

LATERAL BENDING OF SYMMETRICALLY LOADED CONICAL DISCS*

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1. Introduction. The general theory of lateral bending for thin circular plates of variable thickness is given in Timoshenko's book,¹ "Theory of plates and shells," where also may be found numerous references to the literature of the subject. Of particular interest is a reference to Föppl who indicated the analogy existing between the rotating disc problem and that of lateral bending in a circular plate of variable thickness. Comparison of the solution for the rotating conical disc problem with the corresponding one for lateral bending shows that the basic differential equations involved, and the expressions for the stresses, are analogous. Therefore, previously described methods² for obtaining solutions of the former problem in terms of hypergeometric functions are applicable to the latter problem. It will appear later that the special type of hypergeometric differential equation associated with the lateral bending problem has solutions which give the stress coefficients with less labor than in the case of the rotating conical disc.

The stress coefficients have been arranged conveniently for numerical calculation of conical discs, which are component parts of a wide variety of engineering structures. The head of a large poppet valve provides a particular example where the principal stress member can be approximated by a system of incomplete conical discs. In order to illustrate an application of the theory, stress coefficients for conical discs subject to lateral bending as well as for rotating conical discs will be used to estimate stress distributions in a steel valve head of constant weight and various proportions. Since the coefficients are obtained from solutions of differential equations for thin discs, the approximate method breaks down in the neighborhood of the valve stem. These limitations have little effect near the periphery, which makes it possible to calculate valve proportions corresponding to approximately uniform stress distribution throughout the head. The description of the illustrative example at the end of the paper explains the method of calculation in detail.

2. Derivation of differential equation. Let M_r and M_t denote radial and tangential bending moments per unit length acting on an element of a circular plate at distance r from the center; then if Q is the corresponding circumferential shearing force per unit length, the equation of equilibrium is

$$M_r + rdM_r/dr - M_t = -Qr. \quad (1)$$

If w denotes downward deflection of the middle surface, then

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¹ Timoshenko, *Theory of plates and shells*, McGraw-Hill, 1st Edition 1940, Art. 54, p. 282.

² K. E. Bisshopp, *Stress coefficients for rotating discs of conical profile*, Journal of Applied Mechanics, Vol. 11, No. 1, March 1944, pp. A1-A9.

$$M_r = -D \left(\frac{d^2 w}{dr^2} + \frac{\sigma}{r} \frac{dw}{dr} \right) = D \left(\frac{d\varphi}{dr} + \frac{\sigma}{r} \varphi \right),$$

$$M_t = -D \left(\frac{1}{r} \frac{dw}{dr} + \sigma \frac{d^2 w}{dr^2} \right) = D \left(\frac{\varphi}{r} + \sigma \frac{d\varphi}{dr} \right),$$
(2)

where σ is Poisson's ratio, $\varphi = -dw/dr$ and $D = Eh^3/12(1-\sigma^2)$, E being Young's modulus and h the thickness of the plate; D is called the flexural rigidity.

In the case of a conical profile, h is a linear function of r , so that substitution from Eqs. (2) in Eq. (1) gives

$$D \frac{d}{dr} \left(\frac{d\varphi}{dr} + \frac{\varphi}{r} \right) + \frac{dD}{dr} \left(\frac{d\varphi}{dr} + \sigma \frac{\varphi}{r} \right) = -Q.$$
(3)

In order to reduce this equation to non-dimensional form, we introduce the radius R to the knife edge of the disc and the thickness h_0 at the center. If $r/R = x$, then $h = h_0(1-x)$, and Eq. (3) becomes

$$\frac{d^2 \varphi}{dx^2} + \left(\frac{1}{x} - \frac{3}{1-x} \right) \frac{d\varphi}{dx} - \left(\frac{1}{x^2} + \frac{3\sigma}{x(1-x)} \right) \varphi = - \frac{12QR^2(1-\sigma^2)}{Eh_0^3(1-x)^3}.$$
(4)

For any particular type of symmetrical loading the shearing force Q is a function of x alone. The maximum radial and tangential bending stresses S_r and S_t are obtained from the general solution of Eq. (4) with the aid of Eqs. (2) and the relations

$$S_r = 6M_r/h^2, \quad S_t = 6M_t/h^2.$$
(4a)

The problem of a conical disc supporting a concentrated vertical load P at the center has some interesting practical applications. In this case $Q = P/2\pi r = P/2\pi R x$, and Eq. (4) becomes

$$x(1-x) \frac{d^2 \varphi}{dx^2} + (1-4x) \frac{d\varphi}{dx} - \left(\frac{1-x}{x} + 3\sigma \right) \varphi = - \frac{6PR(1-\sigma^2)}{\pi E h_0^3 (1-x)^2}.$$
(5)

It can be verified by substitution that a particular integral of Eq. (5) is

$$\varphi_s(x) = - \frac{2PR(1+\sigma)}{\pi E h_0^3 (1-3\sigma)} \left(\frac{2-3\sigma}{(1-x)^2} + \frac{1}{x} + \frac{1}{1-x} \right).$$
(6)

The auxiliary equation, the solutions of which are independent of the type of loading, is obtained by setting the right hand side of Eq. (5) equal to zero. After making the substitution $\varphi = xF$, we obtain

$$x(1-x) \frac{d^2 F}{dx^2} + 3(1-2x) \frac{dF}{dx} - 3(1+\sigma)F = 0,$$
(7)

which is recognized to be of hypergeometric type.

