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EXPANSION OF DETERMINANTAL EQUATIONS INTO POLYNOMIAL FORM*

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I. INTRODUCTION

Determinantal equations of the form

$$|A_0\lambda^n + A_1\lambda^{n-1} + \cdots + A_n| = 0,$$

where the coefficients A_i are square matrices of order m, arise in quantum mechanics, electrical circuit theory, the theory of small vibrations, and in many other branches of physics and engineering. In the process of solving such equations, it is frequently desirable to expand them into polynomial form. Since most of the techniques for such expansion are not discussed in the standard works on numerical computation, it has seemed advisable to make a critical comparison of the methods available. The most important methods will be described in sufficient detail to aid the non-professional computer.

1. Basic techniques of solution of determinantal equations. Three basic techniques for solution of determinantal equations must be considered: (1) Direct solution of the equation by numerical methods [1]. (2) Direct solution by the method of matrix multiplication [2–6] (directly applicable only to the case $|A - I\lambda| = 0$). (3) Expansion into polynomial form [6–14] and solution of the polynomial equation by standard methods.

In spite of the fact that much effort has been spent in developing techniques to avoid expansion into polynomial form, that very technique frequently proves to be most economical of effort. Let us consider the numerical solution of the simplest case:

$$D(\lambda) = |A - I\lambda| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0. \tag{1}$$

If we assign a value to λ and evaluate $D(\lambda)$ using Chiò's expansion [15] of the numerical determinant, we shall make the following number of operations:

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$$(1/3)(n^3 + 2n - 3) \qquad \text{M-D (multiplications and divisions)},$$
$$(n/6)(n + 1)(2n + 1) \qquad \text{A-S (additions and subtractions)}.$$

The number of additions and subtractions includes the n subtractions required to evaluate the diagonal terms of (1). If we have $D(\lambda)$ in polynomial form, we can evaluate it by synthetic division with only n multiplications and n additions and subtractions. We can obtain the polynomial expansion of (1) by Danielewsky's method (§2c) with n^3-2n+1 multiplications and divisions and $n(n-1)^2$ additions and subtractions; hence to obtain k different values of $D(\lambda)$ by first obtaining the polynomial expansion, and then evaluating the polynomial, we need

$$(n^3 - 2n + 1) + kn \qquad \text{M-D},$$
$$n(n - 1)^2 + kn \qquad \text{A-S}.$$

To obtain k values of $D(\lambda)$ from the determinant will require

$$(k/3)(n^3 + 2n - 3) \qquad \text{M-D},$$
$$(kn/6)(n + 1)(2n + 1) \qquad \text{A-S}.$$

A comparison of these results shows that it will be quicker to obtain the polynomial expansion first if we need more than three values of $D(\lambda)$.

If we use iterative methods to solve (1), we form the sequence of matrices

$$AX,\ A^2X,\ A^3X,\ \cdots, \tag{2}$$

where X is an arbitrary column matrix,*

$$X = \{X_1, \cdots, X_n\}.$$

To form each member of the sequence (2) requires n^2 multiplications and divisions and $n(n-1)$ additions and subtractions; hence for k iterations we need

$$kn^2 \qquad \text{M-D},$$
$$kn(n - 1) \qquad \text{A-S}.$$

If we have the polynomial form, each iteration requires only n multiplications and divisions and $n-1$ additions and subtractions. Adding these to the operations required to expand (1) into polynomial form by the modified Danielewsky method (§2c) we need

$$(n^3 - 2n + 1) + kn \qquad \text{M-D},$$
$$n(n - 1)^2 + k(n - 1) \qquad \text{A-S}.$$

Hence if $k \geqq n+1$ the expansion to polynomial form represents a net saving, still using the powerful iterative method [3, 4, 16, 17, 18].

II. EXPANSION OF A DETERMINANTAL EQUATION INTO POLYNOMIAL FORM

2. Methods applicable to the case $|A - I\lambda| = 0$. Direct expansion is tedious except for the very lowest orders, although sometimes it is desirable because all the elements need not be given numerically. Purely numerical methods, such as the method of undetermined coefficients or the use of an interpolation formula, will be described further

* In this paper, a row of quantities enclosed in braces will denote a column matrix.

on, as they are applicable to the most general case. Of the various methods particularly applicable to the present case (Eq. 1), five will be discussed. These will be taken up in chronological order of discovery.

a) *The method of Leverrier* [7, 12, 19]. Until recently, Leverrier's method was probably the best general method for obtaining the polynomial expansion of the characteristic equation of the matrix A. Let the characteristic values of this matrix, i.e., the roots of the equation $|A - I\lambda| = 0$, be $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then from well known relations between the coefficients of a polynomial equation and its roots [20], we have

$$\begin{aligned} |A - I\lambda| = &(-1)^n\lambda^n + (-1)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} \\ &+ (-1)^{n-2}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n)\lambda^{n-2} + \cdots \\ &+ (\lambda_1\lambda_2 \cdots \lambda_n). \end{aligned} \tag{3}$$

But direct expansion of the determinant gives

$$|A - I\lambda| = (-1)^n\lambda^n + (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \cdots,$$

whence we have the relation

$$a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n. \tag{4}$$

If we consider the kth power of the matrix A, the characteristic values of the matrix A^k will be $\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k$ [21]. From this we obtain

$$\begin{aligned} |A^k - I\mu| &= (-1)^n\mu^n + (-1)^{n-1}(\lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k)\mu^{n-1} + \cdots \\ &= (-1)^n\mu^n + (-1)^{n-1}(a_{11}^{(k)} + a_{22}^{(k)} + \cdots + a_{nn}^{(k)})\mu^n + \cdots, \end{aligned}$$

which yields the relation

$$s_k = a_{11}^{(k)} + a_{22}^{(k)} + \cdots + a_{nn}^{(k)} = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k, \tag{5}$$

where the terms $a_{ii}^{(k)}$ are taken from the principal diagonal of A^k.

If we write our original equation as

$$|A - I\lambda| = (-1)^n[\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots + p_{n-1}\lambda + p_n] = 0.$$

we have, from (3), (4) and (5),

$$\begin{aligned} p_1 &= -(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = -(a_{11} + a_{22} + \cdots + a_{nn}) = -s_1, \\ p_1^2 &= -s_1p_1 = (\lambda_1 + \lambda_2 + \cdots + \lambda_n)^2 \\ &= \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2 + 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots + \lambda_{n-1}\lambda_n) = s_2 + 2p_2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned}$$

or

$$p_1 = -s_1, \ p_2 = -\tfrac{1}{2}(s_1p_1 + s_2), \cdots,$$

which are of a set of n simultaneous linear equations for the coefficients p_k ($k = 1, 2, \cdots, n$).

Leverrier's method can be summarized as follows: We form the powers of the matrix A, i.e., A^k ($k = 1, 2, \cdots, n$), and add the diagonal terms of each matrix to obtain

$$s_k = a_{11}^{(k)} + a_{22}^{(k)} + \cdots + a_{nn}^{(k)}.$$

We then set up the set of n simultaneous linear equations

$$p_k = -(1/k)(s_1 p_{k-1} + s_2 p_{k-2} + \cdots + s_{k-1} p_1 + s_k) \qquad (k = 1, 2, \cdots, n), \tag{6}$$

which can be solved for p_k $(k=1, 2, \cdots, n)$. We can then write the equation

$$D(\lambda) = |A - I\lambda| = (-1)^n(\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots + p_n).$$

Leverrier's method is rather tedious because of the labor of forming the powers of the matrix A. Each element of a product matrix will require n multiplications and $n-1$ additions and subtractions, and each matrix will have n^2 terms, except that one need form only the n diagonal terms for the last matrix. The fastest way to solve the simultaneous equations is to solve them successively, i.e., to use the solution of the first to solve the second, the solutions of the first two to solve the third, and so on. This will require $\frac{1}{2}(n^2+n-2)$ multiplications and divisions and $\frac{1}{2}n(n-1)$ additions and subtractions. Consequently, Leverrier's method will require, in the general case,

$$\begin{aligned} &\tfrac{1}{2}(n-1)(2n^3 - 2n^2 + n + 2) && \text{M-D,} \\ &\tfrac{1}{2}n(n-1)(2n^2 - 4n + 3) && \text{A-S.} \end{aligned} \tag{7}$$

Example. Let us consider the matrix

$$A = \begin{bmatrix} 6 & -3 & 4 & 1 \\ 4 & 2 & 4 & 0 \\ 4 & -2 & 3 & 1 \\ 4 & 2 & 3 & 1 \end{bmatrix}.$$

The sums of the elements of the principal diagonals of A, A^2, A^3, A^4 are found to be $s_1=12$, $s_2=56$, $s_3=288$, $s_4=1504$. Thus Eqs. (6) take the form

$$\begin{aligned} p_1 &= -12, \\ p_2 &= -(1/2)(12p_1 + 56) = 44, \\ p_3 &= -(1/3)(12p_2 + 56p_1 + 288) = -48, \\ p_4 &= -(1/4)(12p_3 + 56p_2 + 288p_1 + 1504) = 16, \end{aligned}$$

and we have

$$|A - I\lambda| = \lambda^4 - 12\lambda^3 + 44\lambda^2 - 48\lambda + 16.$$

b) *The method of Krylov* [7]. Even as originally formulated by Krylov, this method represents a considerable saving of effort over the method of Leverrier when the order of the determinant is greater than four. As modified by Fraser, Duncan and Collar [22], the saving is even greater. The modified form of this method is as follows.

