POWER SERIES EXPANSIONS OF THE VELOCITY POTENTIAL IN COMPRESSIBLE FLOW*

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1. The equation for compressible flow. The equation describing the two-dimensional irrotational flow of a compressible fluid is

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = \tfrac{1}{2}M\left[\frac{\partial \varphi}{\partial r}\frac{\partial q^2}{\partial r} + \frac{\partial \varphi}{r\partial \theta}\frac{\partial q^2}{r\partial \theta}\right][1 - \tfrac{1}{2}(\gamma - 1)(q^2 - 1)]^{-1}, \quad (1.1)$$

where r, θ are polar coordinates, M is the Mach number of the flow at some distance from the origin where the velocity is assumed to be uniform, γ is the ratio of specific heats of the fluid and φ, q are the velocity potential and velocity, respectively, at the point r, θ. All distances are referred to some typical dimension in the field, and velocities are referred to the speed of the uniform stream. The boundary conditions to be imposed on φ are (1) $\partial \varphi/\partial r \to \cos \theta$ as $r \to \infty$, where θ is measured from the direction of the uniform stream, and (2) the normal derivative of φ equals zero at the boundary of a fixed obstacle in the stream.

Many attempts have been made to solve this equation analytically, but none has met with complete success. Most of the progress has been through some linearizing process, such as writing φ as a power series in M^2, the non-linear terms on the right hand side of (1.1) then being of higher order in M^2 than the linear terms on the left. Again, a transformation to new independent variables, the magnitude and direction of the local fluid velocity, gives a linear equation for φ. In this paper, use is made of an expansion of φ as a power series in r^{-1} and it is shown that the equation for each of the coefficients in the series is linear. Some light is thrown on the form of the expression for φ which may be expected in the two cases of zero and finite circulation about the obstacle.

2. Case of zero circulation. For the case of flow past an obstacle about which there is no circulation we make the assumption that φ can be written in the following form:

$$\varphi = r \cos \theta + r^{-1}f_1(\theta) + r^{-2}f_2(\theta) + r^{-3}f_3(\theta) + \cdots. \quad (2.1)$$

It will be seen later that the term independent of θ reduces to a constant when the circulation is zero, and thus may be ignored. Whether or not this expansion for φ is valid will be apparent from the subsequent solutions which are obtained for the functions $f_n(\theta)$. Invalidity of the assumption (2.1) may be made manifest either as a lack of convergence of the power series in r^{-1}, or as a lack of periodicity in θ of any of the $f_n(\theta)$. In the former case, the series could presumably be made convergent by choosing r large enough, so that any conclusions would apply only to regions far from the origin. If, as seems probable, there is no specially favoured region, we may regard the expansion (2.1) as valid generally provided that the resulting equations for the $f_n(\theta)$ give periodic solutions.

The expression for q^2 corresponding to (2.1) is

$$q^2 = 1 + r^{-2}(-2f_1 \cos \theta - 2f_1' \sin \theta) + \cdots$$
$$+ r^{-n-1}(-2nf_n \cos \theta - 2f_n' \sin \theta + \cdots) + \cdots \qquad (2.2)$$

where dashes denote differentiation with respect to θ, and, in the general coefficient, only terms involving f_n have been retained. Substituting both (2.1) and (2.2) in the equation (1.1) and equating coefficients of r^{-n-2} we obtain, after some reduction,

$$f_n''(1 - M^2 \sin^2 \theta) - f_n'(n+1)M^2 \sin 2\theta + f_n[n^2 + nM^2 - (n^2 + 2n)M^2 \cos^2 \theta]$$
$$= M^2 E_n(f_{n-2}, f_{n-3}, \cdots, f_1, \theta), \qquad (2.3)$$

where E_n is an undetermined function.

The equation (2.3) may be normalized by writing

$$f_n(\theta) = \frac{h_n(\theta)}{(1 - M^2 \sin^2 \theta)^{\frac{1}{2}(n+1)}} \qquad (2.4)$$

which gives for $h_n(\theta)$

$$h_n'' + h_n \left\{ 1 + \frac{(n^2 - 1)(1 - M^2)}{(1 - M^2 \sin^2 \theta)^2} \right\} = M^2 E_n(1 - M^2 \sin^2 \theta)^{\frac{1}{2}(n+1)}. \qquad (2.5)$$

The homogeneous equation may also be written in the form

$$h_n' + h_n \left(1 + u_0 + \sum_{p=1}^{\infty} 2u_p \cos 2p\theta \right) = 0, \qquad (2.6)$$

where

$$u_p = (n^2 - 1)(1 - M^2)(-1)^p \sum_{m=p}^{\infty} \frac{(2m)!(m+1)}{(m-p)!(m+p)!} \left(\frac{M}{2} \right)^{2m}. \qquad (2.7)$$

