$$G_\mu(a, b, r; t) = F^*(a, b, r; \mu) - \pi \sum_{i=1}^{\infty} \alpha_i^2 \frac{e^{-k(\alpha_i^2+\mu^2)t}}{\alpha_i^2 + \mu^2} \cdot \frac{J_0(\alpha_i a) J_0(\alpha_i b)}{J_0^2(\alpha_i a) - J_0^2(\alpha_i b)} \cdot \{J_0(\alpha_i r) Y_0(\alpha_i a) - J_0(\alpha_i a) Y_0(\alpha_i r)\}. \quad (20)$$

The complete solution of the system A is given by (17) in conjunction with (20).

In the particular case where $\varphi(z, t)$ is a function of z only, (16) becomes

$$T(r, z; t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty G_\mu(a, b, r; t) \varphi(\alpha) \sin \mu z \sin \mu \alpha d\mu d\alpha,$$

which is in agreement with Tranter's solution.

In a similar manner it is possible to treat the more complicated case where the boundary condition (4) is replaced by

$$\left(\alpha \frac{\partial}{\partial r} + \beta\right) T(r, z; t) = 0 \quad \text{for} \quad r = b$$

The formal solution corresponding to this boundary condition is in fact given once more by (17), with the function $G_\mu(a, b, r; t)$ satisfying the integral equation

$$p \int_0^\infty e^{-pt} G_\mu(a, b, r; t) dt = \frac{Y(p)}{Z(p)}, \quad (21)$$

where

$$Y(p) = I_0(\lambda r) K_0(\lambda a) - I_0(\lambda a) K_0(\lambda r),$$

$$Z(p) = \alpha\lambda \{K_0(\lambda a) I_0'(\lambda b) - I_0(\lambda a) K_0'(\lambda b) + \beta\{I_0(\lambda b) K_0(\lambda a) - I_0(\lambda a) K_0(\lambda b)\}.$$

The inversion of (21) proceeds in accordance with formula (19).

EFFECT OF A SMALL HOLE ON THE STRESSES IN A UNIFORMLY LOADED PLATE*

By VLADIMIR MORKOVIN (*Bell Aircraft Corporation*)

In a paper of the same title Martin Greenspan recently[1] determined the stress distribution in a large, uniformly loaded plate weakened by a small hole of an approximately ovaloid shape. Greenspan employed a rather laborious method of piecing together particular solutions of the biharmonic equation for the stress function until all the boundary conditions could be satisfied. This process would become prohibitive in case of more complicated boundary conditions. It is the purpose of this note to apply to the same problem the elegant and more general, yet not well known, method for solving plane problems of elasticity which is most often associated with the name of N. I. Mushelisvili.[2]

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[1] This Quarterly, **2**, 60–71 (1944).

[2] See for instance N. I. Mushelisvili, Math. Annalen, **107**, 282–312 (1932). For detailed English exposition see I. S. Sokolnikoff's *Mathematical theory of elasticity*, (mimeographed lecture notes, Brown University, 1941), pp. 243–318.

This method rests on the representation[3] of the general biharmonic stress function U in terms of two analytic functions of the complex variable $z=x+iy$, $\varphi(z)$ and $\psi(z)$,

$$U = \operatorname{Re}\left\{\bar{z}\varphi(z) + \int^{z} \psi(z)dz\right\}. \tag{1}$$

As a consequence of (1), any quantity characterizing the state of stress can be expressed in terms of $\varphi(z)$ and $\psi(z)$. Thus:

$$\sigma_y + \sigma_x = 2[\varphi'(z) + \bar{\varphi}'(\bar{z})] = 4\operatorname{Re}\{\varphi'(z)\}, \tag{2}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\varphi''(z) + \psi'(z)], \tag{3}$$

where the stresses bear the usual designation. Consider an arc PQ of some curve in the plane of the plate, *e.g.*, the boundary curve, and let $X_n ds$ and $Y_n ds$ be the x- and y-components of the force acting on the element ds of this arc (from a direction lagging behind the positive tangent by some positive angle $\epsilon \leqq 180°$). Then the quantity

$$\int_P^Q (X_n + iY_n)ds = -i[\varphi(z) + z\bar{\varphi}'(\bar{z}) + \bar{\psi}(\bar{z})]_P^Q \tag{4}$$

represents the resultant force on the arc PQ. When forces are prescribed on a boundary curve, equation (4) expresses the boundary condition in terms of the functions $\varphi(z)$ and $\psi(z)$.

In the case at hand, the equation of the boundary Γ is

$$x = p \cos \beta + r \cos 3\beta, \qquad y = q \sin \beta - r \sin 3\beta, \tag{5}$$

and the forces applied to it simply vanish so that the left-hand side of (4) becomes zero.

