THE MATHEMATICS OF WEIR FORMS*

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1. Introduction. This paper aims at making more readily available the results of a study of the mathematics of weir forms, a subject in Hydraulics to which higher mathematics can be applied. Section 2 covers the general application of Abel's integral equation to the forms of weirs by Brenke.[1] Section 3 deals with sectionally analytic weir forms, particularly the Stout-Sutro weir. The writer believes he has made the original application of Abel's integral equation to this corrected weir form. Section 4 deals with cases when the quantity of flow can be expressed as a convergent series.

2. Abel's integral equation. One method of solution of the problem of weir forms involves Abel's integral equation. The natural conditions found in the flow of water through weirs satisfy all the requirements of this integral equation, so it proves a superior mathematical tool in handling the general problem. In 1922 Brenke studied the problem of the weir form when the flow was proportional to some power of the depth. He made the original application of Abel's integral equation. This equation has the form

$$\phi(x) = \int_a^x \frac{f(s)ds}{(x-s)^\lambda}, \qquad 0 < \lambda < 1 \tag{1}$$

and its solution is, under certain conditions,

$$f(x) = \frac{\sin \lambda\pi}{\pi} \int_a^x \frac{\phi'(s)ds}{(x-s)^{1-\lambda}}. \tag{2}$$

To obtain (2) from (1) use is made of two fundamental formulas, namely[2,3]

$$\frac{\pi}{\sin \lambda\pi} = \int_s^z \frac{dx}{(z-x)^{1-\lambda}(x-s)^\lambda}, \qquad 0 < \lambda < 1 \tag{3}$$

$$\int_a^z \int_s^z \frac{\phi'(s)dx}{(z-x)^{1-\lambda}(x-s)^\lambda} ds = \int_a^z \frac{1}{(z-x)^{1-\lambda}} \int_a^x \frac{\phi'(s)ds}{(x-s)^\lambda} dx. \tag{4}$$

$\phi(s)$ is assumed to be continuous and have a continuous derivative in the closed

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[1] W. C. Brenke, *An application of Abel's integral equation*, Am. Math. Monthly, **29**, 58 (1922).

[2] E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th Ed., Cambridge Univ. Press, London, 1927, pp. 211, 229.

[3] Maxime Bôcher, *An introduction to the study of integral equations*, Cambridge Tracts in Math. and Math. Physics, No. 10, Cambridge Univ. Press, London, 1926, p. 8.

interval, a to b. Formula (4) is known as Dirichlet's generalized formula.[4] Multiply (3) by $\phi'(s)ds$ and integrate from a to z, $(a \leqq z \leqq b)$, which gives

$$\frac{\pi}{\sin \lambda\pi}[\phi(z) - \phi(a)] = \int_a^z \int_s^z \frac{\phi'(s)dx}{(z-x)^{1-\lambda}(x-s)^\lambda} ds. \tag{5}$$

If (4) is applied to the right hand member of (5), we have

$$\phi(z) - \phi(a) = \frac{\sin \lambda\pi}{\pi}\int_a^z \frac{1}{(z-x)^{1-\lambda}}\int_a^x \frac{\phi'(s)ds}{(x-s)^\lambda} dx. \tag{6}$$

Then, if $\phi(a)=0$ and if we replace the inner integral on the right of (6) by its value from (2), we see that (6) becomes (1). Hence (2) is a solution of (1).

The weir is actually symmetrically constructed as in Fig. 3, but for purposes of the present calculation a half section is used (Fig. 1). Letting $y=f(x)$ express the

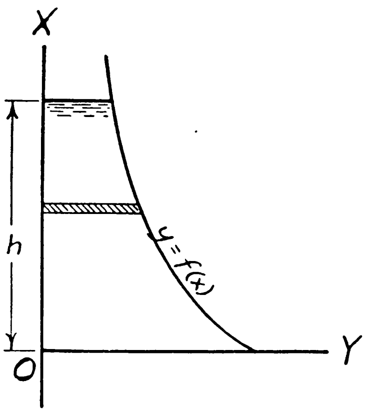

FIG. 1.