The expression for u_p is convergent provided $M^2 < 1$ and $\sum u_p$ is absolutely convergent for the same condition. The equation (2.6) will be recognized as being identical with Hill's equation which arises in a determination of the motion of the lunar perigee.[1] As is already known,[2] Hill's equation may be solved by the assumption of a series expansion of h_n; for instance, we may write

$$h_n = e^{\mu \theta} \sum_{q=-\infty}^{\infty} b_q e^{2qi\theta} \qquad (2.8)$$

and determine the constants b_q, and μ, which governs the periodicity of the solution. However, the method is tedious and we shall find it more expedient to derive a solution in powers of M^2 from the equation (2.3).

We may also note that if only terms of order M^0 and M^2 are retained in (2.6), the homogeneous equation for h_n degenerates to Mathieu's equation, viz.,

$$h_n' + h_n[n^2 - M^2(n^2 - 1) \cos 2\theta] = 0. \qquad (2.9)$$

It is known that Mathieu's equation yields a periodic solution under certain condi-

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[1] G. W. Hill, Acta. Math., **8**, 1–36 (1886).

[2] See for example, Whittaker and Watson, *Modern analysis*, Camb. Univ. Press, Cambridge, 1940, p. 413.

tions; a proof that equation (2.9) gives a periodic solution (with consistent approxima-
tion) is demonstrated in the Appendix.

3. Periodicity of f_n. Consider first the homogeneous equation for f_n, viz.,

$$f_n''(1 - M^2 \sin^2\theta) - f_n'(n+1)M^2 \sin 2\theta + f_n[n^2 + nM^2 - (n^2 + 2n)M^2 \cos^2\theta] = 0. \quad (3.1)$$

In order to show that this equation gives a periodic solution for f_n, we write the solu-
tion in the form

$$f_n = F_{n,0} + M^2 F_{n,1} + M^4 F_{n,2} + \cdots. \quad (3.2)$$

Substituting in (3.1) and equating coefficients of powers of M^2 we obtain a series of
equations, each of the same type, for $F_{n,0}$, $F_{n,1}$, $\cdots$. The typical equation is

$$F_{n,m+1}'' + n^2 F_{n,m+1} = F_{n,m}'' \sin^2\theta + F_{n,m}'(n+1)\sin 2\theta - F_{n,m}[n - (n^2 + 2n)\cos^2\theta] \quad (3.3)$$

the solution of which is periodic unless a term $\cos n\theta$ appears on the right-hand side.
Suppose that $F_{n,m}$ consists of the sum of a number of cosine terms; if, for instance,

$$F_{n,m} = C \cos(p\theta + \delta)$$

then (3.3) becomes

$$F_{n,m+1}'' + n^2 F_{n,m+1} = C\{\tfrac{1}{4}(p+n)(p+n+2)\cos[(p+2)\theta + \delta]$$

$$+ \tfrac{1}{2}(n^2 - p^2)\cos(p\theta + \delta) + \tfrac{1}{4}(n-p)(n-p+2)\cos[(p-2)\theta + \delta]\}. \quad (3.4)$$

$F_{n,m+1}$ will thus be unperiodic only if $p = n-2$; the cases $p = n$ and $p = n+2$ do not
give finite terms in $\cos n\theta$ on the right of (3.4). Now the value of $F_{n,0}$ is

$$A_n \cos(n\theta + \epsilon_n) \qquad (A_n, \epsilon_n \text{ arbitrary}).$$

$F_{n,1}$ therefore contains terms in $\cos(n+2)\theta$ and $\cos n\theta$. Similarly, $F_{n,2}$ will contain
terms in $\cos(n+4)\theta$, $\cos(n+2)\theta$ and $\cos n\theta$. Generally, we see that a term $\cos(n-2)\theta$
cannot arise in any of the $F_{n,m}$. Hence each of the $F_{n,m+1}$ will be periodic, and the solu-
tion of (3.1) is periodic.

Unfortunately, a proof that the particular solution of the equation (2.3) for f_n is
periodic is not as simple, since the function E_n is a very complicated one. For both
$n = 1$ and $n = 2$, $E_n = 0$ so that the complete solutions of f_1 and f_2 are periodic.