It is more convenient to deal with the boundary conditions after mapping the region exterior to the ovaloid Γ in the z plane conformally into the region exterior to a circle γ of unit radius in the ζ plane by means of the mapping function

$$z = \omega(\zeta) = s\zeta + t/\zeta + r/\zeta^3, \tag{6}$$

where $s=(p+q)/2$ and $t=(p-q)/2$. The values of ζ on γ shall be denoted by σ. It is noted that γ corresponds to Γ and that $\bar{\sigma}=1/\sigma$. Greenspan's curvilinear coordinate lines are obtained by setting $|\zeta|$ and $\arg \zeta$ equal to constants. For the sake of simplicity, the notation $\varphi(z)\equiv\varphi[\omega(\zeta)]\equiv\varphi(\zeta)$; $\psi(z)\equiv\psi[\omega(\zeta)]\equiv\psi(\zeta)$, will be used. Then,

$$\varphi'(z) = \varphi'(\zeta)/\omega'(\zeta), \qquad \psi'(z) = \psi'(\zeta)/\omega'(\zeta), \tag{7}$$

and the boundary condition (4) reads

$$\varphi(\sigma) + \frac{s\sigma^4 + t\sigma^2 + r}{\sigma^3(s - t\sigma^2 - 3r\sigma^4)}\bar{\varphi}'(1/\sigma) + \bar{\psi}(1/\sigma) = 0. \tag{8}$$

Two further results from the general theory of the method are needed before the functions $\varphi(\zeta)$ and $\psi(\zeta)$ can be determined from (8). First, it is known that when there is no unbalanced force acting on the hole and when the stresses at infinity are uniform, say $\sigma_x=S_x$, $\sigma_y=S_y$, $\tau_{xy}=T_{xy}$, the functions $\varphi(\zeta)$ and $\psi(\zeta)$ take the form

[3] Derivations of formulae (1)–(4) and a formula leading to (9) are given by I. S. Sokolnikoff, *loc. cit.*

$$\varphi(\zeta) = sB\zeta + \sum_1^\infty \frac{a_n}{\zeta^n}, \qquad \psi(\zeta) = s(B' + iT_{xy}) + \sum_1^\infty \frac{b_n}{\zeta^n}, \tag{9}$$

where $4B = S_y + S_x$ and $2B' = S_y - S_x$. Secondly, one may easily derive the following theorems:

(I) *If $f(\zeta)$ is an analytic function within γ, except perhaps at $\zeta = 0$, where it has a pole with a principal part of $\sum_1^n A_k/\zeta^k$, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\sigma)d\sigma}{\sigma - \zeta} = -\sum_1^n \frac{A_k}{\zeta^k}, \quad \text{for} \quad |\zeta| > 1.$$

(II) *If $f(\zeta)$ is an analytic function outside of γ, except perhaps at $\zeta = \infty$ where it has a pole with a principal part of $\sum_0^n A_k \zeta^k$, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\sigma)d\sigma}{\sigma - \zeta} = -f(\zeta) + \sum_0^n A_k \zeta^k, \quad \text{for} \quad |\zeta| > 1.$$

The actual process of solving the problem consists of multiplying equation (8) and its conjugate by $d\sigma/2\pi i(\sigma - \zeta)$ and integrating around γ with the aid of theorems I and II. The first integration yields the equation

$$-\varphi(\zeta) + sB\zeta + \frac{\bar{a}_1}{\zeta}\frac{r}{s} - \frac{Br}{\zeta^3} - \frac{Bt}{\zeta}\left(1 + \frac{r}{s}\right) - \frac{s}{\zeta}(B' - iT_{xy}) = 0 \tag{10a}$$

which determines $\varphi(\zeta)$ except for $\bar{a}_1$. The coefficient a_1 is found by multiplying (8) by $\sigma d\sigma/2\pi i(\sigma - \zeta)$ and repeating the integration around γ. One is led to the following equations:

$$-\zeta\varphi(\zeta) + sB\zeta^2 + a_1 - Br/\zeta^2 = 0, \tag{10b}$$

$$a_1 = Bt\frac{r + s}{r - s} + \frac{B's^2}{r - s} + \frac{iT_{xy}s^2}{r + s}, \tag{11}$$

$$\varphi(\zeta) = sB\zeta + a_1/\zeta - Br/\zeta^3. \tag{12}$$

The integration of the conjugate of equation (8) furnishes the form of the remaining unknown function:

$$\psi(\zeta) = -\left(\frac{s}{\zeta} + t\zeta + r\zeta^3\right)\left(\frac{sB\zeta^4 - a_1\zeta^2 + 3rB}{s\zeta^4 - t\zeta^2 - 3r}\right) - \frac{sB}{\zeta}$$
$$+ s(B' + iT_{xy})\zeta - a_1\frac{r}{s}\zeta + Br\zeta^3 + Bt\left(1 + \frac{r}{s}\right)\zeta. \tag{13}$$

Whenever, in plane problems of elasticity, the mapping function $\omega(\zeta)$ is rational, the solution can be carried out in a manner similar to that above.

It remains to verify Greenspan's results. According to equations (2), (7), and (12), one has

$$\sigma_x + \sigma_y = 2\left[\frac{sB\zeta^4 - a_1\zeta^2 + 3rB}{s\zeta^4 - t\zeta^2 - 3r} + \frac{sB\bar{\zeta}^4 - \bar{a}_1\bar{\zeta}^2 + 3rB}{s\bar{\zeta}^4 - t\bar{\zeta}^2 - 3r}\right], \tag{14}$$

which readily reduces to equation (26) of Greenspan when ζ is on γ. The equality of stresses at the hole and at infinity identifies the two solutions completely.