3. Complementary functions. Equation (7) is of the form

$$x(1-x) \frac{d^2 F}{dx^2} + [c - (a+b+1)x] \frac{dF}{dx} - abF = 0,$$

where $c=3$, $a+b=5$, and $ab=3(1+\sigma)$. The first solution can be represented by a power series; the integral exponent difference³ $1-c=-2$ shows that the second solution contains a logarithm.* In the notation of the hypergeometric function, the first solution is

$$F_1(x) = F(a, b, c, x) = 1 + \frac{ab}{1 \cdot c} x + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} x^2 + \dots, \tag{8}$$

which converges absolutely and uniformly when $|x| < 1$. The asymptotic behavior of the hypergeometric function in the neighborhood of its poles is given by⁴

$$F(a, b, c, x) \underset{x \rightarrow 1}{\sim} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-x)^{c-a-b}, \tag{9}$$

whenever $c-a-b$ is an integer less than zero, $\Gamma(z)$ being the well known gamma function. Thus $F_1(x)$ has a second order singularity at $x=1$ such that,

$$F_1(x) \underset{x \rightarrow 1}{\sim} \frac{\Gamma(3)\Gamma(2)}{\Gamma(a)\Gamma(b)} (1-x)^{-2} = \frac{2(1-x)^{-2} \sin a\pi}{\pi(a-1)(a-2)(a-3)(a-4)}, \tag{10}$$

which may be used to approximate the function for values of x near unity. The presence of singularities of lower order in the remainder term for $F_1(x)$ makes this method unsuitable for accurate numerical work. Better approximations for similar functions with second and third order singularities are given in Ref. (2).

The logarithmic solution⁵ of Eq. (7) is†

$$F_2(x) = -\frac{(ab-4)(ab-6)}{2} F_1(x) \log_e x + \frac{1}{x^2} - \frac{ab-6}{x} - g(x), \tag{11}$$

where

$$g(x) = \sum_{n=2}^{\infty} \frac{(n+a-3) \dots (a-2)(n+b-3) \dots (b-2)}{n!(n-2)!} x^{n-2} \Phi_n, \tag{12}$$

and

$$\begin{aligned} \Phi_n &= \psi(n-3+a) + \psi(n-3+b) - \psi(n) - \psi(n-2) \\ &= \frac{1}{a-2} + \dots + \frac{1}{a+n-3} + \frac{1}{b-2} + \dots + \frac{1}{b+n-3} \\ &\quad - 1 - \frac{1}{2} - \dots - \frac{1}{n} - 1 - \frac{1}{2} - \dots - \frac{1}{n-2}. \end{aligned} \tag{13}$$

The principal part of expansion (11) shows that $F_2(x)$ has a second order singularity at the origin. The nature of the singularity at $x=1$ can be recognized by observing

³ Whittaker and Watson, *A course of modern analysis*, Cambridge, England, 4th Edition, 1927, p. 198.

* When $\sigma=1/3$ both solutions can be expressed in terms of rational algebraic functions.

⁴ Titchmarsh, *Theory of functions*, Oxford, England, 1932, p. 224.

⁵ Forsyth, *Theory of differential equations*, Cambridge, England, 1902, vol. 4, part 3, p. 147.

† The numerical value of c is used since it is independent of Poisson's ratio σ .

the limiting form of the n th term of $g(x)$ which is proportional to that of $F_1(x)$ provided $\lim_{n \rightarrow \infty} \Phi_n$ remains finite. That this is the case can be shown with the aid of the logarithmic derivative⁶ of the Gamma function, which gives

$$\lim_{n \rightarrow \infty} \Phi_n = 2 \left(\frac{1}{a-2} + \frac{1}{a-1} + \frac{1}{a} \right) - 2\gamma - 2\Psi(a) - \pi \cot a\pi, \quad (14)$$

where $\Psi(a) = \Gamma'(a+1)/\Gamma(a+1)$ and γ is Euler's constant. Thus $F_2(x)$ has a second order singularity at $x=1$ of magnitude

$$F_2(x) \sim - \lim_{n \rightarrow \infty} \Phi_n \frac{\sin a\pi}{\pi} / (1-x)^2. \quad (15)$$

The slow convergence of the power series near the singularities of $F_1(x)$ and $F_2(x)$ makes numerical evaluation of the stress coefficients for all values of x between zero and unity exceedingly difficult, in spite of available asymptotic approximations. One scheme for removing this difficulty would be to construct from the transformed differential equation (7) two new solutions of argument $1-x$ and combine them linearly with $F_1(x)$ and $F_2(x)$, as described in Ref. (2). This differential equation is invariant under transformation by $1-x$, which brings about added convenience of calculation; however, considerable further reduction in computation can be accomplished by expressing $F_1(x)$ and $F_2(x)$ in terms of symmetrical hypergeometric functions.

4. Solutions in terms of even and odd functions. Whenever $2c = a + b + 1$, which condition is satisfied by Eq. (7), the transformation $(1-2x)^2 = t$ reduces the standard form of the hypergeometric equation to

$$t(1-t) \frac{d^2F}{dt^2} + \left[\frac{1}{2} - \left(\frac{a}{2} + \frac{b}{2} + 1 \right) t \right] \frac{dF}{dt} - \frac{ab}{4} F = 0. \quad (16)$$

The solutions of this equation as functions of x are⁷

$$F \left\{ \frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}, (1-2x)^2 \right\} \equiv G_1(x), \quad (1-2x)F \left\{ \frac{1}{2}(a+1), \frac{1}{2}(b+1), \frac{3}{2}, (1-2x)^2 \right\} \equiv G_2(x).$$

This shows that $G_1(x) = G_1(1-x)$ and $G_2(x) = -G_2(1-x)$. Since only functions of x are involved,

$$\begin{aligned} G_1(x) &= C_1F_1(x) + C_2F_2(x), \\ G_2(x) &= D_1F_1(x) + D_2F_2(x), \end{aligned} \quad (17)$$

where C_1, C_2, D_1, D_2 are constants; $G_1(x)$ and $G_2(x)$ are respectively even and odd relative to the point $x = \frac{1}{2}$. The series for $G_1(x)$ and $G_2(x)$ are very convenient for computation when $.25 \leq x \leq .50$, while those for $F_1(x)$ and $F_2(x)$ are equally so when $0 \leq x \leq .25$. Since the G 's are symmetrical it is necessary to compute only one half as many fundamental values for constructing tables of stress coefficients as would be required with the F 's. From this point on therefore, the F 's are subordinated to the role of "helping functions," while the G 's form the basis of all subsequent calculations.