However we can show generally that to the order of M^2, at least, f_n is periodic.
To this order the denominator of the right hand side of equation (1.1) may be treated
as unity. It is also possible to write, in the numerator of this expression,

$$f_n = A_n \cos(n\theta + \epsilon_n) \quad (3.5)$$

where A_n and ϵ_n are constants independent of M^2. Considering only the terms of E_n
which can lead to a term in $\cos n\theta$, we find that

$$E_n = -2\cos\theta \sum_{p=1}^{n-2} p(n-p)(n-p-1)f_p f_{n-p-1} - 2\cos\theta \sum_{1}^{n-2}(n-p)f_p' f_{n-p-1}'$$

$$-2\sin\theta \sum_{1}^{n-2} p(n-p)f_p f_{n-p-1}' + 2\sin\theta \sum_{1}^{n-2}(n-p)(n-p-1)f_p' f_{n-p-1}$$

$$+ \text{ unimportant terms.} \quad (3.6)$$

By the use of (3.5), this becomes

$$E_n = -2 \cos \theta \sum_{1}^{n-2} p(n-p)(n-p-1)A_p A_{n-p-1} \cos \left[(n-2p-1)\theta+\epsilon_{n-p-1}-\epsilon_p\right]$$

$$+ 2 \sin \theta \sum_{1}^{n-2} p(n-p)(n-p-1)A_p A_{n-p-1} \sin \left[(n-2p-1)\theta+\epsilon_{n-p-1}-\epsilon_p\right]+ \cdots$$

$$= -2 \sum_{1}^{n-2} p(n-p)(n-p-1)A_p A_{n-p-1} \cos \left[(n-2p)\theta+\epsilon_{n-p-1}-\epsilon_p\right]+ \cdots, \tag{3.7}$$

showing that no terms in $\cos n\theta$ occur in E_n. Hence f_n is periodic at least as far as the order of M^2.

By carrying out in detail the iteration process for a solution of (2.3) in powers of M^2 for f_3 and f_4, and also for f_5 for the case of a doubly symmetrical body (for which $f_2=f_4=\cdots=0$), the author has been able to show that non-periodicity does not arise when terms of the order of M^4 are considered, nor does it arise when the order of M^6 in f_3 is considered. In addition, the solution of equation (1.1) appropriate to flow past a circular cylinder has been found[3] to be of the form (2.1), at least as far as the order of M^6.

Hence, although not yet proved, it seems probable that the particular solution to the equation (2.3) for f_n is periodic. If this be the case, then the complete solution for f_n is periodic and the assumption of a series expansion (2.1) for φ may be regarded as valid for the case of zero circulation.

4. Numerical values of the constants in f_1. In the case $n=1$, the normalized equation (2.5) becomes

$$h_1'' + h_1 = 0 \tag{4.1}$$

which gives

$$f_1 = \frac{A \cos (\theta + \alpha)}{1 - M^2 \sin^2 \theta} \tag{4.2}$$

where A and α are arbitrary constants, to be determined by the inner boundary layer condition. This appears to be the only case for which the complete solution for f_n is obtainable, and has also been put foreard by Imai.[4]

In general, A and α are functions of M^2 and we may write, for small values of M^2,

$$A = a_0 + a_1 M^2 + a_2 M^4 + \cdots, \tag{4.3}$$

$$\alpha = \alpha_0 + \alpha_1 M^2 + \alpha_2 M^4 + \cdots. \tag{4.4}$$

Expanding f_1 in powers of M^2 we then have

$$f_1 = a_0 \cos (\theta+\alpha_0)+M^2\left[a_0 \cos (\theta+\alpha_0) \sin^2 \theta+a_1 \cos(\theta+\alpha_0) - a_0\alpha_1 \sin(\theta+\alpha_0)\right]+ \cdots. \tag{4.5}$$

The value of f_1 correct to the order M^2 is known for several different obstacle shapes from the work of previous investigators. For instance, Kaplan[5] has determined the first approximation to compressible flow past an elliptic cylinder at an angle of incidence β to the uniform stream. Comparing his result with (4.5) we find, after some reduction,

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[3] I. Imai, *On the flow of a compressible fluid past a circular cylinder*, II, Proc. Phys.-Math. Soc. Japan, **23**, 180 (1941).

[4] I. Imai, loc. cit.