The Cayley-Hamilton theorem [23] states that a square matrix satisfies its own characteristic equation when interpreted as a matrix equation, i.e., if

$$\lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots + p_n = 0$$

is the characteristic equation of the matrix A, then

$$A^n + p_1A^{n-1} + p_2A^{n-2} + \cdots + p_nI = 0. \tag{8}$$

If we postmultiply (8) by an arbitrary column matrix

$$C(0) = \{c_{10}, c_{20}, \cdots, c_{n0}\},$$

and define the sequence

$$\begin{aligned} C(1) &= AC(0) = \{c_{11}, c_{21}, \cdots, c_{n1}\}, \\ C(2) &= AC(1) = A^2C(0) = \{c_{12}, c_{22}, \cdots, c_{n2}\}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned}$$

we obtain the matrix equation

$$C(0)p_n + C(1)p_{n-1} + \cdots + C(n-2)p_2 + C(n-1)p_1 = -C(n).$$

This is equivalent to the set of n simultaneous linear equations in the n unknowns $p_1, p_2, \cdots, p_n$,

$$\sum_{k=0}^{n-1} c_{ik}p_{n-k} = -c_{in} \qquad (i = 1, 2, \cdots, n). \tag{9}$$

Solving this set of equations for the p_k's, we can readily write down the polynomial equation.

The formation of each of the column matrices $C(k)$ requires n^2 multiplications and $n(n-1)$ additions and subtractions. The solution of Eqs. (9) by Aitken's method [28] requires $(1/2)n^2(n+3)$ multiplications and divisions and $(n/2)(n^2-1)$ additions and subtractions. Consequently, we require

$$\begin{aligned} &(3/2)n^2(n+1) && \text{M-D}, \\ &(n/2)(n-1)(3n+1) && \text{A-S}. \end{aligned} \tag{10}$$

Krylov's original method can be shown to require

$$\begin{aligned} &(1/3)(n^4 + 4n^3 + 2n^2 - n - 3) && \text{M-D}, \\ &(1/6)n(n-1)(2n^2 + 7n - 1) && \text{A-S}. \end{aligned} \tag{11}$$

Example. Let us again consider the matrix of the previous example. We have

$$C(0) = \{1, 1, 1, 1\}, \qquad C(1) = \{8, 10, 6, 10\}, \qquad C(2) = \{52, 76, 40, 80\},$$
$$C(3) = \{324, 520, 256, 560\}, \qquad C(4) = \{1968, 3360, 1584, 3664\}.$$

Thus (9) becomes

$$\begin{aligned} p_4 + 8p_3 + 52p_2 + 324p_1 + 1968 &= 0, \\ p_4 + 10p_3 + 76p_2 + 520p_1 + 3360 &= 0, \\ p_4 + 6p_3 + 40p_2 + 256p_1 + 1584 &= 0, \\ p_4 + 10p_3 + 80p_2 + 560p_1 + 3664 &= 0, \end{aligned}$$

whence

$$p_4 = 16, \qquad p_3 = -48, \qquad p_2 = 44, \qquad p_1 = -12.$$

The characteristic equation of A is then

$$16 - 48\lambda + 44\lambda^2 - 12\lambda^3 + \lambda^4 = 0.$$

c) *The method of Danielewsky* [8]. The essence of Danielewsky's method is the transformation of the expression $D(\lambda)=|A-I\lambda|$ to the Frobenius standard form

$$D(\lambda)=\begin{vmatrix} p_1-\lambda & p_2 & p_3\cdots & p_n \\ 1 & -\lambda & 0\cdots & 0 \\ 0 & 1 & -\lambda\cdots & 0 \\ \cdot & \cdot & \cdot\cdots & \cdot \\ 0 & 0 & 0\cdots & -\lambda \end{vmatrix} = (-1)^n(\lambda^n-p_1\lambda^{n-1}-p_2\lambda^{n-2}-\cdots-p_n).$$

This gives the polynomial expansion directly.

Danielewsky starts with the $(n-1)$th element of the nth row (a constant term), reduces it to unity, and then uses this to eliminate the constant terms from the other elements of the nth row. This process introduces extraneous terms in λ in the $(n-1)$th row, which can then be removed by multiplying the other rows by appropriate constants and adding to the $(n-1)$th row. A similar procedure is then followed with the $(n-2)$th element of the $(n-1)$th row, and the reduction is continued until the standard form is reached.

This process of elimination can readily be carried out by matrix multiplications. Let us consider a matrix of order 6 which has already had two rows reduced. It is then of the form

$$C=\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and the determinant has been reduced to the form

$$D(\lambda) = |C - I\lambda|.$$

We now postmultiply the matrix C by the matrix E,

$$E=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -c_{41}/c_{43} & -c_{42}/c_{43} & 1/c_{43} & -c_{44}/c_{43} & -c_{45}/c_{43} & -c_{46}/c_{43} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

to obtain

$$C' = \begin{bmatrix} c'_{11} & c'_{12} & c'_{13} & c'_{14} & c'_{15} & c'_{16} \\ c'_{21} & c'_{22} & c'_{23} & c'_{24} & c'_{25} & c'_{26} \\ c'_{31} & c'_{32} & c'_{33} & c'_{34} & c'_{35} & c'_{36} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

After this transformation we have

$$D(\lambda)E = |C' - E\lambda|.$$

If this expression is now premultiplied by E^{-1}, we return to a form which is equal to our original expression, but one step closer to the standard form,

$$D(\lambda) = |E^{-1}C' - E^{-1}E\lambda| = |C'' - I\lambda|.$$

Fortunately, E^{-1} can be written down directly,

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This particular premultiplication changes only the third row of C', hence the determinant has been transformed to the form

$$D(\lambda) = \begin{vmatrix} c'_{11} - \lambda & c'_{12} & c'_{13} & c'_{14} & c'_{15} & c'_{16} \\ c'_{21} & c'_{22} - \lambda & c'_{23} & c'_{24} & c'_{25} & c'_{26} \\ c''_{31} & c''_{32} & c''_{33} - \lambda & c''_{34} & c''_{35} & c''_{36} \\ 0 & 0 & 1 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & -\lambda \end{vmatrix}$$

A continuation of this process will eventually yield the normal form.

If the matrix method is followed strictly, it involves an undue amount of writing. With only a slight increase in the number of operations, it can be abridged to give greater ease of calculation with a calculating machine, and to permit checking at every stage of the computation. A numerical example will best illustrate the method and the check.

Example. For the matrix of the two previous examples, the scheme of calculations will run as follows:

						Σ	Σ′
(1)		6	−3	4	1	8	
(2)		4	2	4	0	10	
(3)		4	−2	3	1	6	
(4)		4	2	*3*	1	10	
(4′)		(1.3333	0.6667	*1*	0.3333	3.3333)	
(5)	4	0.6667	−5.6667	1.3333	−0.3333	−4	−5.3333
(6)	2	−1.3333	−0.6667	1.3333	−1.3333	−2	−3.3333
(7)	3	0	−4	1	0	−3	−4
(8)	1	0	0	1	0	1	
(9)		0	*−36*	12	−4	−28	
(9′)		(0	*1*	−0.3333	0.1111	0.7778)	
(10)	0	0.6667	0.1574	−0.5556	0.2963	0.5648	0.4074
(11)	−36	−1.3333	0.01852	1.1111	−1.2593	−1.4630	−1.4815
(12)	12	0	1	0	0	1	
(13)	−4	0	0	1	0	1	
(14)		*48*	11.3333	−44	45.3333	60.6667	
(14′)		(*1*	0.2361	−0.9167	0.9444	1.2638)	
(15)	48	0.01389	0	0.05556	−0.3333	−0.2639	−0.2778
(16)	11.3333	1	0	0	0	1	
(17)	−44	0	1	0	0	1	
(18)	45.3333	0	0	1	0	1	
(19)		12	−44	48	−16	0	

Hence

$$D(\lambda) = \lambda^4 - 12\lambda^3 + 44\lambda^2 - 48\lambda + 16.$$

The explanations of this scheme are as follows. We first postmultiply the matrix whose elements are given in lines 1, 2, 3 and 4, by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4/3 & -2/3 & 1/3 & -1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is accomplished by dividing the elements of row 4 by the element in the third column, 3, yielding row 4′. The second-order minors of the unit element in the third column of row 4′ are then formed with rows 1, 2, and 3, the unit element always taking the leading position. This is easily done by writing row 4′ on a card, and forming the cross products with the various rows. These minors are entered in rows 5, 6 and 7, under the column corresponding to the elements with which the cross product is formed. The elements in column 3 are formed by dividing the corresponding elements of column 3, rows 1, 2 and 3 by the italicized element in row 4. Row 8 is immediately written as shown. Thus, the element -5.6667 in row 5, column 2, comes from $1(-3)-(0.6667)(4)$, and the element 1.333 in row 5 column 3 comes from dividing the element 4 in row 1 column 3 by 3.

We next premultiply the matrix with the elements given in rows 5, 6, 7 and 8, by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is accomplished by writing the elements of row 4 in a column on a card (here shown written to the left of rows 5, 6, 7 and 8), and forming the sum of the products of these numbers with the elements of the columns of the rows 5, 6, 7, and 8. This yields row 9, which is the transformed form of row 7 after the matrix multiplication. Since the rest of the matrix is unchanged, it is unnecessary to rewrite it.