Returning to Eq. (17), we employ the familiar method of comparison of singularities for evaluation of the linear factors. It is apparent from the character of the F 's that the G 's have second order singularities at zero and unity whose values may be deduced from Eq. (9). After some reduction, we obtain

⁶ Ref. 3, p. 246.

⁷ Ref. 3, p. 297, Example 7.

$$\begin{aligned}
 C_1 &= \frac{\sqrt{\pi} \Gamma(a)\Gamma(b)}{32\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)} \left[1 + \lim_{n \rightarrow \infty} \Phi_n \frac{\sin a\pi}{\pi} \right], \\
 C_2 &= \sqrt{\pi}/[16\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)], \\
 D_1 &= - \frac{\sqrt{\pi} \Gamma(a)\Gamma(b)}{64\Gamma\{\frac{1}{2}(a+1)\}\Gamma\{\frac{1}{2}(b+1)\}} \left[1 - \lim_{n \rightarrow \infty} \Phi_n \frac{\sin a\pi}{\pi} \right], \\
 D_2 &= \sqrt{\pi}/[32\Gamma\{\frac{1}{2}(a+1)\}\Gamma\{\frac{1}{2}(b+1)\}].
 \end{aligned}
 \tag{18}$$

The functions $G_1(x)$, $G_2(x)$, and their derivatives are tabulated in Table 1.

Since the F 's and G 's are linearly dependent, $xG_1(x)$ and $xG_2(x)$ are fundamental solutions of Eq. (5), from which, by use of Eqs. (2) and (4a), the stress coefficients follow immediately.

5. Determination of the deflection functions. The deflection $w(x)$ can be expressed in the form

$$w(x) = R[w_1(x) + w_2(x) + \bar{w}_3(x)], \tag{18a}$$

where $w_1(x)$ and $w_2(x)$ arise from the complementary functions respectively, and $\bar{w}_3(x)$ arises from the particular integral. The calculation of $\bar{w}_3(x)$ presents no difficulty, since only elementary functions with known integrals are involved. Direct integration of Eq. (6) gives

$$\bar{w}_3(x) = - \int \varphi_3(x)dx = \frac{2PR(1 + \sigma)}{\pi E h_0^3(1 - 3\sigma)} \left[\frac{2 - 3\sigma}{1 - x} + \log_e \frac{x}{1 - x} \right]. \tag{19}$$

The construction of the deflection functions $w_1(x)$ and $w_2(x)$ is considerably more difficult, since it is necessary to evaluate integrals of the type $\int xG_i(x)dx$ ($i=1, 2$), which involves additional infinite series. For purposes of computation, a convenient procedure, that also has the advantage of being easy to check, is to use a combination of analytical and numerical methods. A prerequisite for this calculation is a fairly extensive and accurate tabulation of the G 's.

A straightforward step by step numerical integration process is seen to fail near the poles of the G 's, due to the presence of ordinary singularities in the integrands. The procedure for constructing the functions $w_1(x)$ and $w_2(x)$ in tabular form consists of removing these singularities analytically and integrating the resulting functions numerically.

6. Removal of singularities from the integrands of $\int xG_1(x)dx$ and $\int xG_2(x)dx$. Let us consider a "subtracting off" function $H_1(x)$ which has the property that $G_1(x) - H_1(x)$ is bounded uniformly, i.e., without finite jumps, throughout the interval of existence of $G_1(x)$. It is necessary that $H_1(x)$ be continuous except for poles which are of the same order as, and coincide with, those of $G_1(x)$. This specification is not sufficient however, since at every point of the interval the difference $G_1(x) - H_1(x)$ is finite, which requires the principal parts of $G_1(x)$ and $H_1(x)$ to be identical. The principal parts of $G_1(x)$ at zero and unity are readily obtainable from Eqs. (11) and (17) together with the relation $G_1(x) = G_1(1-x)$. Since $G_2(x) = -G_2(1-x)$ and the F 's and G 's are linearly dependent, the corresponding principal parts of $G_2(x)$ may be found by the same process.

The integral parts of $G_1(x)$ and $G_2(x)$ can be approximated by polynomials of low degree, which makes it possible to reduce the differences $G_1(x) - H_1(x)$ and $G_2(x) - H_2(x)$ to uniformly small values throughout the interval by an intelligent choice of the "subtracting off" functions. Incidentally this process provides a convenient check on the accuracy of the tabulated values of the G 's. After some manipulation a pair of suitable "subtracting off" functions were found to be

$$H_1(x) = C_2[-7 + ab - g(0) + C_1/C_2 - \frac{1}{2}(ab - 4)(ab - 6) \log_e x(1 - x) + 1/x^2 + 1/(1 - x)^2 - (ab - 6)/x - (ab - 6)/(1 - x)], \tag{20}$$

$$H_2(x) = D_2[(7 - ab - g(0) + D_1/D_2)(1 - 2x) - \frac{1}{2}(ab - 4)(ab - 6) \log_e \{x/(1 - x)\} + 1/x^2 - 1/(1 - x)^2 - (ab - 6)/x + (ab - 6)/(1 - x)]. \tag{21}$$