[5] C. Kaplan, *On the use of residue theory for treating the subsonic flow of a compressible fluid*, N.A.C.A. Technical Report 728, 1942.

$$a_0^2 = 1 - \tfrac{1}{2}c^2 \cos 2\beta + \tfrac{1}{16}c^4, \quad (4.6) \qquad \tan \alpha_0 = \frac{c^2 \sin 2\beta}{4 - c^2 \cos 2\beta}, \qquad (4.7)$$

$$a_1 = \frac{1}{4a_0} \left[(- \tfrac{1}{16}c^4 - 4 - 16c^{-4}) + (\tfrac{3}{4}c^2 + 20c^{-2}) \cos 2\beta + (\tfrac{1}{16}c^4 - 3)\cos 4\beta \right]$$

$$+ \frac{1}{4a_0} \log \frac{4 + c^2}{4 - c^2} \left[(\tfrac{1}{2}c^2 + 14c^{-2} + 32c^{-6}) + (- \tfrac{1}{32}c^4 - 5 - 40c^{-4})\cos 2\beta \right.$$
$$\left. + (\tfrac{1}{8}c^2 + 6c^{-2})\cos 4\beta \right], \qquad (4.8)$$

$$\alpha_1 = \sin 2\beta \left[\tfrac{1}{4} + \frac{1}{2a_0^2}(1 - \tfrac{1}{4}c^2 \cos 2\beta) + \tfrac{1}{8}(4c^{-2} - \tfrac{1}{4}c^2) \log \frac{4 + c^2}{4 - c^2} \right], \qquad (4.9)$$

where c is the distance between the centre and focus of the ellipse, and the unit of length is the radius of the circle into which the ellipse transforms (equal to the mean of the major and minor axes). At zero incidence, these expressions become

$$a_0 = 1 - \tfrac{1}{4}c^2 = \frac{2t}{1 + t}, \qquad (4.10)$$

$$a_1 = \tfrac{1}{4}(4c^{-2} - 1)(3 - 4c^{-2}) + \tfrac{1}{32}c^2(4c^{-2} - 1)^4 \log \frac{4 + c^2}{4 - c^2}$$

$$= \frac{t(1 - 2t)}{(1 - t)^2} - \frac{2t^4}{(1 - t^2)(1 - t)^2} \log t, \qquad (4.11)$$
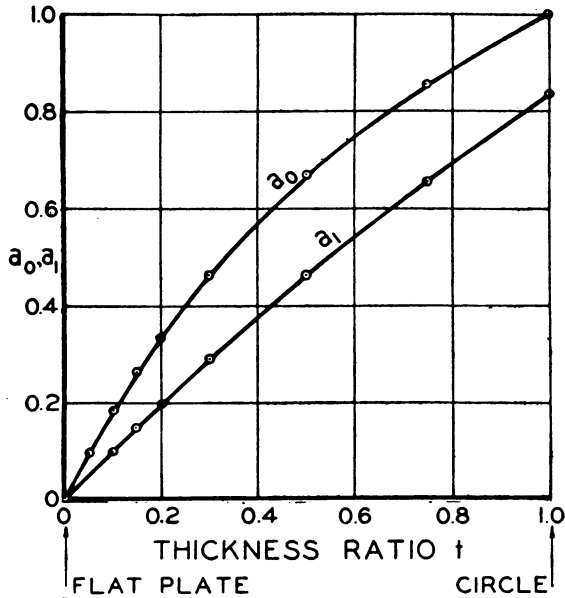
$$\alpha_0 = \alpha_1 = 0, \qquad (4.12)$$

FIG. 1. a_0, a_1, for ellipses (0° incidence).

where t is the ratio of thickness to chord of the ellipse. The expressions (4.10) and (4.11) are shown graphically in Fig. 1. For ellipses of thickness ratio less than $0\cdot2$, a_1 is given accurately by $a_1 = t$.

Using results obtained by Tomotika and Umemoto,[6] it is also possible to derive numerical values for a_0, a_1 corresponding to the flow past a symmetrical Joukowski aerofoil of arbitrary thickness at zero incidence. After considerable reduction, we obtain from a comparison of these authors' formulae with (4.5)

$$a_0 = \tfrac{1}{4}(1 - k)(3 + k), \tag{4.13}$$

$$\begin{aligned} a_1 = &- \tfrac{1}{64}k^{-8}(1 - 3k + 2k^2 + 2k^3 - 3k^4 + k^5)(1 + k)^5 \log (1 - k^2) \\ &+ \tfrac{1}{384}(- 6k^{-6} - 12k^{-5} + 15k^{-4} + 42k^{-3} - 5k^{-2} - 52k^{-1} + 207 - 170k \\ &\qquad\qquad\qquad\qquad\qquad - 61k^2 + 36k^3 + 10k^4 - 4k^5), \end{aligned} \tag{4.14}$$

$$\alpha_0 = \alpha_1 = 0, \tag{4.15}$$

where

$$k = \frac{1 - \epsilon}{1 + \epsilon} \tag{4.16}$$

and ϵ is the parameter controlling thickness-to-chord ratio in the Joukowski transformation as shown in the table. The unit of length is again the radius of the circle

TABLE—Relation between ϵ and t.