The whole process is now repeated starting with row 9, dividing each element of that row by the italicized element (-36) to obtain row 9′. This is written on a card and used to form the cross products with rows 5 and 6, giving the elements in the first, third and fourth columns of rows 10 and 11. The elements in the second column are obtained by dividing the corresponding elements of rows 5 and 6 by -36. Rows 12 and 13 can be written down immediately as shown. The process is continued until row 19 is reached. At this stage it is unnecessary to rewrite the entire matrix, since the desired coefficients appear in only the first row, i.e., row 19. The polynomial can now be written down as shown.

The columns labelled $\sum$ and $\sum'$ are used for checking the work. The elements in the column labelled $\sum$ are obtained by summing the elements of the first four columns of that row, while those in $\sum'$ come from only three columns, omitting that column which contains the element used as the pivot for the previous set of cross multiplications. The cross products formed with the $\sum$ columns should equal the elements of the $\sum'$ column at the next stage of the transformation, e.g., the element -3.333 in row 6, column $\sum'$ comes either from adding the elements $(-1.3333)+(-0.6667)+(-1.3333)$ of row 6, or from the cross product $(1)(10)-(4)(3.3333)$. Since these give equal results, the computation of row 6 is probably correct. This check is not applicable to the row just reduced, so there is no point in forming $\sum'$ for that row, or the rows already in standard form. The accuracy of row 9, and similar rows, is checked by forming the sum of the products of the column (4, 2, 3, 1) and the elements of column $\sum$. Since this product-sum is equal to the sum of the elements of row 9, the accuracy of that row is checked. Compensating errors can occur, so the check is not absolute, but it is a great help in avoiding an accumulation of errors.

We must next consider the exceptional case in which a zero appears for the element with which we expect to divide in making the next reduction, i.e., the element one place to the left of the diagonal. The following two cases arise: (1) There is at least one element in the row under consideration which does not have a vanishing constant term. (2) All of the constant terms in the row under consideration vanish.

Case (1) can be decomposed into two subcases, according as the non-vanishing element is (a) to the left of the diagonal, (b) to the right of the diagonal. In subcase (a), we add the elements of the column containing the non-vanishing element to the column in which we wish to introduce a constant term. (This technique can also be used if the element immediately to the left of the diagonal has a fairly large tabular error, and another term farther to the left is more certain.) This will not only introduce the desired constant term, but in some row it will introduce an unwanted term in λ off the diagonal. This can be removed, however, by subtracting the appropriate row from the row containing the extraneous λ. The reduction can then go ahead as usual. In subcase (b), the determinant is immediately factorable into the product of two determinants, one of which is already in standard form.

These subcases can best be illustrated by examples. In subcase (a), let us suppose that after two reductions we reach the form

$$\begin{vmatrix} 4-\lambda & 3 & -2 & 5 & 3 \\ 1 & 2-\lambda & -1 & 4 & 1 \\ 2 & 0 & 4-\lambda & -1 & 6 \\ 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & -\lambda \end{vmatrix}.$$

If we add column 1 to column 2, we obtain

$$\begin{vmatrix} 4-\lambda & 7-\lambda & -2 & 5 & 3 \\ 1 & 3-\lambda & -1 & 4 & 1 \\ 2 & 2 & 4-\lambda & -1 & 6 \\ 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & -\lambda \end{vmatrix}.$$

This has an extraneous λ in row 1, which we can eliminate by subtracting row 2 from row 1, to obtain

$$\begin{vmatrix} 3-\lambda & 4 & -1 & 1 & 2 \\ 1 & 3-\lambda & -1 & 4 & 1 \\ 2 & 2 & 4-\lambda & -1 & 6 \\ 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & -\lambda \end{vmatrix}.$$

This is now in a form capable of treatment by the general method. In subcase (b), we might reach the form

$$\begin{vmatrix} 4-\lambda & 3 & -2 & 5 & 3 \\ 1 & 2-\lambda & -1 & 4 & 1 \\ 0 & 0 & 4-\lambda & -1 & 6 \\ 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & -\lambda \end{vmatrix},$$

which can be factored into the product

$$\begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} \begin{vmatrix} 4-\lambda & -1 & 6 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix}.$$

The determinant on the right is already in the Frobenius standard form, while that on the left can be expanded immediately, although in the general case it would have to be reduced further by the general method.

In case (2), the vanishing of all constant terms in a given row indicates that λ is a factor of $D(\lambda)$. If the rth row from the bottom has vanishing constant terms, it means that λ^r is a factor of $D(\lambda)$. The determinant for the lower degree polynomial which we have yet to determine can readily be constructed from the elements above the vanishing row. As an example, let us consider

$$\begin{vmatrix} 4-\lambda & 3 & -2 & 5 & 3 \\ 1 & 2-\lambda & -1 & 4 & 1 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & -\lambda \end{vmatrix}.$$

This is equal to

$$\begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} (-\lambda)^3.$$

In its original form, Danielewsky's method requires

$$\begin{aligned} &(n^2-2)(n-1) && \text{M-D}, \\ &n(n-1)^2 && \text{A-S}, \end{aligned} \tag{12}$$

and in the modified form given in detail above, it requires

$$\begin{aligned} &(n-1)(n^2+n-1) && \text{M-D}, \\ &n(n-1)^2 && \text{A-S}. \end{aligned} \tag{13}$$

In spite of the extra operations required, the modified form is to be preferred, because it is better adapted to routine computation with a calculating machine, and because it can be checked at each stage of the computation.

d) *Reiersøl's method.* Reiersøl [14] bases his method of obtaining the coefficients of the determinantal equation

$$D(\lambda) = |A - I\lambda| = (-1)^n(\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \cdots - p_n)$$

on the fact that the coefficients p_k can be calculated as $(-1)^{k+1}$ times the sum of all k-rowed principal minors of the matrix A. The method is powerful for low values of n, but for large n the labor is considerable.

Reiersøl uses a method for computing the principal minors of A based on the same pivotal method used by Aitken in various numerical processes dealing with determinants [28]. In the process of evaluating a determinant by Chiò's method [15], simple quotients of various minors are obtained, and the method is easily extended to give all of the principal minors of the matrix.

This form of calculation is used here since it is uniform with most of the other methods described in this paper, and requires essentially the same number of operations as the application of Reiersøl's recursion formulae. In fact, it is merely a schematization of his method.

In the process of evaluating the determinant $|A|$ by Chiò's method [15], we use the identity

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \cdot & \cdot & \cdot \quad \cdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a'_{22} & a'_{23} \cdots a'_{2n} \\ a'_{32} & a'_{33} \cdots a'_{3n} \\ \cdot & \cdot \quad \cdots \\ a'_{n2} & a'_{n3} \cdots a'_{nn} \end{vmatrix},$$

where $a'_{ij} = a_{ij} - a_{i1}a_{1j}/a_{11}$, $(i, j = 2, 3, \cdots, n)$. If we define our principal minors (minors obtained by striking out the same rows as columns) as $B_{rs\cdots w}$ $(r < s < \cdots < w)$, where $r, s, \cdots, w$ are the indices of the rows (and columns) used to form the minor, then we have

$$B_{1s} = a_{11}a'_{ss}.$$

Carrying the reduction one stage farther, we obtain

$$|A| = a_{11}a'_{22} \begin{vmatrix} a''_{33} & a''_{34} \cdots a''_{3n} \\ a''_{43} & a''_{44} \cdots a''_{4n} \\ \cdot & \cdot \quad \cdots \\ a''_{n3} & a''_{n4} \cdots a''_{nn} \end{vmatrix}$$

where, as before $a''_{ij} = a'_{ij} - a'_{i1}a'_{1j}/a'_{11}$. Then

$$B_{12t} = a_{11}a'_{22}a''_{tt},$$

and so on through the reduction. This gives all the second order principal minors of the form B_{1s}, the third order minors of form B_{12t}, the fourth order minors of the form B_{123u}, and so on, including the value of the determinant itself.

For the minors in which the first two subscripts are 1 and 3, we start with the determinant

$$a_{11} \begin{vmatrix} a'_{33} \cdots a'_{3n} \\ a'_{43} \cdots a'_{4n} \\ \cdot \quad \cdots \\ a'_{n3} \cdots a'_{nn} \end{vmatrix},$$

and carry through the pivotal reduction as before. For minors of the form $B_{2}\ldots$ we start with

$$\begin{vmatrix} a_{22} & a_{23} \cdots a_{2n} \\ a_{32} & a_{33} \cdots a_{3n} \\ \cdot & \cdot \quad \cdots \\ a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix}.$$

In this way all of the principal minors can be built up. Since we always start with a term on the principal diagonal, the algebraic sign presents no problem.

