic functions; therefore they vanish.

Let us now transform the b.c. (4.9b). Let C (Fig. 2) be the boundary curve of the cross section. Let $t_\alpha = dx_\alpha/ds$. Then

$$\begin{aligned} n_\alpha &= \epsilon_{\alpha\beta}t_\beta, & t_\alpha &= \epsilon_{\beta\alpha}n_\beta, \\ dn_\alpha/ds &= \kappa t_\alpha, & dt_\alpha/ds &= -\kappa n_\alpha, \end{aligned} \tag{4.12}$$

where κ is the curvature of C, positive if C bends away from n_α.

We may replace the first of (4.9b) by two invariant equations, obtained by multiplying by n_α, t_α, respectively. Thus

$$\begin{aligned} \sigma V + w^*_{\alpha,\beta}n_\alpha n_\beta + w_{\alpha,\beta}n_\alpha n_\beta &= 0, \\ (w^*_{\beta,\alpha} + w^*_{\alpha,\beta})t_\alpha n_\beta + (w_{\beta,\alpha} + w_{\alpha,\beta})t_\alpha n_\beta &= 0. \end{aligned} \tag{4.13}$$

From (4.10) we have

$$\begin{aligned} (w_{\beta,\alpha} + w_{\alpha,\beta})n_\alpha n_\beta &= 2\phi_{,\alpha\beta}n_\alpha n_\beta - 2\psi_{,\alpha\beta}t_\alpha n_\beta, \\ (w_{\beta,\alpha} + w_{\alpha,\beta})t_\alpha n_\beta &= 2\phi_{,\alpha\beta}t_\alpha n_\beta + \psi_{,\alpha\beta}n_\alpha n_\beta - \psi_{,\alpha\beta}t_\alpha t_\beta. \end{aligned} \tag{4.14}$$

The following identities are easily verified:

$$\begin{aligned} \partial^2\phi/\partial n^2 &= \phi_{,\alpha\beta}n_\alpha n_\beta, & \frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial n}\right) &= \phi_{,\alpha\beta}t_\alpha n_\beta + \kappa\phi_{,\alpha}t_\alpha, \\ \Delta\phi &= \phi_{,\alpha\beta}n_\alpha n_\beta + \phi_{,\alpha\beta}t_\alpha t_\beta, & \partial^2\phi/\partial s^2 &= \phi_{,\alpha\beta}t_\alpha t_\beta - \kappa\phi_{,\alpha}n_\alpha. \end{aligned} \tag{4.15}$$

Hence

$$\begin{aligned} w_{\alpha,\beta}n_\alpha n_\beta &= -\left(\frac{\partial^2}{\partial s^2} + k^2 + \kappa\frac{\partial}{\partial n}\right)\phi - \left(\frac{\partial}{\partial s}\frac{\partial}{\partial n} - \kappa\frac{\partial}{\partial s}\right)\psi, \\ (w_{\beta,\alpha} + w_{\alpha,\beta})t_\alpha n_\beta &= 2\left(\frac{\partial}{\partial s}\frac{\partial}{\partial n} - \kappa\frac{\partial}{\partial s}\right)\phi - 2\left(\frac{\partial^2}{\partial s^2} + \tfrac{1}{2}k^2 + \kappa\frac{\partial}{\partial n}\right)\psi. \end{aligned} \tag{4.16}$$