These functions have the added property that

$$\lim_{x \rightarrow 0, x \rightarrow 1} \{G_1(x) - H_1(x)\} = \lim_{x \rightarrow 0, x \rightarrow 1} \{G_2(x) - H_2(x)\} = 0.$$

Integrals of the type $\int xG(x)dx$ now can be evaluated directly from the identity

$$-w_i(x) = \int xG_i(x)dx \equiv \int x[G_i(x) - H_i(x)]dx + \int xH_i(x)dx, \quad (i = 1, 2). \tag{22}$$

The second integral on the right-hand side is expressible in terms of elementary functions, while the first one behaves like a polynomial which can be computed easily with any numerical integration formula having a suitably small remainder depending on the magnitude of the differences of $x[G_i(x) - H_i(x)]$. Evaluation of the second integral of Eq. (22) with $\sigma = .3$ gives, with the constant of integration chosen so that $w_1(\frac{1}{2}) = w_2(\frac{1}{2}) = 0$,

$$\int xH_1(x)dx = .060,042,74 \left\{ .082,589,7x^2 + .052,500,0x - 2.103,471 + \frac{1}{1 - x} + \log_e x - 1.047,500,0 \log_e (1 - x) - .052,500,0x^2 \log_e x(1 - x) \right\}, \tag{23}$$

$$\int xH_2(x)dx = .040,784,50 \left\{ .080,308,8x^3 - .060,231,6x^2 + 4.147,500x + 1.349,471,6 - \frac{1}{1 - x} + \log_e x + 1.0475,000,0 \log_e (1 - x) + .052,500,0x^2 \log_e \frac{1 - x}{x} \right\}. \tag{24}$$

7. Deflection and stress coefficients. It is convenient to state the actual deflection in the form

$$w = \frac{2R^2(1 - \sigma^2)}{Eh_0} [Aw_1 + Bw_2 + (P/h_0^2)w_3 + C], \tag{25}$$

where, from Eqs. (18a) and (19),

$$w_3 = \frac{1}{\pi(1 - 3\sigma)(1 - \sigma)} \left[\frac{2 - 3\sigma}{1 - x} + \log_e \frac{x}{1 - x} \right],$$

and w_1 and w_2 are non-dimensional functions of x defined by Eq. (22). w_1, w_2, w_3

are deflection coefficients, and are tabulated in Table 2. The constants A , B , and C are seen to have the dimensions of stress.

The bending stresses can be stated in a form entirely analogous to that obtained for the rotating disc problem. With the aid of Eqs. (4a), (2), and (25), we have

$$\begin{aligned} S_r &= A p_1 + B p_2 + (P/h_0^2) p_3, \\ S_t &= A q_1 + B q_2 + (P/h_0^2) q_3, \end{aligned} \quad (26)$$

where

$$\begin{aligned} p_1 &= (1-x)[xdG_1/dx + (1+\sigma)G_1], \\ q_1 &= (1-x)[\sigma xdG_1/dx + (1+\sigma)G_1]; \end{aligned}$$

a similar pair of relations apply to p_2 and q_2 ; when $\sigma = .3$, p_3 and q_3 can be computed directly from the formulas

$$\begin{aligned} p_3 &= -4.547284 \frac{1-x}{x} \left[\frac{.63 + 2.27x - .7x^2}{(1-x)^3} - \frac{.7}{x} \right], \\ q_3 &= -4.547284 \frac{1-x}{x} \left[\frac{2.1 - 2.14x + .7x^2}{(1-x)^3} + \frac{.7}{x} \right]. \end{aligned} \quad (27)$$

p_1 , p_2 , p_3 , q_1 , q_2 , q_3 are the stress coefficients, and are tabulated in Tables 3 and 4.

The tables of coefficients are especially convenient for approximating a plate of variable thickness with a system of conical profiles. Calculations in this type of problem show that it is necessary that the coefficients be accurate to six significant figures in order to obtain four significant figures in the final results. Consequently the tables have been calculated accurately to five parts in two million. Their general usefulness can be extended considerably with the aid of an auxiliary table of interpolation coefficients. It was found that such a table based on Bessel's central difference formula⁸ for six ordinates gives interpolated values of the coefficients as accurately as the tabulated ones, except near the ends of the table where the values are seldom used. In such cases a knowledge of the singularities of the tabulated functions indicates the necessary procedure for applying an interpolation formula.

ILLUSTRATIVE EXAMPLE

Stress distributions in a steel valve of constant weight and various proportions were estimated by an approximate method based on thin conical disc stress coefficients tabulated for both the lateral bending and rotating cases. The valve head is represented by a system of truncated conical shells of variable thickness, whose apex angles are nearly 180° as shown in Fig. 1. The angle of the seat determines the direction of the reaction which imposes two independent stress systems on the valve head. An approximation to these stresses can be made on the assumption that the membrane and bending stresses correspond to those in an equivalent system of conical discs. This assumption is admissible, since it has been demonstrated⁹ for conical shells of constant thickness, that the stress distribution has the same character as that in a

⁸ J. B. Scarborough, *Numerical mathematical analysis*, The Johns Hopkins Press, Baltimore, Md., 1930, p. 64.