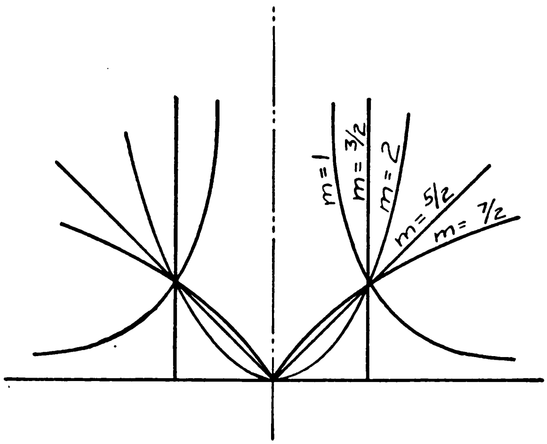

FIG. 2.

distribution of width over depth, h the depth of flow, C_d the coefficient of discharge (approximately 0.6), and assuming that the quantity of flow is proportional to the mth power of the depth of stream, we have

$$C_d\int_0^h [2g(h-x)]^{1/2}f(x)dx = bh^m, \tag{7}$$

or, letting $K=b/C_d(2g)^{1/2}$,

$$\int_0^h (h-x)^{1/2}f(x)dx = Kh^m.$$

Differentiating with respect to h, we have

$$\int_0^h \frac{f(x)dx}{(h-x)^{1/2}} = 2Kmh^{m-1}. \tag{8}$$

This equation has the form of Abel's integral equation.

To find the equation of the weir form when the flow is bh^m, we have

[4] W. A. Hurwitz, *Note on certain iterated and multiple integrals*, Annals of Math., **9**, 183 (1907).

$$f(x) = \frac{\sin \pi/2}{\pi} \int_0^x \frac{2Km(m-1)h^{m-2}}{(x-h)^{1/2}}\, dh,$$

or

$$f(x) = \frac{2Km(m-1)}{\pi} \int_0^x \frac{h^{m-2}}{(x-h)^{1/2}}\, dh. \tag{9}$$

By the use of Gamma Functions,[5]

$$f(x) = \frac{2K\Gamma(m+1)}{\pi^{1/2}\Gamma(m-\frac{1}{2})} x^{m-3/2}, \qquad m \geqq 2. \tag{10}$$

Let n be a positive integer $\geqq 2$. Then the Gamma Functions become simple products when $m=n$ or $m=n+\frac{1}{2}$. When $m=n$,

$$f(x) = \frac{K}{\pi} \frac{2^n n!}{1\cdot 3\cdot 5\cdot \cdots (2n-3)} x^{n-3/2}; \qquad n \geqq 2. \tag{11}$$

When $m=n+\frac{1}{2}$,

$$f(x) = K \frac{1\cdot 3\cdot 5\cdot \cdots (2n+1)}{2^n(n-1)!} x^{n-1}; \qquad n \geqq 2. \tag{12}$$

3. Sectionally analytic weir forms. When m is equal to or greater than $\frac{3}{2}$ one gets continuous forms of weirs (Fig. 2). When m is greater than $\frac{1}{2}$ and less than $\frac{3}{2}$ the weir forms have an infinite width at the bottom, the curve $f(x)$ approaching the X-axis asymptotically. As this is impossible in practice, the necessary correction due to limiting the width of the weir furnishes an interesting mathematical problem which has been studied in the case where $m=1$.

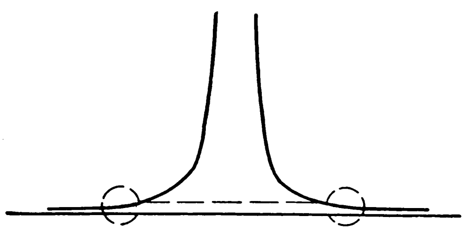

FIG. 3. Copy of Stout's drawing in 1897.

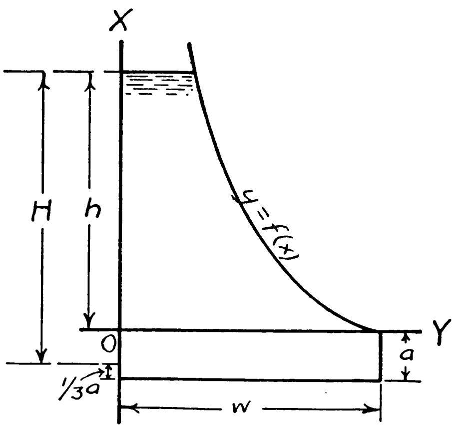

FIG. 4.

The weir in which the flow is proportional to the depth is of engineering value. One of the first records of it is in an article by O. V. P. Stout.[6] Approximate correction was made by circular openings (Fig. 3). A weir of this type was also constructed by Sutro and it is referred to in some texts as the Sutro weir. The modern way to correct

[5] F. S. Woods, *Advanced calculus*, Ginn & Co., 1926, p. 164.

[6] O. V. P. Stout, *A new form of weir notch*, Trans. of the Nebraska Engineering Society, **1**, 13 (1897).

the Stout-Sutro weir is to start with a rectangular cross section of depth a and width w (Fig. 4). The upper section is then designed to give a flow proportional to the first power of the depth when the depth of flow equals or exceeds a.