The number of operations required by this method is:

$$\begin{array}{ll} 5\cdot 2^n - (n^2 + 4n + 5) & \text{M-D}, \\ 4\cdot 2^n - (n^2 + 3n + 4) & \text{A-S}. \end{array} \tag{14}$$

Example. Let us again consider the expression

$$D(\lambda) = \begin{vmatrix} 6-\lambda & -3 & 4 & 1 \\ 4 & 2-\lambda & 4 & 0 \\ 4 & -2 & 3-\lambda & 1 \\ 4 & 2 & 3 & 1-\lambda \end{vmatrix}.$$

The scheme of calculations will run as follows:

$a_{11} = 6$	1	$-1/2$	$2/3$	$1/6$	$B_1 = 6$
	4	2	4	0	$B_2 = 2$
	4	-2	3	1	$B_3 = 3$
	4	2	3	1	$B_4 = 1$
$a'_{22} = 4$		1	$1/3$	$-1/6$	$B_{12} = 6\cdot 4 = 24$
		0	$1/3$	$1/3$	$B_{13} = 6\cdot 1/3 = 2$
		4	$1/3$	$1/3$	$B_{14} = 6\cdot 1/3 = 2$
$a''_{33} = 1/3$			1	1	$B_{123} = 6\cdot 4\cdot 1/3 = 8$
			-1	1	$B_{124} = 6\cdot 4\cdot 1 = 24$
$a'''_{44} = 2$				1	$B_{1234} = 6\cdot 4\cdot (1/3)\cdot 2 = 16$
$a_{11} = 6$					
$a'_{33} = 1/3$			1	1	
			$1/3$	$1/3$	
				0	$B_{134} = 6\cdot (1/3)\cdot 0 = 0.$

This completes all terms with 1 as the first index. Starting with the third order determinant we obtain by striking out the first row and column,

$a_{22} = 2$	1	2	0	
	-2	3	1	
	2	3	1	
$b'_{33} = 7$		1	$1/7$	$B_{23} = 2\cdot 7 = 14$
		-1	1	$B_{24} = 2\cdot 1 = 2$
$b''_{44} = 8/7$			1	$B_{234} = 2\cdot 7\cdot 8/7 = 16$

$$B_{34} = \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} = 0.$$

Hence we have

$$p_1 = (-1)^2(B_1 + B_2 + B_3 + B_4) = 12,$$
$$p_2 = (-1)^3(B_{12} + B_{13} + B_{14} + B_{23} + B_{24} + B_{34}) = -44,$$
$$p_3 = (-1)^4(B_{123} + B_{124} + B_{124} + B_{234}) = 48,$$
$$p_4 = (-1)^5 \cdot 16 = -16,$$
$$D(\lambda) = \lambda^4 - 12\lambda^3 + 44\lambda^2 - 48\lambda + 16.$$

e) *Samuelson's method.* Samuelson [13] has devised one of the fastest methods yet developed. His method requires a few more operations than Danielewsky's, but the routine involved is extremely simple.

To get the polynomial expansion of

$$D(\lambda) = |A - I\lambda| = (-1)^n(\lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \cdots - p_n),$$

we consider the differential equation

$$D(d/dt)x_1(t) = (-1)^n[x_1^{(n)}(t) - p_1x_1^{(n-1)}(t) - p_2x_1^{(n-2)}(t) - \cdots - p_nx_1(t) = 0, \tag{15}$$

where the superscripts in brackets denote derivatives with respect to t. Equation (15) can be written as a set of n simultaneous first order differential equations

$$x_i'(t) = \sum_{j=1}^{n} b_{ij}x_j(t) \qquad (i = 1, 2, \cdots, n),$$

where

$$[b_{ij}] = \begin{bmatrix} p_1 & p_2 & p_3 \cdots & p_{n-1} & p_n \\ 1 & 0 & 0 \cdots & 0 & 0 \\ 0 & 1 & 0 \cdots & 0 & 0 \\ \cdot & \cdot & \cdot \cdots & \cdot & \cdot \\ 0 & 0 & 0 \cdots & 1 & 0 \end{bmatrix}$$

is the "companion matrix" to the polynomial in question.

Actually we need a scheme to go from a system in many variables to a high order system in one variable. Samuelson accomplishes this in the following manner.

Let us consider the system

$$Ax(t) = x'(t), \tag{16}$$

where $x(t)$ is the column matrix

$$x(t) = \{x_1(t), x_2(t), \cdots, x_n(t)\}.$$

Equation (16) gives us n equations in the $2n$ variables $x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n'$. There are insufficient equations to eliminate all of the variables except those carrying the subscript 1. However we can differentiate (16) $(n-1)$ times with respect to t, obtaining the n^2 equations

$$\begin{aligned} Ax^{(n-1)}(t) &= x^{(n)}(t), \\ Ax^{(n-2)}(t) &= x^{(n-1)}(t), \\ &\vdots \\ Ax(t) &= x'(t). \end{aligned} \tag{17}$$

We now have n^2 linear equations in the n^2+n variables $(x_1, x_2, \cdots, x_n; x_1', x_2', \cdots,$

x_n'; $\cdots$; $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$). We can use all but one equation to eliminate the n^2-1 variables not involving the subscript 1, and substitute in the remaining equation to get the desired high order equation in x_1 and its derivatives.

Let us consider

$$A = \left[\begin{array}{c|c} a_{11} & a_{12} \cdots a_{1n} \\ \hline a_{21} & a_{22} \cdots a_{2n} \\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{array}\right] = \left[\begin{array}{c|c} a_{11} & R \\ \hline S & M \end{array}\right].$$

If we transfer the variables with subscript 1 to the right of (17), we can rearrange and rewrite it in the form

$$W = \left[\begin{array}{cccccc|ccccc} -I & M & 0 \cdots & 0 & 0 & & 0 & -S & 0 \cdots & 0 \\ 0 & -I & M \cdots & 0 & 0 & & 0 & 0 & -S \cdots & 0 \\ \cdots & & & & & & \cdots & & & \\ 0 & 0 & 0 \cdots & -I & M & & 0 & 0 & 0 \cdots & -S \\ 0 & 0 & 0 \cdots & 0 & R & & 0 & 0 & 0 \cdots & -a_{11} \\ 0 & 0 & 0 \cdots & R & 0 & & 0 & 0 & 0 \cdots & 0 \\ \cdots & & & & & & \cdots & & & \\ 0 & 0 & R \cdots & 0 & 0 & & 0 & 1 & -a_{11} \cdots & 0 \\ \hline 0 & R & 0 \cdots & 0 & 0 & & 1 & -a_{11} & 0 \cdots & 0 \end{array}\right]. \quad (18)$$

The elimination of the n^2-1 unwanted variables from the first n^2-1 equations and the subsequent substitution in the remaining equation can be performed by pivotal reduction [28], always using elements of the matrix on the left of (18) until a single row remains on the right.

Reduction down to the first row containing R can be made in the general form, yielding

$$\left[\begin{array}{c|cccccccc} R & 0 & 0 & 0 & 0 \cdots 0 & 0 & 1 & -a_{11} \\ RM & 0 & 0 & 0 & 0 \cdots 0 & 1 & -a_{11} & -RS \\ RM^2 & 0 & 0 & 0 & 0 \cdots 1 & -a_{11} & -RS & -RMS \\ \cdots & \cdots & & & & & & \\ RM^{n-1} & 1 & -a_{11} & -RS & -RMS \cdots & & \cdots & -RM^{n-2}S \end{array}\right]. \quad (19)$$

In practice this is the point at which to start the reduction.

To set up the matrix (19) requires

$$n(n-1)^2 \qquad \text{M-D},$$
$$n(n-1)(n-2) \qquad \text{A-S}.$$

Pivotal reduction of (19) will require

$$4n^2 - 13n + 12 \qquad \text{M-D},$$
$$4n^2 - 13n + 12 \qquad \text{A-S},$$

making totals for the method of

$$\begin{aligned} &n^3 + 2n^2 - 12n + 12 \qquad \text{M-D}, \\ &n^3 + n^2 - 11n + 12 \qquad \text{A-S}. \end{aligned} \tag{20}$$

Samuelson uses a method due to Crout for the reduction of his equations. Crout's method involves forming exactly the same products and sums as are formed in Aitken's method used above, although Crout's formulation involves somewhat less writing than the above method, but also requires keeping in mind somewhat more complicated formulae. For the average engineer or physicist, ease is fully as important as speed.

Example. We again consider the matrix

$$\left[\begin{array}{c|ccc} 6 & -3 & 4 & 1 \\ \hline 4 & 2 & 4 & 0 \\ 4 & -2 & 3 & 1 \\ 4 & 2 & 3 & 1 \end{array}\right] = \left[\begin{array}{c|c} a_{11} & R \\ \hline S & M \end{array}\right].$$

We find that $RM = [-12, 3, 5]$, $RM^2 = [-20, -24, 8]$, $RM^3 = [24, -128, -16]$, $RS = [8]$, $RMS = [-16]$, $RM^2S = [-144]$. The matrix to be reduced then becomes

$$\left[\begin{array}{ccc|ccccc} -3 & 4 & 1 & 0 & 0 & 0 & 1 & -6 \\ -12 & 3 & 5 & 0 & 0 & 1 & -6 & -8 \\ -20 & -24 & 8 & 0 & 1 & -6 & -8 & 16 \\ 24 & -128 & -16 & 1 & -6 & -8 & 16 & 144 \end{array}\right].$$

The reduction then proceeds as follows:

									Σ
-3	1	$-\frac{4}{3}$	$-\frac{1}{3}$	0	0	0	$-\frac{1}{3}$	2	1
	-12	3	5	0	0	1	-6	-8	-17
	-20	-24	8	0	1	-6	-8	16	-33
	24	-128	-16	1	-6	-8	16	144	27
-13		1	$-\frac{1}{13}$	0	0	$-\frac{1}{13}$	$\frac{10}{13}$	$-\frac{16}{13}$	$\frac{5}{13}$
			$\frac{4}{3}$	0	1	-6	$-\frac{44}{3}$	56	-13
		-96	-8	1	-6	-8	24	96	3
$-\frac{1300}{507}$			1	0	$-\frac{507}{1300}$	$\frac{5018}{1300}$	$-\frac{12324}{1300}$	$\frac{3224}{1300}$	$\frac{3289}{1300}$
			$-\frac{200}{13}$	1	-6	$-\frac{200}{13}$	$\frac{1272}{13}$	$-\frac{288}{13}$	$\frac{519}{13}$
				1	-12	44	-48	16	1

The pivotal element for each succeeding reduction is made equal to unity by dividing that row by the value of that element. The column marked $\sum$ is used as a check. For any stage of the reduction, the cross products are formed using the $\sum$ column as if it were any other column, and the values entered as usual. These values should equal the sum of the elements in the row in which they appear. The check is not absolute, but it is very useful.