Thus our eigen-value problem may be stated in the following final general form: *Given a domain in the (x_1, x_2) plane, bounded by a curve C, we seek eigen-values k and eigen-functions V, ϕ, ψ to satisfy the p.d.e.*

$$(\Delta + k^2)V = 0, \qquad (\Delta + k^2)\phi = 0, \qquad (\Delta + k^2)\psi = 0, \tag{4.17a}$$

and the b.c. on C

$$\sigma V + \overset{*}{w}_{\alpha,\beta} n_\alpha n_\beta - \left(\frac{\partial^2}{\partial s^2} + k^2 + \kappa \frac{\partial}{\partial n}\right)\phi - \left(\frac{\partial}{\partial s}\frac{\partial}{\partial n} - \kappa \frac{\partial}{\partial s}\right)\psi = 0,$$

$$\tfrac{1}{2}(\overset{*}{w}_{\alpha,\beta} + \overset{*}{w}_{\beta,\alpha}) t_\alpha n_\beta + \left(\frac{\partial}{\partial s}\frac{\partial}{\partial n} - \kappa \frac{\partial}{\partial s}\right)\phi - \left(\frac{\partial^2}{\partial s^2} + \tfrac{1}{2}k^2 + \kappa \frac{\partial}{\partial n}\right)\psi = 0, \qquad (4.17\text{b})$$

$$(1 - 2\sigma)\frac{\partial V}{\partial n} + k^2 \overset{*}{w}_\beta n_\beta - \overset{*}{w}_{\beta,\beta\gamma} n_\gamma + 2k^2 \frac{\partial \phi}{\partial n} - k^2 \frac{\partial \psi}{\partial s} = 0.$$

Here w_α^* is any particular solution of (4.6).

If we choose w_α^* as in (4.7), we may substitute in (4.17b):

$$\sigma V + \overset{*}{w}_{\alpha,\beta} n_\alpha n_\beta = -\frac{1}{2}(1-2\sigma)V - \frac{1}{4}\frac{\partial}{\partial n}(r^2)\frac{\partial V}{\partial n},$$

$$\frac{1}{2}(\overset{*}{w}_{\beta,\alpha} + \overset{*}{w}_{\alpha,\beta}) t_\alpha n_\beta = -\frac{1}{8}\frac{\partial}{\partial s}(r^2)\frac{\partial V}{\partial n} - \frac{1}{8}\frac{\partial}{\partial n}(r^2)\frac{\partial V}{\partial s}, \qquad (4.18)$$

$$(1-2\sigma)\frac{\partial V}{\partial n} + k^2 \overset{*}{w}_\beta n_\beta - \overset{*}{w}_{\beta,\beta\gamma} n_\gamma = 2(1-\sigma)\frac{\partial V}{\partial n} - \frac{1}{4}k^2 V \frac{\partial}{\partial n}(r^2) + \frac{1}{2}\frac{\partial}{\partial n}(x_\beta V_{,\beta}).$$

On the other hand, if we choose w_α^* as in (4.8), we may substitute for these three quantities, respectively,

$$\sigma V - \partial^2 W/\partial n^2, \qquad -\left(\frac{\partial}{\partial s}\frac{\partial}{\partial n} - \kappa\frac{\partial}{\partial s}\right)W, \qquad 2(1-\sigma)\frac{\partial V}{\partial n} - 2k^2\frac{\partial W}{\partial n}. \qquad (4.19)$$

If we choose w_α^* as in (4.7), we have the following expressions for the displacement in terms of V, ϕ, ψ, k:

$$\begin{aligned} u_0 &= e^{kx_0} v_0, \qquad u_\alpha = e^{kx_0} v_\alpha, \\ k v_0 &= 2(1-\sigma)V + \tfrac{1}{2}x_\alpha V_{,\alpha} + k^2\phi, \\ v_\alpha &= -\tfrac{1}{2}x_\alpha V + \phi_{,\alpha} - \epsilon_{\alpha\beta}\psi_{,\beta}. \end{aligned} \qquad (4.20)$$

It takes only a moment to verify that these expressions satisfy (4.2a). These formulas (in a slightly more general form) are the key to Dougall's treatment of the circular cylinder.[2] However, he gives no indication as to how to he obtained them, or whether they are a general representation of the exponential type of displacement. I have shown that they are in fact a general representation. Given v_i and k, then V, ϕ, ψ are uniquely determined.