$\epsilon = 0$	0·03	0·05	0·07	0·10	0·15	0·20	0·30	0·50	∞
$t = 0$	0·0378	0·0618	0·0849	0·1179	0·1687	0·2150	0·2958	0·4210	1

into which the aerofoil transforms. The values of a_0 and a_1 are also shown in Fig. 2. Most practical aerofoil shapes have a thickness ratio less than 0·2 and it is evident that a_0 and a_1 may be estimated with fair accuracy for these shapes by assuming a linear dependence on t. Using the slope appropriate to $t = 0$, we have

$$a_0 = 1.538t, \qquad a_1 = 0.769t = \tfrac{1}{2}a_0. \tag{4.17}$$

From the closeness of the values of a_0 and a_1 for an ellipse and a Joukowski aerofoil of the same thickness ratio, it seems reasonable to suppose that a_0 and a_1 do not vary greatly with change of shape of the obstacle. These results may then be of use in estimating compressible flow past obstacles of different shapes at points far from the origin.

The same procedure is also possible with the higher coefficients in the series for φ. For instance, the equation for f_2 is

$$f_2' (1 - M^2 \sin^2 \theta) - 3f_2' M^2 \sin 2\theta + f_2(4 + 2M^2 - 8M^2 \cos^2 \theta) = 0 \tag{4.18}$$

of which the solution, in powers of M^2, as far as M^2, is

$$f_2 = b_0 \cos (2\theta + \beta_0) + M^2[b_1 \cos (2\theta + \beta_1) - \tfrac{1}{2}b_0 \cos (4\theta + \beta_0)] + \cdots , \tag{4.19}$$

where b_0, b_1, β_0, β_1 are arbitrary and independent of M^2. These constants could also be determined from a comparison with known first-order solutions of the compressible

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[6] S. Tomotika and H. Umemoto, *On the subsonic flow of a compressible fluid past a symmetrical Joukowski aerofoil*, Tokyo Imp. Univ. Aero. Res. Inst., Report 205, 1941.

The expression for the velocity potential for compressible flow given by Tomotika and Umemoto is slightly in error.

In the original symbols the coefficient of $\cos (2\delta + \alpha)$ in the expression for φ_1 (Eq. 122) is given as $(h^2/k)(2 + \lambda_* + \tfrac{5}{3}\lambda_*^2 + \tfrac{2}{3}\lambda_*^3 - 8\lambda_*^2 \log \lambda_*)$ but should read as $(h^2/k)(2 + \tfrac{8}{3}\lambda_*^2 + \tfrac{2}{3}\lambda_*^4 - 8\lambda_*^2 \log \lambda_*)$.

flow past particular obstacles. We may note that for bodies symmetrical about axes parallel and perpendicular to the uniform stream,

$$b_0 = b_1 = \cdots = 0. \tag{4.20}$$

For these doubly-symmetrical bodies, in fact,
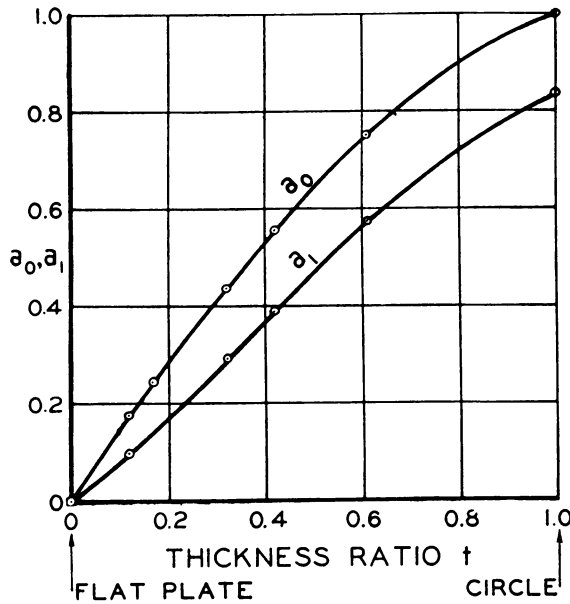
$$f_2 = f_4 = \cdots = 0. \tag{4.21}$$

FIG. 2. a_0, a_1 for Joukowski aerofoils (0° incidence).