3. Methods applicable to the case $|A - B\lambda| = 0$. -a) *The method of reciprocation.* The equation

$$|A - B\lambda| = 0$$

has the same roots as the equation [24]

$$|B^{-1}A - I\lambda| = 0, \tag{21}$$

where B^{-1} is the reciprocal of B, provided only that B is not singular. The matrix product $B^{-1}A$ can readily be formed by Aitken's method [28], and the determinant (21), which contains λ's only along the principal diagonal, can then be expanded by one of the methods of §2.

The formation of the product $B^{-1}A$ requires

$$(n^2/2)(3n - 1) \qquad \text{M-D,}$$
$$(n/2)(n - 1)(3n - 1) \qquad \text{A-S.}$$

Using the modified Danielewsky method to obtain the polynomial form, we shall need all told

$$\begin{matrix} (1/2)(n + 1)(5n^2 - 6n + 2) & \text{M-D,} \\ (n/2)(n - 1)(5n - 3) & \text{A-S.} \end{matrix} \tag{22}$$

Example. Let us consider the determinant

$$D(\lambda) = \begin{vmatrix} -9+2\lambda & -8+3\lambda & -7+\lambda & -7+2\lambda \\ 15-3\lambda & 16-5\lambda & 13-2\lambda & 15-4\lambda \\ -8+\lambda & -8+2\lambda & -7+2\lambda & -8+3\lambda \\ 23-3\lambda & 24-5\lambda & 19-3\lambda & 22-6\lambda \end{vmatrix}.$$

We have

$$A = \begin{bmatrix} -9 & -8 & -7 & -7 \\ 15 & 16 & 13 & 15 \\ -8 & -8 & -7 & -8 \\ 23 & 24 & 19 & 22 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 & 1 & 2 \\ -3 & -5 & -2 & -4 \\ 1 & 2 & 2 & 3 \\ -3 & -5 & -3 & -6 \end{bmatrix},$$

$$B^{-1}A = \begin{bmatrix} -8 & 0 & -2 & 3 \\ 5 & 0 & 1 & -2 \\ 4 & 8 & 4 & 7 \\ -6 & -8 & -5 & -7 \end{bmatrix}, \qquad f(\lambda) = \begin{vmatrix} -8-\lambda & 0 & -2 & 3 \\ 5 & -\lambda & 1 & -2 \\ 4 & 8 & 4-\lambda & 7 \\ -6 & -8 & -5 & -7-\lambda \end{vmatrix},$$

$$B(B^{-1}A) = \begin{bmatrix} 2 & 3 & 1 & 2 \\ -3 & -5 & -2 & -4 \\ 1 & 2 & 2 & 3 \\ -3 & -5 & -3 & -6 \end{bmatrix} \begin{bmatrix} -8 & 0 & -2 & 3 \\ 5 & 0 & 1 & -2 \\ 4 & 8 & 4 & 7 \\ -6 & -8 & -5 & -7 \end{bmatrix} = \begin{bmatrix} -9 & -8 & -7 & -7 \\ 15 & 16 & 13 & 15 \\ -8 & -8 & -7 & -8 \\ 23 & 24 & 19 & 22 \end{bmatrix} = A.$$

Thus

$$f(\lambda) = \lambda^4 + 11\lambda^3 + 33\lambda^2 + 8\lambda + 8.$$

The calculation of B^{-1} can be carried through conveniently by means of Aitken's method [28] of obtaining the reciprocal of a matrix. The product $B(B^{-1}A)$ can be formed as a check.

b) *The Danielewsky-Masuyama method.* An extension of Danielewsky's method of transforming a λ determinant to the Frobenius standard form has been made by Masuyama [6] for the case $|A - B\lambda| = 0$. This method requires

$$\begin{aligned} &(1/24)n(n-1)(7n^2 + 13n + 66) && \text{M-D,} \\ &(1/24)(n-1)(7n^3 + 5n^2 + 58n - 48) && \text{A-S.} \end{aligned} \tag{23}$$

For low orders (through the fifth) this represents a small saving in the number of operations over the method given in §3a, but it is not as well adapted to machine computation. It is applicable, however, to the case in which the determinant of the matrix B vanishes, but in this case the method of §4b, using Newton's interpolation formula, is to be preferred. For these reasons, we shall not consider the method in more detail here.

4. Methods applicable to the case $|A_0\lambda^n + A_1\lambda^{n-1} + \cdots + A_n| = 0$. -a) *Transformation to the form* $|A - I\lambda| = 0$. It is possible to transform an mth order determinant, the terms of which are polynomials at most of degree n in λ, into a determinant of order mn with terms linear in λ, provided that the matrix of the coefficients of λ^n is not singular. This can be done in more than one way, but the following seems most convenient [25].

If the determinant we wish to transform is

$$D(\lambda) = |A_0\lambda^n + A_1\lambda^{n-1} + \cdots + A_{n-1}\lambda + A_n| = 0, \tag{24}$$

we consider the related set of simultaneous linear differential equations

$$(A_0D^n + A_1D^{n-1} + \cdots + A_{n-1}D + A_n)x = 0 \qquad (D = d/dt), \tag{25}$$

where x is the column matrix

$$x = \{x_1, x_2, \cdots, x_n\},$$

and

$$|A_0| \neq 0. \tag{26}$$

Because of the condition (26) the reciprocal matrix A_0^{-1} exists. Consequently we can premultiply Eq. (25) by A_0^{-1}, obtaining

$$(I_mD^n + A_0^{-1}A_1D^{n-1} + \cdots + A_0^{-1}A_{n-1}D + A_0^{-1}A_n)x = 0, \tag{27}$$

where I_m is a square matrix of order m with units on the principal diagonal and zeroes everywhere else. We can write (27) in the form

$$I_mDx^{(n-1)} + A_0^{-1}A_1x^{(n-1)} + A_0^{-1}A_2x^{(n-2)} + \cdots + A_0^{-1}A_{n-1}x^{(1)} + A_0^{-1}A_nx^{(0)} = 0, \tag{28}$$

where the superscripts in brackets denote derivatives with respect to t. Equation (28) represents a set of m simultaneous differential equations. There are n of the column matrices $x^{(k)}$. Thus to make the set equivalent to Eq. (27) we need the $m(n-1)$ additional equations of the form

$$x^{(r)} = Dx^{(r-1)} = D^2x^{(r-2)} = \cdots = D^r x^{(0)} \qquad (r = 1, 2, \cdots, n-1). \tag{29}$$

If in Eqs. (28), (29) we now set

$$x^{(k)} = q^{(k)} e^{-\lambda t} = \{q_1^{(k)}, q_2^{(k)}, \cdots, q_m^{(k)}\} e^{-\lambda t} \qquad (k = 0, 1, 2, \cdots, n-1),$$

we get the following set of mn homogeneous simultaneous linear equations in the q's:

$$\begin{aligned}
(A_0^{-1}A_1 - I_m\lambda)q^{(n-1)} + A_0^{-1}A_2q^{(n-2)} + \cdots + A_0^{-1}A_{n-1}q^{(1)} + A_0^{-1}A_nq^{(0)} &= 0,\\
- I_mq^{(n-1)} \qquad - I_mq^{(n-2)}\lambda \qquad\qquad &= 0,\\
- I_mq^{(n-2)} - I_mq^{(n-3)}\lambda \qquad &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\\
- I_mq^{(1)} - I_mq^{(0)}\lambda &= 0.
\end{aligned}$$

The condition of compatability for this set of equations is the vanishing of the compound determinant

$$\begin{vmatrix}
A_0^{-1}A_1 - I_m\lambda & A_0^{-1}A_2 \cdots & A_0^{-1}A_{n-1} & A_0^{-1}A_n \\
- I_m & - I_m\lambda \cdots & 0 & 0 \\
0 & - I_m \cdots & 0 & 0 \\
0 & 0 \cdots & - I_m & - I_m\lambda
\end{vmatrix}.$$

Since the λ's appear only along the principal diagonal, this can be expanded into polynomial form by any of the methods of §2.

This method involves the calculation of the reciprocal of A_0 and the formation of n matrix products of the form $A^{-1}B$. This can be done by Aitken's method [28]. The special form of the expression makes the expansion to polynomial form by Danielewsky's method considerably faster than in the general case.

