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NOTE ON FLOW IN CANALS*

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Introduction. The design of irrigation systems to service a certain area sometimes necessitates the installation of pumping stations between each of which is a canal section with water being pumped in at one end and out at the other at constant rates. Starting or stopping the pumps produces long waves which are sometimes referred to as "surge waves" or "bore waves," with associated changes in height of the water surface. Such waves have been considered by Massé[1] and Deymié.[2] It is the prediction of these changes in height of the water surface, under constant inflow, constant outflow, or any combination of the two, which is the problem solved in this note. This is done by first considering an infinite canal with one source of constant inflow and using a method of "image" sources to produce the effect of reflections at the ends of a finite canal. Simplifying assumptions[3] are as follows:

1). The canal is of constant cross section throughout its length.
2). The effect of fluid velocity in the canal on the wave velocity is negligible.
3). Vertical accelerations of water are negligible.
4). The frictional resistance to flow is proportional to the velocity.
5). The height of water surface above normal depth, due to wave action, is small compared to normal depth.

Fundamental wave equations. The usual equations of motion and continuity are:[4]

$$-\frac{\partial \eta}{\partial x}=\frac{\partial u}{g\partial t}+\frac{B}{g}u, \qquad (1); \qquad\qquad -\frac{\partial u}{\partial x}=\frac{\partial \eta}{h\partial t}, \qquad (2)$$

respectively, where x is the horizontal distance along the canal (ft.), t is the time (sec.), h is the normal depth of the water in the canal (ft.), η is the height of the water surface above normal (ft.), u is the horizontal velocity of the fluid (ft./sec.), g is the acceleration of gravity (ft./sec.^{2}) and B is a constant to be determined. In the computations on a special case for a finite canal, B was taken as igA/Q where A is the area of a vertical cross section of the canal (sq. ft.) and i is the slope of the canal computed from Chezy's formula to give a flow of Q cu. ft. per sec. One alternative procedure is to find the frictional force resisting flow in the form[4] $X=cu^2$ and approximate this

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[1] P. Massé, *Hydrodynamique fluviale, régimes variables*, Hermann, Paris, 1935.

[2] Ph. Deymié, *Proc. 5th Int. Congress Appl. Mech., Cambridge, Mass., 1938*, Wiley, 1939, pp. 537–543.

[3] Lamb, *Hydrodynamics* (6th ed.), Cambridge University Press, 1932, p. 254, et seq.

[4] Lamb, loc. cit.

force by 0.75 cuQ/A. This gives the "least square fit" to the force X in the range of velocities $0 \leqq u \leqq Q/A$.

It is easily seen that, for the infinite canal of uniform cross section with constant inflow at a point, the boundary condition is $u(0, t) = Q/A$ = constant, where Q is the constant inflow.

In order to reduce the problem to dimensionless form, it is convenient to make the substitutions

$$\rho = \frac{B}{2}, \qquad \rho t = \alpha, \qquad \frac{\rho x}{a} = \beta, \tag{3}$$

$$\frac{u(x, t)}{u(0, 0)} = \frac{u(\beta a/\rho, \alpha/\rho)}{u_0} = U(\beta, \alpha), \tag{4}$$

$$\frac{\eta(x, t)}{\eta(0, 0)} = \frac{\eta(\beta a/\rho, \alpha/\rho)}{\eta_0} = H(\beta, \alpha), \tag{5}$$

where a is the wave velocity ($=\sqrt{gh}$ ft./sec.) and

$$u_0 = \sqrt{g/h}\,\eta_0. \tag{6}$$

Equations (1) and (2) become

$$-\frac{\partial H}{\partial \beta} = \frac{\partial U}{\partial \alpha} + 2U, \quad (7); \qquad -\frac{\partial U}{\partial \beta} = \frac{\partial H}{\partial \alpha}. \quad (8)$$