5. Case of finite circulation. In the case of flow past a body about which a finite circulation exists, the assumption of a power series expansion for φ becomes

$$\varphi = r \cos \theta + f_0(\theta) + r^{-1}f_1(\theta) + r^{-2}f_2(\theta) + \cdots , \tag{5.1}$$

where f_0 is not periodic, but $f_1, f_2, \cdots$ must be so. Once again we can take the criterion of validity of this assumption to be the periodicity of the resulting solutions for $f_1, f_2, \cdots$. Substituting (5.1) in the original equation (1.1) and equating coefficients of r^{-n-2}, the equation for f_n is obtained as

$$f_n' (1 - M^2 \sin^2 \theta) - f_n'(n + 1)M^2 \sin 2\theta + f_n[n^2 + nM^2 - (n^2 + 2n)M^2 \cos^2 \theta]$$
$$= M^2 E_n'(f_{n-1}, f_{n-2}, \cdots, f_0, \theta), \tag{5.2}$$

where

$$E_n' = E_n - 2(n + 1) \cos \theta f_0' f_{n-1}' - 2 \sin \theta f_0'' f_{n-1}' - 2 \sin \theta f_0' f_{n-1}''$$

$$+ \tfrac{1}{2}f_0' \sum_{p=0}^{n-2} [p(n - p - 2)(f_p' f_{n-p-2} + f_p f_{n-p-2}') + f_p'' f_{n-p-2}' + f_p' f_{n-p-2}'']$$

$$- (\gamma - 1)f_0'[(n - 1)^2 f_{n-1} + f_{n-1}'']$$

$$+ (\gamma - 1)f_0' \sum_{p=0}^{n-2} f_p'[(n - p - 2)^2 f_{n-p-2} + f_{n-p-2}''] \qquad (n > 1). \tag{5.3}$$

As before, the appearance of a term $\cos n\theta$ in E_n' at any stage of the solution in powers of M^2 will indicate that f_n is non-periodic and hence that the assumption (5.1) is invalid. We shall in fact show that in this case, non-periodic solutions do arise. However, a more appropriate form for the function φ presents itself, and this is shown to lead to a solution which is self-consistent.

Confining attention to the order M^2, we may write

$$f_n = F_{n,0} + M^2 F_{n,1}. \tag{5.4}$$

$F_{n,0}$, of course, is the incompressible solution

$$F_{n,0} = A_n \cos (n\theta + \epsilon_n). \tag{5.5}$$

Substituting (5.4) and (5.5) in (5.2) and equating coefficients of powers of M^2 we obtain the equation for $F_{n,1}$ as

$$
\begin{aligned}
F_{n,1}'' + n^2 F_{n,1} = {} & A_n n(n+1) \cos \left[(n+2)\theta + \epsilon_n\right] + A_{n-1}\big\{2n(n-1)f_0' \sin (n\theta + \epsilon_{n-1}) \\
& + 2(n-1)f_0' \sin \left[(n-2)\theta + \epsilon_{n-1}\right] \\
& + 2(n-1)f_0'' \sin \theta \sin \left[(n-1)\theta + \epsilon_{n-1}\right] \\
& - \tfrac{1}{2}f_0' \sum_{p=0}^{n-2} A_p A_{n-p-2}(n-p-2) \sin \left[(n-2p-2)\theta + \epsilon_{n-p-2} - \epsilon_p\right]\big\} \\
& + \text{contribution from } E_n.
\end{aligned}
\tag{5.6}
$$

It will be shown later that f_0'' is of order M^2 and may thus be neglected here. As shown previously E_n does not give rise to a term $\cos n\theta$ or $\sin n\theta$ in this equation. Hence there is one term on the right of (5.6) which gives a non-periodic particular solution for $F_{n,1}$; this is

$$2n(n-1)f_0' A_{n-1} \sin (n\theta + \epsilon_{n-1}) \tag{5.7}$$

and the power series expansion (5.1) for φ is not valid.

A form for φ which will avoid an inconsistent solution is suggested by the relation

$$\nabla^2 \left[\log r \cdot G(r, \theta)\right] = \frac{2}{r} \frac{\partial G}{\partial r} + \log r \cdot \nabla^2 G, \tag{5.8}$$

where G is an arbitrary function of r and θ, and ∇^2 is the Laplacian operator. If we choose G to be a solution of Laplace's equation, in particular,

$$G = B_n \cdot r^{-n} \cos (n\theta + \epsilon_n'), \tag{5.9}$$

where B_n and ϵ_n are arbitrary constants, then

$$\nabla^2(G \log r) = -2n B_n r^{-n-2} \cos (n\theta + \epsilon_n'). \tag{5.10}$$

If now, in place of (5.1) we write

$$\varphi = r \cos \theta + f_0 + \sum_{n=1}^{\infty} r^{-n}\left[f_n + \log r \cdot M^2 B_n \cos (n\theta + \epsilon_n')\right], \tag{5.11}$$

the equation (5.6) for $F_{n,1}$ is unaffected except for the addition to the right-hand side of a term

$$2n B_n \cos (n\theta + \epsilon_n').$$

The particular solution for $F_{n,1}$ will thus be periodic if we choose

$$B_n = (n - 1)f_0' A_{n-1}, \qquad \epsilon_n' = \epsilon_{n-1} + \pi/2 \qquad (n > 1). \tag{5.12}$$