In the quadratic case, which is the most general usually met with in physical problems, the transformation to diagonal form requires

$$3m^3 \qquad \text{M-D,}$$
$$m(m-1)(3m-1) \qquad \text{A-S,}$$

while the reduction to polynomial form requires

$$2(3m-1)(m^2-1) \qquad \text{M-D,}$$
$$m(6m^2-9m+5)+2 \qquad \text{A-S.}$$

The total number of operations in the quadratic case will be

$$\begin{aligned} &9m^3 - 2m^2 - 6m + 2 \qquad \text{M-D,}\\ &9m^3 - 13m^2 + 6m + 2 \qquad \text{A-S.}\end{aligned} \tag{30}$$

Example. Let us consider the determinant

$$\begin{vmatrix} \lambda^3+5\lambda^2+2\lambda+3 & -\lambda^3+2\lambda^2-\lambda-4 & 4\lambda^3+\lambda^2+3 \\ -\lambda^3+2 & \lambda^2+\lambda & 5\lambda^3+4\lambda^2+3\lambda \\ 5\lambda^3-4\lambda^2-\lambda+1 & 2\lambda^3+3\lambda^2+\lambda+2 & 3\lambda^3-5\lambda^2+4\lambda+4 \end{vmatrix} = 0.$$

We have

$$A_0 = \begin{bmatrix} 1 & -1 & 4 \\ -1 & 0 & 5 \\ 5 & 2 & 3 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 1 & 4 \\ -4 & 3 & -5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 3 \\ -1 & 1 & 4 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 3 & 4 & 3 \\ 2 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix},$$

$$A_0^{-1} = \frac{1}{46}\begin{bmatrix} 10 & -11 & 5 \\ -28 & 17 & 9 \\ 2 & 7 & 1 \end{bmatrix}, \qquad A_0^{-1}A_1 = \frac{1}{46}\begin{bmatrix} 30 & 24 & -59 \\ -176 & -12 & -5 \\ 6 & 14 & 25 \end{bmatrix},$$

$$A_0^{-1}A_2 = \frac{1}{46}\begin{bmatrix} 15 & -16 & -13 \\ -65 & 54 & 87 \\ 3 & 6 & 25 \end{bmatrix}, \qquad A_0^{-1}A_3 = \frac{1}{46}\begin{bmatrix} 13 & 50 & 50 \\ -41 & -94 & -48 \\ 21 & 10 & 10 \end{bmatrix}.$$

Consequently we can write our determinantal equation in the diagonal form $|A - I\lambda|$, where A is the square matrix

$$\begin{bmatrix} 30/46 & 24/46 & -59/46 & 15/46 & -16/46 & -13/46 & 13/46 & 50/46 & 50/46 \\ -176/46 & -12/46 & -5/46 & -65/46 & 54/46 & 87/46 & -41/46 & -94/46 & -48/46 \\ 6/46 & 14/46 & 25/46 & 3/46 & 6/46 & 25/46 & 21/46 & 10/46 & 10/46 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

b) *Interpolation-formula method.* The mth order determinant (24) will, on expansion, give a polynomial of degree not greater than $r = mn$ in λ. Such a polynomial will contain $(r+1)$ numerical coefficients of the various powers of λ, and if we evaluate $D(\lambda)$ for $(r+1)$ numerical values of λ, we will have sufficient data to determine the coefficients. This can be done either by using an interpolation formula or by solving a set of simultaneous linear equations for the coefficients.

If the successive values chosen for λ differ by a constant difference, the Gregory-Newton interpolation formula is well adapted to obtaining the polynomial expansion. This requires the formation of a difference table. However, the difference table is easy to compute, and in addition it indicates the order of polynomial to expect, which is particularly useful when the matrix of the coefficients of λ^n is singular ($|A_0| = 0$).

The Gregory-Newton interpolation formula is usually written in the form [26]

$$D(\lambda) = D(a) + x\Delta D(a) + \frac{x(x-1)}{2!}\Delta^2 D(a) + \cdots + \frac{x(x-1)\cdots(x-r+1)}{r!}\Delta^r D(a), \tag{31}$$

where $\lambda = a + xh$, and the difference functions $\Delta^r D(a)$ are formed as follows:

$$\begin{array}{lllll}
D(a) & & & & \\
 & \Delta D(a) & & & \\
D(a+h) & & \Delta^2 D(a) & & \\
 & \Delta D(a+h) & & \Delta^3 D(a) & \\
D(a+2h) & & \Delta^2 D(a+h) & & \cdot \\
 & \Delta D(a+2h) & & \cdot & \\
D(a+3h) & & \cdot & & \cdot \\
 & \cdot & & \cdot & \\
\cdot & & \cdot & & \cdot \\
 & \cdot & & \cdot & \\
\cdot & & \cdot & & \\
 & \cdot & & & \\
D(a+rh) & & & &
\end{array}$$

where $\Delta D(a) = D(a+h) - D(a)$, $\Delta^2 D(a) = \Delta D(a+h) - \Delta D(a)$, and so on. The kth differences of a kth degree polynomial will be constant, so this gives us immediately the degree of the polynomial. It will always pay to calculate one or two extra values of the determinant to check the constancy of the last differences as a check on one's work.

Equation (31) does not yield the polynomial form directly, since each term is a polynomial in λ. The transformation to descending powers in λ can be made once for all, so that the polynomial form can readily be calculated once a difference table is worked out [11]. We wish to put $D(\lambda)$ in the form

$$D(\lambda) = \sum_{t=0}^{r} p_t(\lambda - a)^t = p_0 + p_1(\lambda - a) + \cdots + p_r(\lambda - a)^r. \tag{32}$$

This can be done by using the relation

$$\left.\begin{aligned}
p_t &= (1/h^t)\sum_{s=t}^{r}\frac{\Delta^s D(a) a_t(s)}{s!} = (1/h^t)\sum_{s=t}^{r} b_t(s)\Delta^s D(a) \qquad (t = 1, 2, \cdots, n),\\
p_0 &= D(a),
\end{aligned}\right\} \tag{33}$$

where the terms $a_t(s)$ are defined by the equation

$$s!\frac{r!}{s!(r-s)!} = \sum_{k=1}^{s} a_k(s)r^k.$$

For the purpose of calculation it is easier to use the recursion relation

$$a_k(s+1) = a_{k-1}(s) - sa_k(s).$$

Table I gives numerical values of the function

$$b_k(s) = a_k(s)/s!$$

which, when used with the tabular differences and Eq. (33) will give the polynomial equation very quickly in the form (32). If a has not been chosen equal to zero, the equation can readily be rearranged in descending powers of λ, instead of $\lambda - a$, by synthetic division.

TABLE I. $b_k(s) = a_k(s)/s!$

s	$k=$ 1	2	3	4	5
1	1.00000 00000				
2	−.50000 00000	+.50000 00000			
3	+.33333 33333	−.50000 00000	+.16666 66667		
4	−.25000 00000	+.45833 33333	−.25000 00000	+.04166 66667	
5	+.20000 00000	−.41666 66667	+.29166 66667	−.08333 33333	+.008333 33333
6	−.16666 66667	+.38055 55556	−.31250 00000	+.11805 55556	−.020833 33333
7	+.14285 71429	−.35000 00000	+.32222 22222	−.14583 33333	+.034722 22222
8	−.12500 00000	+.32410 71429	−.32569 44444	+.16788 19444	−.048611 11111
9	+.11111 11111	−.30198 41270	+.32551 80776	−.18541 66667	+.061863 42593
10	−.10000 00000	+.28289 68245	−.32316 46825	+.19942 68078	−.074218 75000

s	$k=$ 6	7	8	9	10
1					
2					
3					
4					
5					
6	+.0013888 88889				
7	−.0041666 66667	+.0001984 12698			
8	+.0079861 11111	−.0006944 44444	+.00002480 158730		
9	−.0125000 00000	+.0015046 29630	−.00009920 634921	+.00000275 573192	
10	+.0174363 42593	−.0026041 66667	+.00023974 867725	−.00001240 079365	+.00000027 55731922

If the successive values chosen for λ do not differ by a constant quantity, the Newton interpolation formula cannot be used. The Lagrange interpolation formula [27] is available, but the method of undetermined coefficients will usually prove more satisfactory.

Both the interpolation formula method and the method of undetermined coefficients require, first of all, the evaluation of $r+1$ numerical determinants of the mth order. This will require

$$(r+1)(1/3)(m-1)(m^2+m+3) \qquad \text{M-D},$$

$$(r+1)(1/6)m(m-1)(2m-1) \qquad \text{A-S}.$$

Each numerical determinant will require the evaluation of m^2 polynomials of degree n, each evaluation requiring n multiplications and n additions and subtractions, if we use the method of synthetic division. If we assume that 0 is chosen as one value of λ, so that we need evaluate the elements of only r determinants, the total number of operations to obtain the $r+1$ values of $D(\lambda)$ will be

$$\begin{aligned} &mr^2 + (r+1)(1/3)(m-1)(m^2+m+3) && \text{M-D,} \\ &mr^2 + (r+1)(1/6)m(m-1)(2m-1) && \text{A-S.} \end{aligned} \tag{34}$$

The interpolation formula method will require an additional $(\frac{1}{2}r)(r+1)$ subtractions to form the difference table, and $(\frac{1}{2}r)(r+1)$ multiplications and $(\frac{1}{2}r)(r-1)$ additions and subtractions to form the coefficients, if the difference between successive assumed values of λ is 1. If not, we shall need an additional r divisions by powers of h, and $r-1$ multiplications to form the powers of h. For the case $h=1$, the total for the interpolation method (including the evaluation of the numerical determinants) is

$$\begin{aligned} &\tfrac{1}{2}(2m+1)r^2 + (r/6)(2m^3+4m-3) + (1/3)(m^3+2m-3) && \text{M-D,} \\ &(m+1)r^2 + (r+1)(m/6)(m-1)(2m-1) && \text{A-S.} \end{aligned} \tag{35}$$

