# NOTE ON THE PROBLEM OF HEAT CONDUCTION IN A SEMI-INFINITE HOLLOW CYLINDER*

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In a recent article[1] C. J. Tranter determined the heat conduction in a semi-infinite cylinder, in the non-steady case, by means of a combination of a Fourier transform and a Laplace transform. Tranter's problem, and generalizations of it involving more complicated boundary conditions, may be solved by a method which was employed in an earlier paper[2] and involves one Laplace transform only.

Let us consider the following generalization of Tranter's problem:

$$\mathrm{A}\begin{cases} k\left(\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{\partial^2}{\partial z^2}\right)T(r,z;t)=\dfrac{\partial}{\partial t}T(r,z;t), \quad a\leqq r\leqq b,\ z>0,\ t>0; & (1)\\ T(r,z;t)=0 \quad \text{for} \quad z=0,\ a\leqq r\leqq b,\ t>0; & (2)\\ T(r,z;t)=0 \quad \text{for} \quad r=a,\ t>0,\ z>0; & (3)\\ T(r,z;t)=\varphi(z,t) \quad \text{for} \quad r=b,\ t>0; & (4)\\ T(r,z;t)=0 \quad \text{for} \quad t=0,\ a\leqq r\leqq b;\ z=0. & (5)\end{cases}$$

The difference between $A$ and Tranter's problem lies in the fact that in $A$ the boundary condition (4) involves a variable temperature.

Let us write

$$L\{T(r,z;t)\}=\int_0^\infty e^{-pt}T(r,z;t)dt=T^*(r,z;p), \tag{6}$$

$$L\{\varphi(z,t)\}=\int_0^\infty e^{-pt}\varphi(z;t)dt=\varphi^*(z;p). \tag{7}$$

If the system $A$ is acted upon by the Laplace operator $L$ defined in (6) and (7), it is readily seen[3] that the Laplace transform $T^*(r, z; p)$ of the unknown temperature $T(r, z; t)$ must satisfy the system

$$\mathrm{A}^*\begin{cases}\left(\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{\partial^2}{\partial z^2}-\dfrac{p}{k}\right)T^*(r,z;p)=0, & (8)\\ T^*(r,z;p)=0 \quad \text{for} \quad z=0 \quad \text{and for} \quad r=a, & (9)\\ T^*(r,z;p)=\varphi^*(z;p) \quad \text{for} \quad r=b. & (10)\end{cases}$$

If we write

$$F^*(a,b,r;\lambda)=\frac{I_0(\lambda r)K_0(\lambda a)-I_0(\lambda a)K_0(\lambda r)}{I_0(\lambda b)K_0(\lambda a)-I_0(\lambda a)K_0(\lambda b)}, \tag{11}$$

where $I_0$, $K_0$ denote Bessel functions, it is easily seen that $F^*(a, b, r; \lambda)\sin \mu z$ is a solution of the differential equation (8), satisfying the boundary conditions (9) when

$$\lambda^2-\mu^2=p/k. \tag{12}$$

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[1] C. J. Tranter, Phil. Mag. (7) **35**, 102–105 (1944).

[2] A. N. Lowan, Phil. Mag. (7) **24**, 410–424 (1937). This article will be referred to as ANL.

[3] For details, see ANL.

In view of the identity

$$\varphi^*(z;\,p) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \varphi^*(\alpha;\,p)\sin\mu z\sin\mu\alpha d\mu d\alpha,$$

and since $F^*(a, b, b; \lambda) = 1$, for $r = b$ it follows that the expression

$$T^*(r, z;\,p) = \frac{2}{\pi}\int_0^\infty\int_0^\infty F^*(a, b, r;\,\lambda)\varphi^*(\alpha;\,p)\sin\mu z\sin\mu\alpha d\mu d\alpha \tag{13}$$

is a solution of the system $A^*$. It now remains to subject (13) to the inverse Laplace operator. If

$$\int_0^\infty e^{-pt}F_\mu(a, b, r;\,t)dt = F^*(a, b, r;\,\lambda) = F^*\left(a, b, r;\,\sqrt{\mu^2 + \frac{p}{k}}\right), \tag{14}$$

then by Borel's theorem[4] the inversion of (14) leads to

$$T(r, z;\,t) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin\mu z\sin\mu\alpha d\mu d\alpha \int_0^t F_\mu(a, b, r;\,\eta)\varphi(\alpha, t-\eta)d\eta. \tag{15}$$

If

$$G_\mu(a, b, r;\,t) = \int_0^t F_\mu(a, b, r;\,\eta)d\eta, \tag{16}$$

the solution (15) may be written in the alternative form

$$T(r, z;\,t) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin\mu z\sin\mu\alpha d\mu d\alpha \int_0^t \frac{\partial}{\partial\eta}G_\mu(a, b, r;\,\eta)\varphi(\alpha;\,t-\eta)d\eta. \tag{17}$$