A solution of Eqs. (7) and (8), adapted from Heaviside's work in electromagnetic theory,[5] is

$$U(\beta, \alpha) = e^{-\alpha} I_0\left\{\sqrt{\alpha^2 - \beta^2}\right\}, \tag{9}$$

$$H(\beta, \alpha) = e^{-\alpha}\left[e^{\alpha} - \beta\left\{I_0(\alpha) + I_1(\alpha)\right\} + \frac{\beta^3}{3!}\,\frac{\left\{I_1(\alpha) + I_2(\alpha)\right\}}{\alpha} - \frac{1 \cdot 3}{5!}\,\frac{\beta^5\left\{I_2(\alpha) + I_3(\alpha)\right\}}{\alpha^2} + \cdots\right], \tag{10}$$

where $I_n(\alpha)$ is the modified Bessel function of order n. This solution is fundamental to the following and corresponds to the case of an infinite canal with a barrier at the origin initially such that $\eta(x, 0) = 2\eta_0$ = const. for $x \leqq 0$ and $\eta(x, 0) = 0$ for $x > 0$. Upon removal of the barrier at $t = 0$, η immediately drops to η_0 and remains there. It is this property of $H(\beta, \alpha)$ that makes it valuable in studying other flow conditions with arbitrary velocity at the origin, that is, $u(0, t)$. In particular, to complete the solution for the infinite canal problem as proposed, it is merely necessary to choose $H(0, \alpha) = F(\alpha)$ so that $U(0, \alpha) \equiv 1$, and then find $H(\beta, \alpha)$ or $U(\beta, \alpha)$ corresponding to $H(0, \alpha)$ by an integration.

Now an increment $\Delta F(\alpha)$ put in at $\alpha = \xi$ for $\beta = 0$ produces an increment ΔU at $\beta = 0$, $\alpha = \alpha$ of

$$\Delta U(0, \alpha) \cong \Delta F e^{-(\alpha - \xi)} I_0(\alpha - \xi). \tag{11}$$

[5] O. Heaviside, *Electromagnetic theory*, vol. 2, The Electrician Printing and Publishing Co., Ltd., London, 1899, p. 303 et seq.

Hence, after integrating and adding the initial effect, we have

$$U(0, \alpha) = \int_0^\alpha e^{-(\alpha-\xi)} I_0(\alpha - \xi) F'(\xi) d\xi + e^{-\alpha} I_0(\alpha) = 1. \tag{12}$$

The solution of this integral equation is easily shown to be[6]

$$F'(\alpha) = e^{-\alpha}\{I_0(\alpha) + I_1(\alpha)\}. \tag{13}$$

The effect of applying increments of $H(0, \alpha)$ may be represented approximately by

$$\Delta U(\beta, \alpha) \cong e^{-(\alpha-\xi)} I_0\{\sqrt{(\alpha-\xi)^2 - \beta^2}\} F'(\xi) d\xi, \tag{14}$$

$$\Delta H(\beta, \alpha) = e^{-(\alpha-\xi)} \Bigg[e^{\alpha-\xi} - \beta\{I_0(\alpha-\xi) + I_1(\alpha-\xi)\} + \frac{\beta^3}{3!} \frac{\{I_1(\alpha-\xi) + I_2(\alpha-\xi)\}}{\alpha-\xi}$$
$$- \frac{1\cdot 3}{5!} \frac{\beta^5\{I_2(\alpha-\xi) + I_3(\alpha-\xi)\}}{(\alpha-\xi)^2} + \cdots \Bigg] F'(\xi) d\xi. \tag{15}$$

Thus, integration and the addition of the initial effect yields the solution of the problem as

$$e^\alpha U(\beta, \alpha) = \int_0^{\alpha-\beta} I_0\{\sqrt{(\alpha-\xi)^2 - \beta^2}\}\{I_0(\xi) + I_1(\xi)\} d\xi + I_0\{\sqrt{\alpha^2 - \beta^2}\}, \tag{16}$$