The stress corresponding to (4.20) is

$$\begin{aligned} &E_{ij} = e^{kx_0} F_{ij}, \qquad F_{\alpha 0} = -k^{-1} F_{\alpha\beta,\beta}, \qquad F_{00} = k^{-2} F_{\alpha\beta,\alpha\beta}, \\ &2(1+\sigma)F_{\alpha\beta} = -(1-2\sigma)\delta_{\alpha\beta}V - \tfrac{1}{2}(x_\alpha V_{,\beta} + x_\beta V_{,\alpha}) + 2\phi_{,\alpha\beta} - (\epsilon_{\alpha\gamma}\psi_{,\gamma\beta} + \epsilon_{\beta\gamma}\psi_{,\gamma\alpha}), \\ &2k(1+\sigma)F_{\alpha 0} = 2(1-\sigma)V_{,\alpha} + \tfrac{1}{2}(x_\beta V_{,\beta})_{,\alpha} - \tfrac{1}{2}k^2 x_\alpha V + 2k^2\phi_{,\alpha} - k^2\epsilon_{\alpha\beta}\psi_{,\beta}, \\ &2(1+\sigma)F_{00} = 2(2-\sigma)V + x_\alpha V_{,\alpha} + 2k^2\phi. \end{aligned} \qquad (4.21)$$

5. The case of the circular section. The circular section has been dealt with so thoroughly by Dougall[2] that there is little more to be said about it. However, it is interesting to see how his method of solution connects up with the preceding approach.

Let the boundary be $r=a$, and let ω be the polar angle. Then the following are wave functions

$$(V, \phi, \psi) = (A, B, C)J_m(kr)e^{im\omega}, \tag{5.1}$$

where A, B, C are any constants. Each of these functions satisfies on $r=a$ the equations

$$\frac{\partial f}{\partial n} = \frac{Kf}{a}, \qquad \frac{\partial f}{\partial s} = \frac{imf}{a}, \qquad \frac{\partial}{\partial s}\frac{\partial f}{\partial n} = \frac{imKf}{a^2}, \tag{5.2}$$
$$K = \xi J'_m(\xi)/J_m(\xi), \qquad \xi = ka.$$

They also satisfy for $r=a$,

$$\frac{\partial}{\partial n}(x_\beta f_{,\beta}) = (m^2 - \xi^2)f/a. \tag{5.3}$$

Substitution in (4.18) and (4.17b) gives

$$L_{i0}Va^2 + L_{i1}\phi + L_{i2}\psi = 0,$$
$$L_{00} = \tfrac{1}{2}(1 - 2\sigma) + \tfrac{1}{2}K, \qquad L_{01} = -m^2 + \xi^2 + K, \qquad L_{02} = im(K-1),$$
$$L_{01} = -im/4, \qquad L_{11} = im(K-1), \qquad L_{12} = m^2 - \tfrac{1}{2}\xi^2 - K, \tag{5.4}$$
$$L_{20} = (1-\sigma)K + \tfrac{1}{4}m^2 - \tfrac{1}{2}\xi^2, \qquad L_{21} = K\xi^2, \qquad L_{22} = -\tfrac{1}{2}im\xi^2.$$

The characteristic equation is

$$|L_{ij}| = 0, \tag{5.5}$$

which is essentially the determinantal equation of Dougall for the determination of ξ, and hence k.

For modes independent of ω, we put $m=0$. Then (5.5) reduces to

$$(\xi^2 + 2K)[\xi^2(K^2 + \xi^2) - 2K^2(1-\sigma)] = 0, \tag{5.6}$$

and we get the two characteristic equations

$$J_2(\xi) = 0, \qquad \xi^2(J_0^2 + J_0'^2) = 2(1-\sigma)J_0'^2. \tag{5.7}$$

The first equation has an infinite sequence of real roots; the second has an infinite sequence of complex roots.

Dougall states (p. 902) that for every m, (5.5) has an infinite number of real roots and an infinite number of complex roots (but no purely imaginary roots, as pointed out earlier).