From (16) it follows that

$$L\{G_\mu(a, b, r;\,t)\} = \int_0^\infty e^{-pt}G_\mu(a, b, r;\,t)dt = p^{-1}L\{F_\mu(a, b, r;\,t)\} = \frac{1}{p}F^*(a, b, r;\,\lambda),$$

whence in view of (11),

$$p\int_0^\infty e^{-pt}G_\mu(a, b, r;\,t)dt = \frac{I_0(\lambda r)K_0(\lambda a) - I_0(\lambda a)K_0(\lambda r)}{I_0(\lambda b)K_0(\lambda a) - I_0(\lambda a)K_0(\lambda b)} = \frac{Y(p)}{Z(p)}\cdot \tag{18}$$

The formal inversion of (18) is

$$G_\mu(a, b, r;\,t) = \frac{Y(0)}{Z(0)} + \sum_{i=1}^\infty \frac{Y(p_i)e^{p_i t}}{p_i[dZ/dp]_{p=p_i}}, \tag{19}$$

where the summation extends over the roots of $Z(p) = 0$. Making use of (12) and remembering that

$$I_0(x) = J_0(xi), \qquad K_0(x) = -Y_0(xi) - (\tfrac{1}{2}\pi i - \gamma + \log 2)J_0(ix),$$

where $J_0$, $Y_0$ are Bessel functions, we obtain from (19), after some simple transformations,

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[4] See ANL, p. 413.

$$G_\mu(a, b, r; t) = F^*(a, b, r; \mu) - \pi \sum_{i=1}^{\infty} \alpha_i^2 \frac{e^{-k(\alpha_i^2+\mu^2)t}}{\alpha_i^2 + \mu^2} \cdot \frac{J_0(\alpha_i a)J_0(\alpha_i b)}{J_0^2(\alpha_i a) - J_0^2(\alpha_i b)} \cdot \left\{J_0(\alpha_i r)Y_0(\alpha_i a) - J_0(\alpha_i a)Y_0(\alpha_i r)\right\}. \quad (20)$$

The complete solution of the system $A$ is given by (17) in conjunction with (20).

In the particular case where $\varphi(z, t)$ is a function of $z$ only, (16) becomes

$$T(r, z; t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty G_\mu(a, b, r; t)\varphi(\alpha) \sin \mu z \sin \mu \alpha d\mu d\alpha,$$

which is in agreement with Tranter's solution.

In a similar manner it is possible to treat the more complicated case where the boundary condition (4) is replaced by

$$\left(\alpha \frac{\partial}{\partial r} + \beta\right) T(r, z; t) = 0 \quad \text{for} \quad r = b$$

The formal solution corresponding to this boundary condition is in fact given once more by (17), with the function $G_\mu(a, b, r; t)$ satisfying the integral equation

$$p \int_0^\infty e^{-pt} G_\mu(a, b, r; t)dt = \frac{Y(p)}{Z(p)}, \quad (21)$$

where

$$Y(p) = I_0(\lambda r)K_0(\lambda a) - I_0(\lambda a)K_0(\lambda r),$$

$$Z(p) = \alpha\lambda\left\{K_0(\lambda a)I_0'(\lambda b) - I_0(\lambda a)K_0'(\lambda b)\right. + \beta\left\{I_0(\lambda b)K_0(\lambda a) - I_0(\lambda a)K_0(\lambda b)\right\}.$$

The inversion of (21) proceeds in accordance with formula (19).

## EFFECT OF A SMALL HOLE ON THE STRESSES IN A UNIFORMLY LOADED PLATE*

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In a paper of the same title Martin Greenspan recently[1] determined the stress distribution in a large, uniformly loaded plate weakened by a small hole of an approximately ovaloid shape. Greenspan employed a rather laborious method of piecing together particular solutions of the biharmonic equation for the stress function until all the boundary conditions could be satisfied. This process would become prohibitive in case of more complicated boundary conditions. It is the purpose of this note to apply to the same problem the elegant and more general, yet not well known, method for solving plane problems of elasticity which is most often associated with the name of N. I. Mushelisvili.[2]

* Received July 5, 1944.

[1] This Quarterly, **2**, 60–71 (1944).

[2] See for instance N. I. Mushelisvili, Math. Annalen, **107**, 282–312 (1932). For detailed English exposition see I. S. Sokolnikoff's *Mathematical theory of elasticity*, (mimeographed lecture notes, Brown University, 1941), pp. 243–318.