$$\frac{\partial^2\phi}{\partial\alpha^2} + \frac{2\partial\phi}{\partial\alpha} = \frac{\partial^2\phi}{\partial\beta^2}\cdot \tag{21}$$

Thus the problem for the infinite canal, of which Eq. (19) is the solution, is equivalent to that of solving Eq. (21) ($\alpha, \beta \geqq 0$) with the boundary conditions

$$\phi(\beta, 0) = 0, \qquad \phi(0, \alpha) = e^{-\alpha}\left[\tfrac{1}{2}\alpha\left\{3I_0(\alpha) + 4I_1(\alpha) + I_2(\alpha)\right\} + I_0(\alpha) + I_1(\alpha)\right].$$

NOTE ON THE ELLIPTIC WING*

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The acceleration potential method of Prandtl for studying the aerodynamics of a lifting surface is well known.[1] In the applications to specific surfaces one is naturally led to wings with circular or elliptic plan view. Prandtl's theory has been completely elaborated for these two cases by W. Kinner[2] and K. Krienes.[3]

In an effort to extend the class of wings for which numerical results have been obtained, a theory based on the work of Krienes was developed for the semi-elliptic wing by E. R. Lorch. Computations carried out for this case by the author had to be abandoned due to very poor convergence. This has suggested that the question of convergence in Krienes' work, which plays an important role there, be examined more closely. This matter is investigated in the present note.

We base our discussion on Krienes' paper. The reader is referred to it for the details which it is impossible to reproduce here. In this work the pressure p is expressed in terms of a potential function ψ. We have

$$p - p_\infty = -\rho_0 V^2\psi(x, y, z), \qquad \Delta\psi = 0, \tag{1}$$

where p_∞ is the pressure at infinity, ρ_0 is the density, and V is the velocity of the wing. In turn ψ is expanded in a series

$$\psi = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} \overset{m}{a}_n \overset{m}{\psi}_n + \sum_{n=1}^{\infty} C_n\Phi_n, \tag{2}$$

(Krienes, Eq. (81) with $D_n = 0$) where

$$\overset{m}{\psi}_n(\rho, \mu, \nu) = \overset{m}{E}_n(\mu)\overset{m}{E}_n(\nu)\overset{m}{E}_n(\rho)\int_\rho^\infty \frac{d\rho}{[\overset{m}{E}_n(\rho)]^2[(\rho^2-1)(\rho^2-k)]^{1/2}},$$

$$\Phi_n = \sum_m \overset{m}{b}_n \overset{m}{\Phi}_n, \qquad \overset{m}{\Phi}_n = c^{1-n}\frac{d}{dc}\left[c^n \overset{m}{\psi}_n(x, y, z, c)\right],$$

* Received June 5, 1944.

[1] L. Prandtl, *Beitrag zur Theorie der tragenden Fläche*, Zeit. f. angew. Math. u. Mech. **16**, 360–361 (1936).

[2] W. Kinner, *Die kreisförmige Tragfläche auf potentialtheoretischer Grundlage*, Ing.-Archiv, **8**, 47–80 (1937).

[3] K. Krienes, *Die elliptische Tragfläche auf potentialtheoretischer Grundlage*, Zeit. f. angew. Math. u. Mech. **20**, 65–88 (1940).

E_m being Lamé functions and $2c$, $2c(1-k^2)^{1/2}$ being the lengths of the axes of the fundamental ellipse. The Φ_n are needed to give infinite lift density on the leading edge.

Let $z=z(x, y)$ be the equation of the wing surface. We consider here the special case

$$z = - x \tan \alpha_0 \approx - x\alpha_0, \tag{3}$$

which represents a plane wing with a small angle of attack α_0. The boundary condition is

$$\frac{\delta z}{\delta x} = \frac{w(x, y)}{V} = \int_{-\infty}^{x} \frac{\delta \psi}{\delta z} dx, \tag{4}$$

where $w(x, y)$ is the downwash. This in conjunction with (3) leads to the following equations for the coefficients a_n^m and C_n in (2):

$$a_n^m = 0, \tag{5}$$

$$\left.\begin{aligned} &\sum_{r=0}^{\infty} C_{2r+1} \frac{(1-k^2)^{1/2}}{r(2r+1)-\alpha(2\alpha-1)} + C_{2\alpha}\frac{2}{4\alpha-1} = \begin{cases} -\frac{2}{3}\alpha_0 & (\alpha = 1), \\ 0 & (\alpha > 1), \end{cases} \\ &C_{2\alpha-1}I_{2\alpha-1,2\alpha-1} + \sum_{r=1}^{\infty} C_{2r}I_{2r,2\alpha-1} = 0, \qquad (\alpha = 1, 2, \cdots), \end{aligned}\right\} \tag{6}$$

where $I_{\gamma,\delta}$ are certain definite integrals which are either elementary or elliptic.

Only Φ_1 contributes to the lift and only Φ_2 to the pitching moment, so that we are interested in obtaining C_1 and C_2. In Krienes' paper the series (6) are broken off after $\alpha=2$, which leads to four linear equations giving approximations to $C_1, \cdots, C_4$; the results are, for an axes ratio $(1-k^2)^{1/2}=0.2$:

$$\left.\begin{aligned} &\text{Lift} = \tfrac{1}{2} \times 4.55\alpha_0\rho_0 V^2 \times \text{area of ellipse}, \\ &\text{Pitching moment} = -1.98\alpha_0 c(1-k^2)^{1/2}(\rho_0/2)V^2 \times \text{area of ellipse} \\ &\text{Center of pressure at } 28.3\% \text{ of the maximum wing chord.} \end{aligned}\right\} \tag{7}$$

These results seem to be in close agreement with those found experimentally.

As a check on convergence, the computations leading to the results (7) were now extended by carrying the series (6) through $\alpha=3$, which gives six linear equations in $C_1, \cdots, C_6$. This addition of one more term leads to equations of the sixth degree for the coefficients of the Lamé functions (as compared with quadratics for the computations of Krienes); also, the elliptic integrals for this case give more difficulty. To take $\alpha>3$ in (6) would make the computations almost prohibitive.

The results now become

$$\left.\begin{aligned} &\text{Lift} = \tfrac{1}{2} \times 4.54\alpha_0\rho_0 V^2 \times \text{area of ellipse}, \\ &\text{Pitching moment} = -1.98\alpha_0 c(1-k^2)^{1/2}(\rho_0/2)V^2 \times \text{area of ellipse}, \\ &\text{Center of pressure at } 28.4\% \text{ of the maximum wing chord.} \end{aligned}\right\} \tag{8}$$

This agrees closely with (7) (order of $\frac{1}{3}\%$). Thus one is led to believe that convergence of the system (6) is very rapid.