⁹ Ref. 1, p. 477.

circular plate whenever the apex angle of the shell is between 168° and 180° . The loads on the composite disc shown in Fig. 2 are determined by resolving the valve seat reaction (of which the axial force is a component) into two perpendicular components, one of which produces pure compression on a section normal to the middle

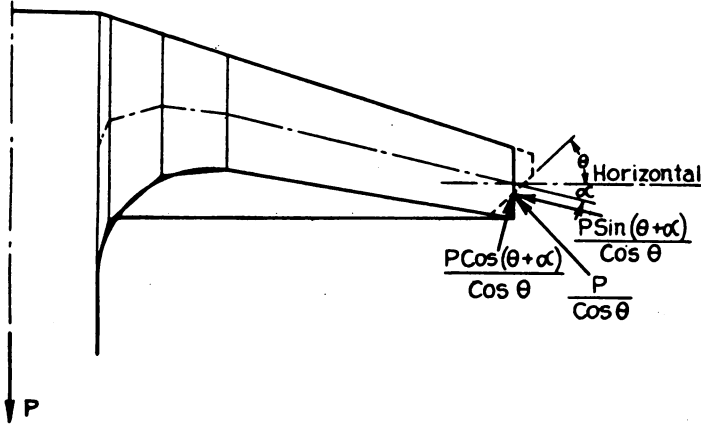


FIG. 1. Half section of valve.

surface at the periphery, and the other of which produces pure bending.* The peripheral forces per unit length are proportional to the resultant vertical force acting on the valve, so that the force resolution in Figs. 1 and 2 has been made in terms of the axial force P , which is considered as a concentrated load, such as would

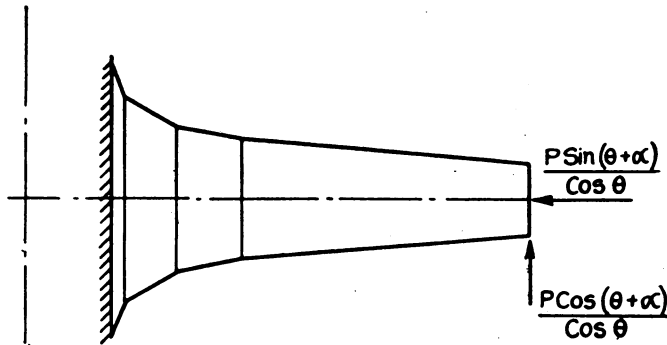


FIG. 2. Half section of composite disc.

be imposed on the valve by impact against its seat. It is safe to assume a concentrated axial load since the impact forces are proportional to the total valve weight, of which approximately 50% is in the stem.

The next step in the calculation is to represent the valve head by a system of equivalent conical discs in the usual manner. The tabular solution for the bending stresses is obtained from the calculation procedure described in Ref. (2), except that

* Variation in the slopes of corresponding generators of middle surfaces belonging to the conical discs of the equivalent system is not considered.

the p 's and q 's now refer to lateral bending coefficients and $\rho R_n^2 \omega^2$ is replaced by P/h_{n0}^2 . The solution for the membrane stresses is unchanged with the exception that the coefficients of p_3 and q_3 are zero, which corresponds to a static stress distribution

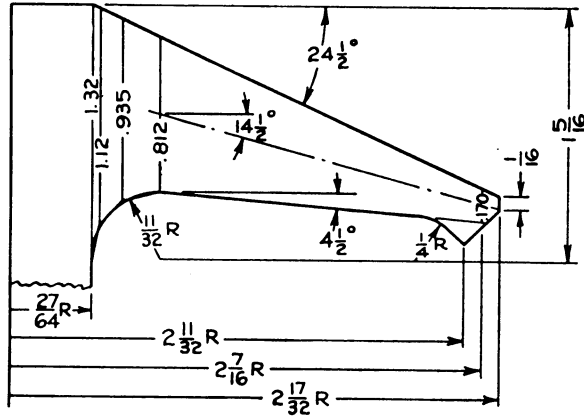


FIG. 3.

in a rotating disc. If \bar{S}_r and \bar{S}_t refer to the corresponding membrane stresses respectively, then the appropriate boundary conditions are: at the boundary between valve stem and head, $S_t = \sigma S_r$, $\bar{S}_t = \bar{S}_r$; at the periphery $S_r = 0$, \bar{S}_r assigned.

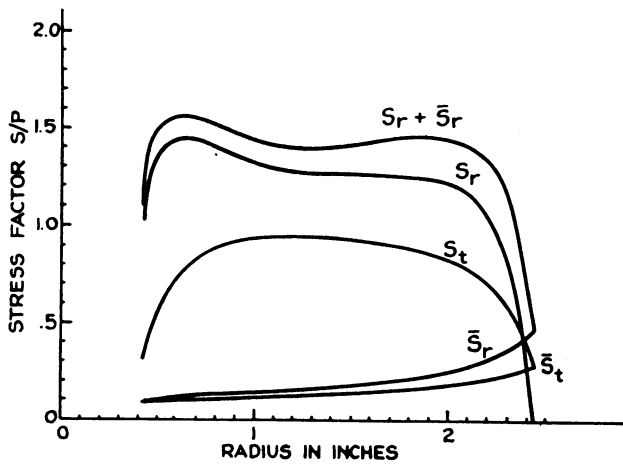


FIG. 4.

The dimensions of the valve head and the results of the stress calculations are shown in Figs. 3 and 4 respectively.

TABLE 1.*—Fundamental solutions of hypergeometric equation, $\sigma = .30$.