For other work bearing on the circular section, reference may be made to Pochhammer,[7] Thomae,[8] Schiff,[9] Chree,[10] Tedone,[11,12,13,14,15] Filon,[16] Purser,[17,18] Timpe,[19]

[7] L. Pochhammer, Journal f. Math., **81**, 33–61 (1876).

[8] J. Thomae, Berichte Verh. K. Sächs. Ges. d. Wiss. zu Leipzig, Math. Phys. Cl., **37**, 399–418 (1885).

[9] Schiff, Journal de Math., **9**, 407–424 (1883).

[10] C. Chree, Cambridge Philosophical Transactions, **14**, 250–369 (1887).

[11] O. Tedone, Atti R. Acc. Lincei Rend. Cl. sci. fis. mat. nat., **10**, 131–137 (1901).

[12] O. Tedone, Atti R. Acc. Lincei Rend. Cl. sci. fis. mat. nat., **13**$_1$, 232–240 (1904).

[13] O. Tedone, Atti R. Acc. Lincei Rend. Cl. sci. fis. mat. nat., **20**$_2$, 617–622 (1911).

Wolf,[20] Barton.[21] Barton makes use of the general Papcovitch[22]-Neuber[23,24] solution of (2.3a), viz.,

$$u_i = \psi_i - \frac{1}{4(1-\sigma)} \frac{\partial}{\partial x_i} (x_j\psi_j + \phi), \tag{5.8}$$

where $\Delta_3\psi_i = 0$, $\Delta_3\phi = 0$. Any solution of (2.3a) may be so expressed, but it seems that for present purposes Dougall's expressions (4.20) are more convenient.

6. Conclusion. The method of Dougall can be extended to a pipe, i.e., a section bounded by two concentric circles. Here are some questions:

1) Are there any other sections which can be solved by simple extensions of the method used for the circle?
2) Do eigen-values exist under reasonably general assumptions regarding the smoothness of the boundary curve?
3) Is there always a set of real eigen-values, or is that a peculiarity of the circle?
4) Write $k = p + iq$, and let m be the least value of $|p|$ in the sequence of eigen-values for a given section. Then m^2S, where S is the area of the section, depends only on the shape of the section. For arbitrary sections, m^2S forms a positive sequence. Is it bounded below, and, if so, what is the lower bound?

This last question is very interesting from a practical point of view, because $|p|$ represents the rate at which end effects decay as we pass along the cylinder. The greater $|p|$, the more rapid the decay. Engineers are worried by end effects, because the Saint Venant solution gives no information about them. The assignment of such a lower bound might be more valuable than the description of a complicated process for the evaluation of eigen-values.

---

[14] O. Tedone, Atti R. Acc. Lincei Rend. Cl. sci. fis. mat. nat., **21**$_1$, 384–389 (1912).
[15] O. Tedone, Encyk. d. math. Wiss., **4**$_4$, p. 150.
[16] L. N. G. Filon, Phil. Trans. Roy. Soc. London, A **198**, 147–233 (1902).
[17] F. Purser, Trans. Roy. Irish Academy, **32** A, 31–60 (1902).
[18] F. Purser, Proc. Roy. Irish Academy, **26** A, 54–60 (1906).
[19] A. Timpe, Mathematische Annalen, **71**, 480–509 (1912).
[20] K. Wolf, K. Akad. d. Wiss. Wien, Math.-naturwiss. Kl., Abt. IIa, **125**, 1149–1166 (1916).
[21] M. V. Barton, J. App. Mechanics, **8**, A-97–A-104 (1941).
[22] P. F. Papcovitch, Comptes rendus Acad. des Sci. Paris, **195**, 513–515 (1932).
[23] H. Neuber, Zeit. angew. Math. u. Mech., **14**, 203–212 (1934).
[24] H. Neuber, *Kerbspannungslehre*, Berlin, 1937.