Example. Let us consider the determinant

$$D(\lambda) = \begin{vmatrix} 4 - 3\lambda + \lambda^2 + \lambda^3 & 2 - \lambda + 3\lambda^2 - 2\lambda^3 \\ -2 + \lambda - \lambda^2 + 4\lambda^3 & 4 + 2\lambda - 2\lambda^2 - 8\lambda^3 \end{vmatrix}.$$

	Δ^0	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
$D(0)$	20						
		−36					
$D(1)$	−16		−476				
		−512		−2628			
$D(2)$	−528		−3104		−4968		
		−3616		−7596		−2880	
$D(3)$	−4144		−10700		−7848		0
		−14316		−15444		−2880	
$D(4)$	−18460		−26144		−10728		0
		−40460		−26172		−2880	
$D(5)$	−58920		−52316		−13608		
		−92776		−39780			
$D(6)$	−151696		−92096				
		−184872					
$D(7)$	−336568						

The accompanying difference table shows that the polynomial will be of the fifth degree instead of the sixth. An investigation of the determinant of the coefficients of λ^3 shows that it vanishes; hence we should expect the polynomial to be of degree less than six.

Since we have taken $a=0$ and $h=1$, Eqs. (32) and (33) will simplify to

$$D(\lambda) = p_0 + p_1\lambda + p_2\lambda^2 + p_3\lambda^3 + p_4\lambda^4 + p_5\lambda^5,$$

$$p_t = \sum_{s=t}^{5} \Delta^s D(0) b_t(s).$$

If we now write the differences in a column, such that they are spaced the same as the elements of Table I, we can readily calculate the coefficients p_t. Placing this column beside the first column of the table, so that $\Delta D(0)$ is opposite $b_1(1)$, and $\Delta^5 D(0)$ is opposite $b_1(5)$, we multiply across and add. This gives p_1. (If we were dealing with a general case in which $h \neq 1$, we should have to divide by h.) Moving to the next column, keeping the same relative vertical position, the operation is repeated to get p_2. (In the general case we would divide by h^2.) Blanks in the table are treated as zeros. The calculation runs as follows (the individual products would not normally be written down, but merely accumulated in the calculating machine):

s	$b_1(s)$	$\Delta^s D(0)$	$[b_1(s)\Delta^s D(0)]$	$b_2(s)$	$\Delta^s D(0)$	$[b_2(s)\Delta^s D(0)]$
1	1.00000	−36	−36		−36	0
2	−0.50000	−476	238	0.50000	−476	−238
3	0.33333	−2628	−876	−0.50000	−2628	1314
4	−0.25000	−4968	1242	0.45833	−4968	−2277
5	0.20000	−2880	−576	−0.41667	−2880	1200
			$p_1 = -8$			$p_2 = -1$

Similarly $p_3 = -36$, $p_4 = 33$, $p_5 = -24$, and $p_0 = D(0) = 20$. Hence $D(\lambda) = 20 - 8\lambda - \lambda^2 - 36\lambda^3 + 33\lambda^4 - 24\lambda^5$.

c) *Method of undetermined coefficients.* We assume an expansion of the form

$$D(\lambda) = p_0 + p_1\lambda + p_2\lambda^2 + \cdots + p_r\lambda^r.$$

If we assign $r+1$ values to λ, we will obtain $r+1$ simultaneous linear equations for the determination of the p's: (If we use $\lambda=0$ as one value, $D(0)=p_0$, so we need solve only r equations.)

$$p_0 + p_1\lambda_i + p_2\lambda_i^2 + \cdots + p_r\lambda_i^r = D(\lambda_i) \qquad (i = 1, 2, \cdots, r+1).$$

This set of equations can then be solved by Aitken's method [28].

In addition to the number of operations (34) for the evaluation of the numerical determinants, the method of undetermined coefficients will require

$$\tfrac{1}{2}r^2(r+3) \qquad \text{M-D},$$
$$\tfrac{1}{2}r(r^2-1) \qquad \text{A-S},$$

to solve the simultaneous linear equations. This makes a total for this method of

$$\begin{aligned} &\tfrac{1}{2}r^2(r+3) + mr^2 + (r+1)(\tfrac{1}{3})(m^3+2m-3) && \text{M-D}, \\ &\tfrac{1}{2}r(r^2-1) + mr^2 + (r+1)\left(\frac{m}{6}\right)(m-1)(2m-1) && \text{A-S}. \end{aligned} \tag{36}$$

It is apparent that the interpolation formula method requires fewer operations than the method of undetermined coefficients, but if irregularly spaced values of the determinants are already at hand, it is better to finish the expansion by the solution of the simultaneous equations than to have to evaluate several new numerical determinants.

In the special case in which the interval between successive values of λ is unity, the set of simultaneous equations can be written in the form

$$AP = \begin{bmatrix} 1 & 0 & 0 \cdots 0 \\ 1 & 1 & 1 \cdots 1 \\ 1 & 2 & 4 \cdots 2^r \\ 1 & 3 & 9 \cdots 3^r \\ \cdot & \cdot & \cdot \cdots \cdot \\ r^0 & r^1 & r^2 \cdots r^r \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ \cdot \\ p_r \end{bmatrix} = \begin{bmatrix} D(0) \\ D(1) \\ D(2) \\ D(3) \\ \cdot \\ D(r) \end{bmatrix}.$$

Since the matrix A is independent of the numerical values of $D(\lambda)$, we can calculate the reciprocal matrix A^{-1} once for all, and then the solution of such a set of equations reduces to the formation of the matrix product

$$\{p_0, p_1, \cdots, p_r\} = A^{-1}\{D(0), D(1), \cdots, D(r)\}.$$

The matrix multiplicaton on the right will require

$$r(r+1) \qquad \text{M-D,}$$
$$r^2 \qquad \text{A-S.}$$

(The first row of A^{-1} will be $[1\ 0\ 0\ \cdots\ 0]$, since $p_0 = D(0)$.) Comparing these figures with the number of operations required for the interpolation formula method, it is seen that the latter method will require the same number of additions and subtractions, but $(\frac{1}{2}r)(r+1)$ fewer multiplications and divisions. Consequently it has not seemed worth while to tabulate the matrices A^{-1} in this paper, especially when we take into consideration the other advantages of the interpolation method, such as the information obtainable from the difference table.

Example. Let us consider the determinant

$$D(\lambda) = \begin{vmatrix} 4 - 3\lambda + \lambda^2 + \lambda^3 & 2 - \lambda + 3\lambda^2 - 2\lambda^3 \\ -2 + \lambda - \lambda^2 + 4\lambda^3 & 4 + 2\lambda - 2\lambda^2 - 8\lambda^3 \end{vmatrix}.$$

We have

$$\begin{aligned}
D(0) &= \quad 20 = p_0, \\
D(1) &= -16 = p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6, \\
D(2) &= -528 = p_0 + 2p_1 + 4p_2 + 8p_3 + 16p_4 + 32p_5 + 64p_6, \\
D(3) &= -4144 = p_0 + 3p_1 + 9p_2 + 27p_3 + 81p_4 + 243p_5 + 729p_6, \\
D(4) &= -18460 = p_0 + 4p_1 + 16p_2 + 64p_3 + 256p_4 + 1024p_5 + 4096p_6, \\
D(5) &= -58920 = p_0 + 5p_1 + 25p_2 + 125p_3 + 625p_4 + 3125p_5 + 15625p_6, \\
D(6) &= -151696 = p_0 + 6p_1 + 36p_2 + 216p_3 + 1296p_4 + 7776p_5 + 46656p_6.
\end{aligned}$$

The solution of this set of equations will give the same results as were obtained by use of the interpolation formula, but will require somewhat more labor. p_6 vanishes, as can easily be seen by substituting the column of constants for the p_6 column in the determinant of the coefficients, yielding a vanishing determinant and indicating that the polynomial is of fifth degree.

5. A comparison of methods for obtaining the polynomial expansion. -a) *Methods applicable to the case* $|A - I\lambda| = 0$. In choosing the best method for the expansion of a λ determinant into polynomial form, it is necessary to consider direct expansion for the lower orders. The number of operations required to reach the polynomial form by successive expansion in terms of minors of a row or column is given by the recursion relations

$$\begin{aligned} M(n) &= nM(n-1) + n(n-1), \qquad M(2) = 2, \\ A(n) &= nA(n-1) + 2(n-1), \qquad A(2) = 2, \end{aligned} \tag{37}$$

where $M(n)$ represents the number of multiplications required to expand an nth order determinant of the form $|A - I\lambda|$, and $A(n)$ the number of additions and subtractions for such an expansion.