x	$G_1(x) = G_1(1-x)$	$G'_1(x) = -G'_1(1-x)$	$G_2(x) = -G_2(1-x)$	$G'_2(x) = G'_2(1-x)$
.00	∞	$-\infty$	∞	$-\infty$
.01	613.258	-121,346.7	416.297	-82,426.1
.02	156.631,5	-15,325.94	106.126,0	-10,410.62
.03	71.137,6	-4,587.63	48.050,0	-3,116.55
.04	40.900,5	-1,955.005	27.507,5	-1,328.327
.05	26.762,0	-1,010.946	17.900,04	-687.077
.06	19.005,02	-590.775	12.627,16	-401.683
.07	14.282,02	-375.613	9.415,02	-255.543
.08	11.187,24	-254.002	7.308,76	-172.948,9
.09	9.045,62	-180.034,7	5.849,83	-122.717,7
.10	7.499,77	-132.421,0	4.795,46	-90.387,7
.11	6.345,89	-100.354,8	4.007,22	-68.619,0
.12	5.460,74	-77.948,5	3.401,39	-53.412,3
.13	4.766,18	-61.804,5	2.924,89	-42.459,9
.14	4.210,70	-49.867,4	2.542,72	-34.365,6
.15	3.759,18	-40.842,7	2.231,01	-28.250,3
.16	3.386,98	-33.887,3	1.973,046	-23.541,2
.17	3.076,44	-28.435,8	1.756,797	-19.854,16
.18	2.814,56	-24.098,7	1.573,446	-16.924,95
.19	2.591,67	-20.602,3	1.416,397	-14.567,50
.20	2.400,38	-17.749,86	1.280,639	-12.648,25
.21	2.235,01	-15.397,53	1.162,300	-11.069,57
.22	2.091,12	-13.438,55	1.058,356	-9.758,98
.23	1.965,196	-11.792,44	.960,410	-8.661,87
.24	1.854,433	-10.397,74	.884,547	-7.736,55
.25	1.756,562	-9.206,96	.811,215	-6.950,83
.26	1.669,738	-8.182,97	.745,149	-6.279,56
.27	1.592,445	-7.296,48	.685,309	-5.702,90
.28	1.523,429	-6.524,11	.630,829	-5.205,08
.29	1.461,645	-5.847,10	.580,987	-4.773,43
.30	1.406,220	-5.250,21	.535,174	-4.397,71
.31	1.356,416	-4.721,02	.492,874	-4.069,55
.32	1.311,608	-4.249,27	.453,647	-3.782,13
.33	1.271,267	-3.826,49	.417,115	-3.529,80
.34	1.234,939	-3.445,58	.382,950	-3.307,88
.35	1.202,236	-3.100,60	.350,869	-3.112,48
.36	1.172,824	-2.786,52	.320,623	-2.940,33
.37	1.146,417	-2.499,05	.291,994	-2.788,70
.38	1.122,767	-2.234,53	.264,788	-2.655,29
.39	1.101,661	-1.989,781	.238,834	-2.538,15
.40	1.082,915	-1.762,067	.213,976	-2.435,65
.41	1.066,371	-1.548,980	.190,076,6	-2.346,40
.42	1.051,893	-1.348,397	.167,007,9	-2.269,25
.43	1.039,367	-1.158,434	.144,654,6	-2.203,20
.44	1.028,695	-.977,394	.122,909,6	-2.147,46
.45	1.019,795	-.803,742	.101,673,3	-2.101,34
.46	1.012,600	-.636,068	.080,852,4	-2.064,31
.47	1.007,058	-.473,063	.060,358,2	-2.035,94
.48	1.003,128	-.313,494	.040,105,8	-2.015,90
.49	1.000,780	-.156,186,3	.020,013,2	-2.003,96
.50	1.000,000	0	0	-2.000,00

* The tables were compiled with the aid of the staff of the Calculation Department of Fairbanks Morse & Co., Beloit, Wis., to whom acknowledgement hereby is made.

TABLE 2.—Deflection coefficients for lateral bending of conical discs, $\sigma = .30$.

r/R	w_1	w_2	w_3	r/R	w_1	w_2	w_3
.00	∞	∞	$-\infty$.50	0	0	10.00402
.01	.341490	.1727156	-15.84278	.51	-.00505132	.0000506834	10.39010
.02	.298578	.1436068	-12.59312	.52	-.01021073	.000205606	10.78484
.03	.272917	.1262435	-10.65009	.53	-.01548680	.000469405	11.18891
.04	.254305	.1136970	-9.24108	.54	-.0208887	.000847188	11.60306
.05	.239546	.1037962	-8.12392	.55	-.0264263	.001344576	12.02809
.06	.227215	.0955742	-7.19072	.56	-.0321101	.001967757	12.46484
.07	.216551	.0885157	-6.38390	.57	-.0379516	.00272355	12.91425
.08	.207102	.0823131	-5.66908	.58	-.0439632	.00361945	13.37730
.09	.1985728	.0767682	-5.02404	.59	-.0501582	.00466373	13.85508
.10	.1907641	.0717460	-4.43361	.60	-.0565514	.00586555	14.34880
.11	.1835327	.0671501	-3.88695	.61	-.0631588	.00723498	14.85973
.12	.1767727	.0629098	-3.37604	.62	-.0699978	.00878323	15.38931
.13	.1704034	.0589712	-2.89476	.63	-.0770879	.01052271	15.93909
.14	.1643621	.0552928	-2.43835	.64	-.0844501	.01246720	16.51082
.15	.1585986	.0518418	-2.00300	.65	-.0921083	.01463208	17.10641
.16	.1530724	.0485919	-1.585657	.66	-.1000884	.01703454	17.72799
.17	.1477501	.0455217	-1.183777	.67	-.1084197	.01969381	18.37793
.18	.1426040	.0426135	-.795248	.68	-.1171349	.0226315	19.05890
.19	.1376105	.0398528	-1.418284	.69	-.1262706	.0258720	19.77389
.20	.1327494	.0372275	-.0513587	.70	-.1358681	.0294430	20.5263
.21	.1280035	.0347271	.306849	.71	-.1459743	.0333756	21.3199
.22	.1233577	.0323431	.657492	.72	-.1566424	.0377057	22.1591
.23	.1187984	.0300681	1.001586	.73	-.1679333	.0424741	23.0488
.24	.1143140	.0278957	1.340034	.74	-.1799171	.0477280	23.9948
.25	.1098938	.0258206	1.673648	.75	-.1926744	.0535219	25.0038
.26	.1055282	.0238382	2.00316	.76	-.206299	.0599190	26.0833
.27	.1012084	.0219446	2.32924	.77	-.220901	.0669937	27.2424
.28	.0969264	.0201366	2.65250	.78	-.236610	.0748334	28.4918
.29	.0926748	.01841116	2.97352	.79	-.253578	.0835418	29.8439
.30	.0884466	.01676613	3.29283	.80	-.271990	.0932430	31.3139
.31	.0842354	.01519954	3.61092	.81	-.292065	.1040868	32.9200
.32	.0800348	.01370985	3.92829	.82	-.314074	.1162557	34.6842
.33	.0758391	.01229585	4.24537	.83	-.338347	.1299744	36.6339
.34	.0716426	.01095665	4.56262	.84	-.365296	.1455227	38.8030
.35	.0674398	.00969164	4.88046	.85	-.395443	.1632539	41.2345
.36	.0632252	.00850050	5.19930	.86	-.429454	.1836207	43.9833
.37	.0589937	.00738318	5.51956	.87	-.468197	.207213	47.1212
.38	.0547400	.00633986	5.84165	.88	-.512828	.234815	50.7436
.39	.0504589	.00537098	6.16596	.89	-.564915	.267491	54.9800
.40	.0461453	.00447722	6.49292	.90	-.626646	.306728	60.0115
.41	.0417938	.00365951	6.82293	.91	-.701169	.354663	66.0987
.42	.0373993	.00291902	7.15642	.92	-.793190	.414495	73.6312
.43	.0329561	.00225716	7.49380	.93	-.910079	.491227	83.2197
.44	.0284588	.001675592	7.83553	.94	-1.064072	.593171	95.8789
.45	.0239016	.001176243	8.18206	.95	-1.277112	.735232	113.4294
.46	.01927865	.000761311	8.53386	.96	-1.592926	.947112	139.5018
.47	.01458362	.000433277	8.89143	.97	-2.11315	1.297850	182.5406
.48	.00981010	.000194922	9.25528	.98	-3.14154	1.993762	267.798
.49	.00495128	.000049349	9.62595	.99	-6.19036	4.06209	521.097
.50	0	0	10.00402	1.00	$-\infty$	∞	∞