The case of $n = 1$ warrants special attention since we cannot write $F_{n-1,0} = A_{n-1} \cos[(n-1)\theta + \epsilon_{n-1}]$ when $n = 1$. When $n = 1$, the "dangerous" term on the right of (5.6) is $-2f_0'^2 \cos\theta$ so that we must choose $B_1 = f_0'^2$, $\epsilon_1' = 0$. Hence,

$$\sum_{n=1}^{\infty} r^{-n} B_n \cos(n\theta + \epsilon_n') = f_0'^2 \frac{\cos\theta}{r} - f_0' \sum_{n=2}^{\infty} \frac{(n-1)A_{n-1}\sin(n\theta + \epsilon_{n-1})}{r^n}$$

$$= f_0'^2 \frac{\cos\theta}{r} + f_0' \sum_{1}^{\infty} \left\{ \cos\theta \frac{\partial}{r\partial\theta}\left[\frac{A_n \cos(n\theta + \epsilon_n)}{r^n}\right] \right.$$

$$\left. + \sin\theta \frac{\partial}{\partial r}\left[\frac{A_n \cos(n\theta + \epsilon_n)}{r^n}\right] \right\}. \tag{5.13}$$

But

$$r\cos\theta + \kappa_0\theta + \sum_{n=1}^{\infty} r^{-n} A_n \cos(n\theta + \epsilon_n) = \varphi_0, \tag{5.14}$$

where φ_0 is the velocity potential for incompressible flow and $2\pi\kappa_0 (= 2\pi f_0'$, neglecting terms in M^2) is the corresponding circulation. Thus

$$\sum_{1}^{\infty} r^{-n} B_n \cos(n\theta + \epsilon_n') = \kappa_0\left(\cos\theta\frac{\partial\phi_0}{r\partial\theta} + \sin\theta\frac{\partial\phi_0}{\partial r}\right) = \kappa_0\frac{\partial\phi_0}{\partial y}, \tag{5.15}$$

where y is the space ordinate at right angles to the direction of the free stream.

We have then, that when a finite circulation exists, the expression for φ, correct to the order M^2 at least, is

$$\varphi = r\cos\theta + f_0 + \sum_{n=1}^{\infty} r^{-n}\cdot f_n + M^2\kappa_0 \log r\cdot\frac{\partial\phi_0}{\partial y} \tag{5.16}$$

and this expression is confirmed by the known results for a circular cylinder.[7] We may also note, as a matter of convenience, that the expression for f_n in (5.16) may be obtained by formally ignoring the log term when substituting in (1.1), and rejecting any non-periodic solutions.

6. **Solutions for f_0 and f_1.** Putting $n = 0$ in (5.2) and noting that $E_0' = 0$, we have

$$f_0'' = \frac{M^2 \sin 2\theta}{1 - M^2 \sin^2\theta} f_0', \tag{6.1}$$

integration of which gives

$$f_0' = \frac{\kappa}{1 - M^2 \sin^2\theta}, \tag{6.2}$$

where κ is a constant which may depend on M^2. This equation shows that the circulation about the body in compressible flow is

$$C = \int_0^{2\pi} f_0' d\theta = \frac{2\pi\kappa}{(1 - M^2)^{1/2}}. \tag{6.3}$$

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[7] S. Tomotika and H. Umemoto, loc. cit., Appendix.

Substituting for κ from (6.3) in (6.2) gives f_0' in terms of the circulation, and an identical equation for the transverse velocity at large r has been found by Glauert[8] using "small perturbation" theory.

The contribution to the velocity potential is found by integration of (6.2) to be

$$f_0 = \kappa(1 - M^2)^{-1/2} \arctan \left[(1 - M^2)^{1/2} \tan \theta\right]$$

or, in terms of the circulation C,

$$f_0 = \frac{C}{2\pi} \arctan \left[(1 - M^2)^{1/2} \tan \theta\right]. \tag{6.4}$$

Just as the magnitude of the circulation about a body in incompressible flow is arbitrary, so in compressible flow, the relation between circulation and Mach number is arbitrary. In certain cases an independent criterion for this relation is available, e.g., for sharp-tailed bodies the circulation should be such as to prohibit an infinite velocity at the tail-point for all Mach numbers. For blunt bodies, a convenient criterion is that the circulation shall be the same at all Mach numbers. In this case, we may treat C as a constant and the expansion of f_0 in powers of M^2 is

$$f_0 = \frac{C}{2\pi} \left[\theta - \tfrac{1}{4}M^2 \sin 2\theta + M^4(-\tfrac{1}{8} \sin 2\theta + \tfrac{1}{32} \sin 4\theta) + \cdots \right]. \tag{6.5}$$

(Actually this expression for f_0 has not been proved valid for powers of M^2 beyond the first. However it seems probable that equation (6.1) for f_0 will hold whatever the form of the remainder of the expression for φ. In the case of a circular cylinder with circulation, it is not hard to show that (6.5) is correct to the order M^4 at least.)