$$e^\alpha H(\beta, \alpha) = \int_0^{\alpha-\beta} \Bigg[e^{\alpha-\xi} - \beta\{I_0(\alpha-\xi) + I_1(\alpha-\xi)\} + \frac{\beta^3}{3!} \frac{\{I_1(\alpha-\xi) + I_2(\alpha-\xi)\}}{\alpha-\xi}$$
$$- \frac{1\cdot 3}{5!} \frac{\beta^5\{I_2(\alpha-\xi) + I_3(\alpha-\xi)\}}{(\alpha-\xi)^2} + \cdots \Bigg] \{I_0(\xi) + I_1(\xi)\} d\xi$$
$$+ e^\alpha - \beta\{I_0(\alpha) + I_1(\alpha)\} + \frac{\beta^3}{3!} \frac{\{I_1(\alpha) + I_2(\alpha)\}}{\alpha} - \frac{1\cdot 3}{5!} \frac{\beta^5\{I_2(\alpha) + I_3(\alpha)\}}{\alpha^2} \cdot \tag{17}$$

Equation (17), on which attention is now focused, may be transformed into slightly better form for computation, as follows. We write

$$e^{-\alpha} \int_0^{\alpha-\beta} \frac{\{I_n(\alpha-\xi) + I_{n+1}(\alpha-\xi)\}}{(\alpha-\xi)^n} \{I_0(\xi) + I_1(\xi)\} d\xi$$
$$= e^{-\alpha} \int_\beta^\alpha \frac{\{I_n(s) + I_{n+1}(s)\}}{s^n} \{I_0(\alpha - s) + I_1(\alpha - s)\} ds = G_n(\beta, \alpha). \tag{18}$$

Therefore,

$$H(\beta, \alpha) = 1 + \int_0^{\alpha-\beta} e^{-\xi}\{I_0(\xi) + I_1(\xi)\} d\xi - \beta\{G_0(\beta, \alpha) + K_0(\alpha)\}$$
$$+ \frac{\beta^3}{3!}\{G_1(\beta, \alpha) + K_1(\alpha)\} - \frac{1\cdot 3\beta^5}{5!}\{G_2(\beta, \alpha) + K_2(\alpha)\} + \cdots, \tag{19}$$

[6] H. T. Davis, *A survey of methods for the inversion of integrals of Volterra type*, Indiana University Studies, Nos. 76 and 77, 1927, p. 51.

where

$$K_n(\alpha) = e^{-\alpha} \frac{\{I_n(\alpha) + I_{n+1}(\alpha)\}}{\alpha^n}. \tag{20}$$

$H(\beta, \alpha)$ as given by Eq. (19) is tabulated in Table 1 for the range of values $0 \leqq \beta \leqq \alpha \leqq 1$, three terms of each series involved in Eq. (19) giving the results to three decimal places. Tables of the modified Bessel function,[7] in conjunction with a Simpson's rule for five intervals, were used in making the computations. The results were checked by graphical integration. It is to be noted that a horizontal row in Table 1 gives the history of the height above normal at a fixed time, while a vertical column gives the height history at a fixed point.

TABLE 1. $H(\beta, \alpha)$

α \ β	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.000										
0.1	1.098	0.905									
0.2	1.191	0.998	0.819								
0.3	1.280	1.086	0.907	0.741							
0.4	1.365	1.171	0.991	0.824	0.670						
0.5	1.446	1.253	1.072	0.904	0.749	0.606					
0.6	1.525	1.331	1.150	0.981	0.825	0.681	0.549				
0.7	1.601	1.407	1.225	1.056	0.898	0.753	0.619	0.496			
0.8	1.674	1.480	1.298	1.127	0.969	0.822	0.687	0.562	0.449		
0.9	1.745	1.550	1.368	1.197	1.037	0.889	0.752	0.626	0.511	0.406	
1.0	1.813	1.619	1.436	1.264	1.104	0.954	0.816	0.688	0.571	0.464	0.366

Note: The table was computed to more figures and in the range $0 \leqq \beta \leqq \alpha \leqq 2$, but since it is used here for illustrative purposes only, the abbreviated form is given.

Finite canal section. The flow condition in a finite canal section of length L with constant flow Q at one end (origin) is simulated by considering an infinite canal with sources of constant inflow Q located at points 0, $\pm 2L$, $\pm 4L$, $\cdots$. In the type of investigation for which this problem was solved, the maximum height in the canal was the prime consideration, and only a few reflections are needed to determine this maximum. Table 1 suffices to carry out the necessary computations. For the outflow case it is merely necessary to reverse results for inflow.

