ON THE TREATMENT OF DISCONTINUITIES IN BEAM DEFLECTION PROBLEMS*

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In a note on the treatment of discontinuities in beam deflection problems (Quarterly of Applied Mathematics, **1**, 349–351) Mr. C. L. Brown suggests the use of Heaviside's unit step function. He thus avoids what he calls the "sectionalizing" treatment in the integration of the differential equation for the deflection of a beam with discontinuous transversal loading.

Mr. Brown's method appears to be equivalent to the procedure which seems to have been first developed by R. Macaulay[1] and has since been included in several British textbooks.[2,3] In order to establish expressions for moments with discontinuous variations, Macaulay introduces terms in twisted brackets, such as $\{x-a\}$, with the convention that these terms be neglected when the quantity within the brackets becomes negative. When integrating the term in question, the quantity in brackets is to be regarded as the independent variable instead of x; the indefinite integral of $\{x-a\}$ would be $\frac{1}{2}\{x-a\}^2$.

Taking Mr. Brown's example, the expression for the bending moment of the beam (l. c., Fig. 1, p. 349) would be with Macaulay's notation:

$$EI\frac{d^2y}{dx^2} = M = -M_1 + R_1x - P\{x-a\}.$$

The first integration would give

$$EI\frac{dy}{dx} = -M_1x + \tfrac{1}{2}R_1x^2 - \tfrac{1}{2}P\{x-a\}^2 + C_1,$$

and the second integration

$$EIy = -\tfrac{1}{2}M_1x^2 + \tfrac{1}{6}R_1x^3 - \tfrac{1}{6}P\{x-a\}^3 + C_1x + C_2.$$

All the above equations hold at all parts of the span so that there are only two constants of integration to be determined from the conditions at both ends of the span, instead of having two for each section of the span as in the classical treatment.

It is therefore apparent that Macaulay's twisted bracket is but another symbol for the multiplication by the unit step function.

An important remark is to be made about the use of this procedure with regard to distributed loads. These must always be made to extend to the right-hand extremity of the beam, introducing negative loads if necessary. An extension of the method

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[1] R. Macauly, *Note on the deflection of beams*, Messenger of Mathematics, **48**, 129 (1919).

[2] R. V. Southwell, *An introduction to the theory of elasticity*, Oxford, 1936, §§194–196.

[3] J. Case, *The strength of materials*, 2nd edition, Arnold & Co., London, 1932, §169.

due to H. A. Webb[3] covers the effect of a concentrated bending couple applied at an intermediate point of the span. As an application, R. V. Southwell suggests (Example 14, l.c.) the derivation of the theorem of three moments for a continuous beam by the same method. The use of the method for a beam with a stepwise variation of bending rigidity seems however to be Mr. Brown's original contribution.

It is hoped that the discussed method, whatever the symbols used, will get more attention from engineers on this side of the Atlantic, as it carries with it a very substantial shortening of the computations.

FORMULAS FOR COMPLEX INTERPOLATION*

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An analytic function of $z=x+iy$ may be approximated by a complex polynomial of degree n passing through $n+1$ points in accordance with the Lagrange-Hermite formula of interpolation. For the important special case when the given $n+1$ points are equidistantly spaced along any straight line in the z-plane, the following tables give the real and imaginary parts of the coefficients $A_k(P)$ of the interpolation polynomial $f(z)=A_k(P)f(z_k)$, where $P=(z-z_0)/h=p+iq$ and h is the complex tabular interval. The formulas cover the cases ranging from complex quadratic (3 points) to complex quintic interpolation (6 points).

Quadratic interpolation (3 points)

$$\Re e A_{-1}(P) = \tfrac{1}{2}[p(p-1)-q^2], \qquad \Im m A_{-1}(P) = q(p-.5),$$

$$\Re e A_0(P) = 1-p^2+q^2, \qquad \Im m A_0(P) = -2pq,$$

$$\Re e A_1(P) = \tfrac{1}{2}[p(p+1)-q^2], \qquad \Im m A_1(P) = q(p+.5).$$

Cubic interpolation (4 points)

$$\Re e A_{-1}(P) = \frac{(1-p)}{6}[p(p-2)-3q^2], \qquad \Im m A_{-1}(P) = \frac{q}{6}[q^2-2+3p(2-p)],$$

$$\Re e A_0(P) = 1+\tfrac{1}{2}[p(p^2-2p-1)+q^2(2-3p)], \qquad \Im m A_0(P) = \frac{q}{2}[p(3p-4)-q^2-1],$$

$$\Re e A_1(P) = -\tfrac{1}{2}[p(p-2)(p+1)+q^2(1-3p)], \qquad \Im m A_1(P) = \frac{q}{2}[p(2-3p)+q^2+2],$$

$$\Re e A_2(P) = \frac{p}{6}(p^2-3q^2-1), \qquad \Im m A_2(P) = \frac{q}{6}[3p^2-q^2-1].$$

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