TABLE 3.—Stress coefficients for lateral bending of conical discs, $\sigma = .30$.

r/R	p_1	$-p_2$	p_3	r/R	p_1	$-p_2$	p_3
.00	$-\infty$	∞	∞	.50	.650000	.500000	-51.4753
.01	-412.069	280.244	31209.9	.51	.676528	.513539	-53.6293
.02	-100.8399	68.8436	7638.76	.52	.704200	.528194	-55.8788
.03	-43.7956	30.1006	3318.31	.53	.733152	.544032	-58.2347
.04	-24.0284	16.67837	1821.085	.54	.763534	.561125	-60.7094
.05	-14.96886	10.52962	1134.831	.55	.795506	.579561	-63.3159
.06	-10.09557	7.22454	765.640	.56	.829243	.599438	-66.0685
.07	-7.18545	5.25310	545.144	.57	.864938	.620867	-68.9831
.08	-5.31459	3.98776	403.367	.58	.902803	.643975	-72.0770
.09	-4.04388	3.13023	307.050	.59	.943074	.668906	-75.3697
.10	-3.14316	2.52420	238.760	.60	.986012	.695824	-77.8828
.11	-2.48255	2.08145	188.6601	.61	1.031911	.724915	-82.6402
.12	-1.984277	1.749150	150.8584	.62	1.081101	.756391	-86.6694
.13	-1.599538	1.494168	121.6581	.63	1.133955	.790495	-91.0010
.14	-1.296465	1.294864	98.6451	.64	1.190896	.827503	-95.6698
.15	-1.053553	1.136640	80.1902	.65	1.252404	.867734	-100.7157
.16	-.855874	1.009365	65.1626	.66	1.319031	.911553	-106.1840
.17	-.692812	.905838	52.7577	.67	1.391409	.959381	-112.1271
.18	-.556649	.820830	42.3909	.68	1.470270	1.011709	-118.6053
.19	-.441674	.750472	33.6293	.69	1.556461	1.069105	-125.6887
.20	-.343586	.691855	26.1469	.70	1.650971	1.132236	-133.4592
.21	-.259098	.642759	19.69457	.71	1.754958	1.201882	-142.0124
.22	-.1856618	.601468	14.07923	.72	1.869789	1.278967	-151.4616
.23	-.1212805	.566640	9.14940	.73	1.997083	1.364585	-161.9406
.24	-.0643692	.537215	4.78487	.74	2.13878	1.460047	-173.6096
.25	-.01365723	.512346	.889247	.75	2.29719	1.566925	-186.6610
.26	.0318840	.491353	-2.61552	.76	2.47513	1.687126	-201.327
.27	.0730951	.473684	-5.79325	.77	2.67604	1.822974	-217.893
.28	.1106690	.458889	-8.69658	.78	2.90412	1.977330	-236.707
.29	.1451814	.446599	-11.36928	.79	3.16461	2.15375	-258.203
.30	.1771155	.436510	-13.84808	.80	3.46408	2.35669	-282.926
.31	.206879	.428369	-16.16404	.81	3.81084	2.59179	-311.566
.32	.234821	.421968	-18.34365	.82	4.21558	2.86631	-345.007
.33	.261238	.417132	-20.4097	.83	4.69218	3.18967	-384.402
.34	.286390	.413718	-22.3820	.84	5.25895	3.57432	-431.270
.35	.310503	.411605	-24.2778	.85	5.94048	4.03696	-487.652
.36	.333776	.410694	-26.1125	.86	6.77038	4.60040	-556.337
.37	.356387	.410903	-27.8996	.87	7.79558	5.29653	-641.223
.38	.378496	.412167	-29.6516	.88	9.08323	6.17096	-747.885
.39	.400248	.414431	-31.3796	.89	10.73220	7.29083	-884.537
.40	.421777	.417655	-33.0941	.90	12.89286	8.75831	-1063.672
.41	.443208	.421806	-34.8048	.91	15.80318	10.73501	-1305.066
.42	.464658	.426865	-36.5208	.92	19.85801	13.48915	-1641.541
.43	.486239	.432816	-38.2510	.93	25.7521	17.49263	-2130.86
.44	.508060	.439656	-40.0040	.94	34.8021	23.6398	-2882.53
.45	.530227	.447386	-41.7881	.95	49.7595	33.7997	-4125.41
.46	.552846	.456017	-43.6117	.96	77.1990	52.4381	-6406.56
.47	.576023	.465566	-45.4835	.97	136.2744	92.5655	-11320.03
.48	.599866	.476057	-47.4119	.98	304.461	206.807	-25315.3
.49	.624487	.487522	-49.4061	.99	1209.305	821.430	-100647.9
.50	.650000	.500000	-51.4753	1.00	∞	∞	$-\infty$