Remarks. Strictly speaking, the results of course apply only to a canal initially at rest, and if in the case of a finite canal with a sloping bottom it is desired to compute heights subsequent to a shut down, these should be referred to the surface in running position which, in the case of a properly designed canal, will be parallel to the bottom.

Heaviside solved the analogous electromagnetic problem for the infinite telegraph cable by use of operational calculus. But so far as the writer knows, the solution as applied to canals is unavailable elsewhere. It is seen that by the elimination of U or H from Eqs. (7) and (8), the differential equation which either one satisfies is

[7] Gray and Mathews, *Treatise on Bessel functions*, Macmillan and Co., London, 1922.

$$\frac{\partial^2\phi}{\partial\alpha^2}+\frac{2\partial\phi}{\partial\alpha}=\frac{\partial^2\phi}{\partial\beta^2}\cdot \tag{21}$$

Thus the problem for the infinite canal, of which Eq. (19) is the solution, is equivalent to that of solving Eq. (21) ($\alpha, \beta \geqq 0$) with the boundary conditions

$$\phi(\beta, 0) = 0, \qquad \phi(0, \alpha) = e^{-\alpha}\left[\tfrac{1}{2}\alpha\left\{3I_0(\alpha) + 4I_1(\alpha) + I_2(\alpha)\right\} + I_0(\alpha) + I_1(\alpha)\right].$$

NOTE ON THE ELLIPTIC WING*

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The acceleration potential method of Prandtl for studying the aerodynamics of a lifting surface is well known.[1] In the applications to specific surfaces one is naturally led to wings with circular or elliptic plan view. Prandtl's theory has been completely elaborated for these two cases by W. Kinner[2] and K. Krienes.[3]

In an effort to extend the class of wings for which numerical results have been obtained, a theory based on the work of Krienes was developed for the semi-elliptic wing by E. R. Lorch. Computations carried out for this case by the author had to be abandoned due to very poor convergence. This has suggested that the question of convergence in Krienes' work, which plays an important role there, be examined more closely. This matter is investigated in the present note.

We base our discussion on Krienes' paper. The reader is referred to it for the details which it is impossible to reproduce here. In this work the pressure p is expressed in terms of a potential function ψ. We have

$$p - p_\infty = -\rho_0 V^2\psi(x, y, z), \qquad \Delta\psi = 0, \tag{1}$$

where p_∞ is the pressure at infinity, ρ_0 is the density, and V is the velocity of the wing. In turn ψ is expanded in a series

$$\psi = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} \overset{m}{a}_n \overset{m}{\psi}_n + \sum_{n=1}^{\infty} C_n\Phi_n, \tag{2}$$

(Krienes, Eq. (81) with $D_n = 0$) where

$$\overset{m}{\psi}_n(\rho, \mu, \nu) = \overset{m}{E}_n(\mu)\overset{m}{E}_n(\nu)\overset{m}{E}_n(\rho)\int_\rho^\infty \frac{d\rho}{[\overset{m}{E}_n(\rho)]^2[(\rho^2-1)(\rho^2-k)]^{1/2}},$$

$$\Phi_n = \sum_m \overset{m}{b}_n \overset{m}{\Phi}_n, \qquad \overset{m}{\Phi}_n = c^{1-n}\frac{d}{dc}\left[c^n \overset{m}{\psi}_n(x, y, z, c)\right],$$

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* Received June 5, 1944.

[1] L. Prandtl, *Beitrag zur Theorie der tragenden Fläche*, Zeit. f. angew. Math. u. Mech. **16,** 360–361 (1936).

[2] W. Kinner, *Die kreisförmige Tragfläche auf potentialtheoretischer Grundlage*, Ing.-Archiv, **8,** 47–80 (1937).

[3] K. Krienes, *Die elliptische Tragfläche auf potentialtheoretischer Grundlage*, Zeit. f. angew. Math. u. Mech. **20,** 65–88 (1940).