We must also consider the application to this case of the interpolation formula method (§4b) and the method of undetermined coefficients (§4c). Since the unknown appears only on the principal diagonal with unit coefficients, the evaluation of the elements of a determinant of nth order will require only n subtractions. Consequently the evaluation of the $n+1$ numerical determinants necessary for either of these methods will require (assuming $\lambda = 0$ as one value used)

$$(1/3)(n^2 - 1)(n^2 + n + 3) \qquad \text{M-D,}$$

$$(1/6)n(n^2 - 1)(2n - 1) + n^2 \qquad \text{A-S.}$$

TABLE II

Eq.	Method	Mult. and Div.	Add. and Sub.
37	Direct Expansion	$M(n) = nM(n-1) + n(n-1)$ $M(2) = 2$	$A(n) = nA(n-1) + 2(n-1)$ $A(2) = 2$
38	Interpolation Formula	$(1/6)(2n^4+2n^3+7n^2+n-6)$	$(n/6)(2n^3-n^2+10n+1)$
39	Undetermined Coefficients	$(1/6)(2n^4+5n^3+13n^2-2n-6)$	$(n/3)(n^3+n^2+2n-1)$
7	Leverrier	$(1/2)(n-1)(2n^3-2n^2+n+2)$	$(n/2)(n-1)(2n^2-4n+3)$
11	Krylov	$(1/3)(n^4+4n^3+2n^2-n-3)$	$(n/6)(n-1)(2n^2+7n-1)$
10	Modified Krylov	$(3/2)n^2(n+1)$	$(n/2)(n-1)(3n+1)$
12	Danielewsky	$(n^2-2)(n-1)$	$n(n-1)^2$
13	Modified Danielewsky	$(n-1)(n^2+n-1)$	$n(n-1)^2$
14	Reiersøl	$(5)(2^n)-(n^2+4n+5)$	$(4)(2^n)-(n^2+3n+4)$
20	Samuelson	$n^3+2n^2-12n+12$	$n^3+n^2-11n+12$

For the case in which $h=1$, the additional operations required to form the difference table and the coefficients by the interpolation formula method run the total to

$$\begin{aligned} &(1/6)(2n^4 + 2n^3 + 7n^2 + n - 6) && \text{M-D,} \\ &(n/6)(2n^3 - n^2 + 10n + 1) && \text{A-S.} \end{aligned} \tag{38}$$

The totals for the method of undetermined coefficients are

$$\begin{aligned} &(1/6)(2n^4 + 5n^3 + 13n^2 - 2n - 6) && \text{M-D,} \\ &(n/3)(n^3 + n^2 + 2n - 1) && \text{A-S.} \end{aligned} \tag{39}$$

For ease of comparison, the expressions for the number of operations required for each of the methods discussed are placed together in Table II. The relative efficiencies of the methods can be seen better from Table III, which gives the actual number of operations required by each method for several orders of determinants.

TABLE III

Order	2		3		4		5		7		9	
Method	M-D	A-S	M-D	A-S	M-D	A-S	M-D	A-S	M-D	A-S	M-D	A-S
Direct Expansion	2	2	12	10	60	46	320	238	13692	10078	986400	725758
Interp. Formula	12	11	46	38	125	102	279	230	972	826	2525	2202
Undet. Coeffi.	19	10	67	41	171	116	364	265	1189	945	2966	2481
Leverrier	6	3	41	27	153	114	414	330	1791	1533	5228	4644
Krylov	17	7	67	38	179	118	389	280	1287	1022	3209	2688
Modified Krylov	18	7	54	30	120	78	225	160	588	462	1215	1008
Danielewsky	2	2	14	12	42	36	92	80	282	252	632	576
Modified Danielewsky	5	2	22	12	57	36	116	80	330	252	712	576
Reiersøl	3	2	14	10	43	32	110	84	558	438	2438	1936
Samuelson	4	2	21	15	62	48	127	107	369	327	795	723

Direct expansion proves to be as fast as any method for the second and third order cases. Even in the fourth order case none of the methods gives a sufficient saving over direct expansion to warrant learning a new technique if only a few equations are to be solved. Danielewsky's method requires the fewest operations from the fifth order up, but it is really harder to use than any of the last three methods, which are about equal in the fifth order case. Above the fifth order, the two most efficient methods are Samuelson's and the Modified Danielewsky. The one to be used will depend a great deal on the habits of the computer.

If we have already started solving the secular equation by the matrix iteration method, we will have computed all or part of the sequence $C(k)$ of Krylov's method, and it will usually be quicker to complete the polynomial expansion by that method.

If we have already evaluated $D(\lambda)$ for several values of λ, as we might do in hunting for a root by the method of false position, it would be preferable to finish by the interpolation formula method or the method of undetermined coefficients, depending on whether the successive values of λ are uniformly spaced or not.

If a machine were available on which matrices could be multiplied with ease, Leverrier's method would be useful, since the set of simultaneous equations for the coefficients is so simple.

b) *Methods applicable to the case* $|A-B\lambda|=0$. The number of operations required for direct expansion of the determinant $|A-B\lambda|$ is given by the recursion relations

$$\begin{aligned} M(n) &= nM(n-1) + 2n^2 \qquad & M(1) &= 0, \\ A(n) &= nA(n-1) + (n-1)(2n+1) \qquad & A(1) &= 0. \end{aligned} \tag{40}$$

The various methods applicable to this case are compared in Tables IV and V.

TABLE IV

Eq.	Method	Mult. and Div.	Add. and Sub.
40	Direct Expansion	$M(n)=nM(n-1)+2n^2$	$A(n)=nA(n-1)+(n-1)(2n+1)$
35	Interpolation Formula	$(1/6)(2n^4+8n^3+7n^2+n-6)$	$(n/6)(n+1)(2n^2+3n+1)$
36	Undetermined Coefficients	$(1/6)(2n^4+11n^3+13n^2-2n-6)$	$(n/3)(n^3+4n^2-n-1)$
22	Reciprocation	$(1/2)(n+1)(5n^2-6n+2)$	$(n/2)(n-1)(5n-3)$
23	Danielewsky-Masuyama	$(1/24)n(n-1)(7n^2+13n+66)$	$(1/24)(n-1)(7n^3+5n^2+58n-48)$

TABLE V

Order	2		3		4		5		7		9	
Method	M-D	A-S	M-D	A-S	M-D	A-S	M-D	A-S	M-D	A-S	M-D	A -S
Direct Expansion	8	5	42	29	200	143	1050	759	44702	32423	3219858	2335679
Interp. Formula	20	15	73	56	189	150	404	330	1315	1120	3254	2850
Undet. Coeff.	27	14	94	59	235	164	489	365	1532	1239	3695	3129
Reciprocation	15	7	58	36	145	102	291	220	820	672	1765	1512
Danielewsky-Masuyama	10	6	42	30	115	89	255	207	875	751	2250	1994

Again direct expansion is to be preferred for second and third order cases. The Danielewsky-Masuyama method requires the fewest operations in the fourth and fifth orders, but because of the large amount of writing necessary with this method, the author prefers the method of reciprocation for all cases from the fourth order up.

In case the determinant $|B|$ vanishes, the interpolation formula method is to be preferred except in the case in which one has already obtained several values of $|A-B\lambda|$ for unequally spaced values of λ, in which case the method of undetermined coefficients is preferable.

c) *Methods applicable to the case* $|A+B\lambda+C\lambda^2|$. If m is the order of the determinant, direct expansion will require the number of operations given by the recursion relations

$$\begin{aligned} M(m) &= mM(m-1) + 3m(2m-1), \qquad & M(1) &= 0, \\ A(m) &= mA(m-1) + (m-1)(6m+1), \qquad & A(1) &= 0. \end{aligned} \tag{41}$$

The various methods applicable to this case are compared in Tables VI and VII.

TABLE VI

Eq.	Method	Mult. and Div.	Add. and Sub.
41	Direct Expansion	$M(m)=mM(m-1)+3m(2m-1)$ $M(1)=0$	$A(m)=mA(m-1)+(m-1)(6m+1)$ $A(1)=0$
30	Transformation	$9m^3-2m^2-6m+2$	$9m^3-13m^2+6m+2$
35	Interpolation Formula	$(1/3)(2m^4+13m^3+10m^2-m-3)$	$(m/6)(4m^3+20m^2+23m+1)$
36	Undetermined Coefficients	$(1/3)(2m^4+25m^3+22m^2-4m-3)$	$(m/6)(4m^3+44m^2-m-5)$

TABLE VII

Order	2		3		4		5		6		7	
Method	M-D	A-S	M-D	A-S	M-D	A-S	M-D	A-S	M-D	A-S	M-D	A-S
Direct Expansion	18	13	99	77	480	383	2535	2039	15408	12419	108129	87191
Trans.	54	34	209	146	522	394	1047	832	1834	1514	2949	2494
Interp. Formula	57	53	199	179	499	446	1039	930	1917	1723	3247	2933
Undet. Coeff.	103	67	340	248	815	634	1634	1325	2919	2437	4808	4102

In this case, direct expansion requires the smallest number of operations through the fourth order, the interpolation method the smallest number for the fifth order, and the method of transformation to the diagonal form the smallest number for all higher orders. The interpolation formula method is convenient, however, even when it requires some extra operations, as it makes possible a preliminary plot of the function. The interpolation formula method is the best method above the fourth order when the determinant $|C|$ vanishes, as the difference table gives the order of polynomial to be expected.

The method of undetermined coefficients is useful primarily when several values of the determinant have already been calculated for unevenly spaced values of λ.

6. **Errors.** Errors of considerable magnitude can readily occur in the various processes described in this paper, due both to errors in the original data and errors caused by rounding off numbers. The study of errors arising in the evaluation of determinants, solution of simultaneous linear equations, iterative methods, etc., is far from complete. A discussion of these errors is beyond the scope of this paper, but the interested computer should consult references [29–34].

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