TABLE 4.—Stress coefficients for lateral bending of conical discs, $\sigma = .30$.

r/R	q_1	q_2	$-q_3$	r/R	q_1	q_2	$-q_3$
.00	∞	∞	∞	.50	.650000	-.1500000	50.2020
.01	428.863	290.968	32477.1	.51	.649206	-.1629858	50.2128
.02	109.4320	73.9901	8285.68	.52	.649426	-.1759765	50.3035
.03	49.6545	33.3836	3758.74	.53	.650664	-.1890247	50.4748
.04	28.5222	19.02706	2158.48	.54	.652935	-.202182	50.7282
.05	18.64509	12.31570	1410.590	.55	.656258	-.215504	51.0654
.06	13.22822	8.63391	1000.466	.56	.660662	-.229044	51.4891
.07	9.93124	6.39200	750.873	.57	.666186	-.242863	52.0026
.08	7.77158	4.92257	587.402	.58	.672875	-.257023	52.6097
.09	6.27752	3.90518	474.331	.59	.680785	-.271589	53.3154
.10	5.19936	3.17022	392.751	.60	.689984	-.286635	54.1253
.11	4.39477	2.62101	331.883	.61	.700553	-.302237	55.0462
.12	3.77767	2.19909	285.210	.62	.712583	-.318481	56.0859
.13	3.29352	1.867385	248.602	.63	.726185	-.335463	57.2535
.14	2.90636	1.601472	219.336	.64	.741486	-.353287	58.5598
.15	2.59166	1.384697	195.5559	.65	.758634	-.372072	60.0169
.16	2.33225	1.205387	175.9608	.66	.777800	-.391950	61.6393
.17	2.11579	1.055157	159.6167	.67	.799185	-.413074	63.4435
.18	1.933233	.927856	145.8386	.68	.823021	-.435615	65.4490
.19	1.777816	.818885	134.1146	.69	.849583	-.459771	67.6783
.20	1.644399	.724748	124.0556	.70	.879189	-.485773	70.1581
.21	1.529018	.642750	115.3617	.71	.912216	-.513887	72.9195
.22	1.428577	.570780	107.7985	.72	.949106	-.544425	75.9993
.23	1.340629	.507172	101.1808	.73	.990389	-.577756	79.4413
.24	1.263215	.450588	95.3605	.74	1.036693	-.614316	83.2977
.25	1.194757	.399950	90.2181	.75	1.088774	-.654629	87.6312
.26	1.133967	.354377	85.6563	.76	1.147548	-.699322	92.5176
.27	1.079790	.313145	81.5951	.77	1.214126	-.749161	98.0492
.28	1.031351	.275652	77.9685	.78	1.289876	-.805081	104.3394
.29	.987924	.241396	74.7215	.79	1.376492	-.868241	111.5287
.30	.948897	.209953	71.8079	.80	1.476092	-.940082	119.7926
.31	.913757	.1809649	69.1890	.81	1.591352	-1.022432	129.3533
.32	.882070	.1541266	66.8319	.82	1.725701	-1.117624	140.4952
.33	.853463	.1291751	64.7086	.83	1.883578	-1.228709	153.5868
.34	.827622	.1058843	62.7951	.84	2.07083	-1.359573	169.1134
.35	.804274	.0840574	61.0712	.85	2.29527	-1.515621	187.7236
.36	.783186	.0635226	59.5192	.86	2.56756	-1.704062	210.302
.37	.764157	.0441290	58.1239	.87	2.90251	-1.934972	238.080
.38	.747014	.0257434	56.8723	.88	3.32128	-2.22272	272.814
.39	.731606	.00824728	55.7532	.89	3.85488	-2.58838	317.080
.40	.717805	-.00846535	54.7569	.90	4.55034	-3.06387	374.786
.41	.705497	-.0244898	53.8749	.91	5.48179	-3.69960	452.095
.42	.694587	-.0399126	53.1001	.92	6.77183	-4.57882	559.197
.43	.684992	-.0548125	52.4264	.93	8.63538	-5.84753	713.963
.44	.676641	-.0692620	51.8486	.94	11.47831	-7.78139	950.147
.45	.669475	-.0833284	51.3624	.95	16.14551	-10.95435	1338.035
.46	.663446	-.0970743	50.9640	.96	24.6485	-16.73272	2045.00
.47	.658511	-.1105590	50.6507	.97	42.8244	-29.0814	3556.83
.48	.654640	-.1238390	50.4203	.98	94.1889	-63.9738	7831.14
.49	.651808	-.1369684	50.2710	.99	368.372	-250.217	30658.4
.50	.650000	-.1500000	50.2020	1.00	∞	$-\infty$	∞