Putting $n = 1$ in equation (5.2), we have

$$f_1''(1 - M^2 \sin^2 \theta) - 2f_1' M^2 \sin 2\theta + f_1(1 + M^2 - 3M^2 \cos^2 \theta) = 0, \tag{6.6}$$

the only finite terms in E_1' being terms which would give rise to non-periodic particular solutions. As shown in section 2, the solution is

$$f_1 = \frac{A \cos (\theta + \alpha)}{1 - M^2 \sin^2 \alpha} \tag{6.7}$$

where we are now justified in using this expression to the order M^2 only; i.e.,

$$f_1 = a_0 \cos (\theta + \alpha_0)$$
$$+ M^2[a_0 \cos (\theta + \alpha_0) \sin^2 \theta + a_1 \cos (\theta + \alpha_0) - a_0 \alpha_1 \sin (\theta + \alpha_0)], \tag{6.8}$$

the constants having the same meaning as in section 2 but different values owing to the finite circulation.

As in the previous work, it is possible to derive the values of a_0, a_1, α_0 and α_1 by comparison of (6.8) with the known results for certain obstacles. Results are available[9] for the flow of a compressible fluid about a circular cylinder of unit radius, with constant circulation C. Comparison with (6.8) shows that

$$a_0 = 1, \qquad a_1 = \tfrac{5}{6} + \left(\frac{C}{2\pi}\right)^2, \tag{6.9}$$

[8] H. Glauert, *The effect of compressibility on the lift on an aerofoil*, Proc. Roy. Soc., A **118**, 113 (1928).
[9] S. Tomotika and H. Umemoto, loc. cit., Appendix.

so that the value of f_1 is not greatly altered by the presence of a small circulation.

We may also note that for a body, symmetrical about axes parallel and perpendicular to the uniform stream

$$f_1'(\theta) = f_1'(\pi - \theta),$$

giving $\alpha_0 = \alpha_1 = 0$ and a corresponding result holds for $f_3, f_5, \cdots$. In the case of even values of n, the angle constants in f_r are equal to π.

7. Appendix; Proof that the equation $y'' + y[n^2 - M^2(n^2 - 1) \cos 2\theta] = 0$ **gives a periodic solution.** The general form of Mathieu's equation[10] may be taken as

$$y'' + y(a + 16q \cos 2\theta) = 0 \qquad (7.1)$$

the solution of which is periodic only if a and q are suitably related. The required relation may be written

$$a = n^2 + a_1 q + a_2 q^2 + \cdots , \qquad (7.2)$$

since a must reduce to the square of an integer when q vanishes. Now, for the particular case of Mathieu's equation with which we are concerned,

$$a = n^2, \qquad 16q = - M^2(n^2 - 1), \qquad (7.3)$$

and (7.2) will be satisfied to the order of M^2 if $a_1 = 0$, or, when $n = 1$, by any value of a_1. Thus we need to show that $a_1 = 0$ for the general Mathieu equation.

This may be done by assuming a solution of (7.1) of the form

$$y = \cos n\theta + q\alpha_1(\theta) + q^2\alpha_2(\theta) + \cdots , \qquad (7.4)$$

where $\alpha_1(\theta), \alpha_2(\theta), \cdots$ are periodic functions of θ, independent of q. (The proof for solutions which reduce to $\sin n\theta$ when $q = 0$ is identical.) Substituting (7.2) and (7.4) in (7.1) and equating coefficients of q we obtain

$$\alpha_1'' + n^2\alpha_1 + \cos n\theta(a_1 + 16 \cos 2\theta) = 0,$$

i.e.,

$$\alpha_1'' + n^2\alpha_1'' = 8 \cos(n + 2)\theta + a_1 \cos n\theta + 8 \cos(n - 2)\theta$$

giving $a_1 = 0$, $(n \neq 1)$ since α_1 is periodic. For the particular case $n = 1$ we have instead $a_1 = -8$, but in this case we no longer require a_1 to be zero.

Further coefficients $a_2, a_3, \cdots$ and the functions $\alpha_1, \alpha_2, \cdots$ may be determined by equating higher powers of q; the general Mathieu function may thus be constructed if desired.

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[10] Whittaker and Watson, loc. cit., p. 409.