The calculations of E. A. Pratt[7] by series solutions gave a mathematically correct form of weir where $h \geqq a$. In this solution a rectangular section of depth a and width w is first assumed. Soundings are made with the zero point $\frac{1}{3}a$ from the bottom,

$$Q = bH = b(h + \tfrac{2}{3}a).$$

The quantity of water discharged through the rectangular portion of the weir is

$$Q_0 = \tfrac{2}{3}wK[(h + a)^{3/2} - h^{3/2}].$$

Therefore

$$Q = \tfrac{2}{3}wK[(h + a)^{3/2} - h^{3/2}] + 2K\int_0^h (h - x)^{1/2}f(x)dx = b(h + \tfrac{2}{3}a).$$

As this equality must hold for $h=0$, $\frac{2}{3}wKa^{3/2} = \frac{2}{3}ab$ and $b = 2wKa^{1/2}$, so

$$\int_0^h (h - x)^{1/2}f(x)dx = \tfrac{2}{3}w[\tfrac{3}{2}ha^{1/2} + a^{3/2} - (h + a)^{3/2} + h^{3/2}].$$

Instead of solving by the use of series, as Pratt did, one may differentiate with respect to h to put the equation in the form of Abel's integral equation; thus

$$\int_0^h \frac{f(x)dx}{(h - x)^{1/2}} = 2w[a^{1/2} - (h + a)^{1/2} + h^{1/2}].$$

For the solution of Abel's integral equation the right hand member must be a continuous function, equal to zero when $h=0$. These conditions being satisfied,

$$\begin{aligned} y = f(x) &= \frac{\sin \pi/2}{\pi} 2w \int_0^x \frac{[-\frac{1}{2}(h + a)^{-1/2} + \frac{1}{2}h^{-1/2}]}{(x - h)^{1/2}} dh \\ &= \frac{w}{\pi}\left[\int_0^x \frac{dh}{[xh - h^2]^{1/2}} - \int_0^x \frac{dh}{[ax + h(x - a) - h^2]^{1/2}}\right] \\ &= \frac{w}{2} + \frac{w}{\pi}\sin^{-1}\frac{a - x}{a + x}, \end{aligned} \tag{13}$$

or

$$y = w - \frac{2w}{\pi}\tan^{-1}\left(\frac{x}{a}\right)^{1/2}. \tag{14}$$

This solution can also be written

$$x = a\tan^2\frac{\pi(w - y)}{2w}. \tag{15}$$

In the design of the Stout-Sutro weir it is now necessary to choose an a for substitution in the above formulas. One will generally know the average depth of flow

[7] E. A. Pratt, *Another proportional-flow weir, Sutro weir*, Engr. News, **72**, 462 (1914).

expected through the weir. It is felt to be better to keep the curve of (13) as close to the curve of the uncorrected weir derived from (10), $y=2Kx^{-1/2}/\pi$, as possible. The scheme is to make the rectangular section of the Stout-Sutro weir have the same dimensions as if the uncorrected weir of (10) were corrected for the average depth of flow by the addition of a rectangular section at the bottom, below the Y-axis, to compensate for limiting its width to $2w$.

One substitutes $y=w$ in the uncorrected formula (10) above and solves for x. This value and that of the h assumed to be average are substituted in

$$\frac{h}{2}\sin^{-1}\frac{h-2x}{h}+\frac{2}{3x^{1/2}}(h+r)^{3/2}-\frac{\pi h}{4}-(hx-x^2)^{1/2}-\frac{2}{3x^{1/2}}(h-x)^{3/2}=0,$$

which is solved for r. One then makes a, the depth of the rectangular section, equal to $x+r$. It must be appreciated that (10) can be corrected for one depth of flow by the addition of a rectangular opening at the bottom, but would not be correct at any other depth of flow. Formula (13) is correct at any depth, $H>\frac{2}{3}a$ (Fig. 4).

4. Series solutions of weir forms. We consider now the forms of weirs when the quantity of flow can be expressed as a convergent series in powers of h. Assume that the quantity of flow, $Q(h)$, can be written

$$Q(h)=\sum_{n=0}^{n=\infty}a_n h^{n+\alpha}, \tag{16}$$

a convergent series not having a constant term, and assume the form of weir to be given by

$$f(x)=\sum_{n=0}^{n=\infty}f_n(x), \tag{17}$$

each term of (17) giving rise to one term of (16).

The general equation is

$$C_d(2g)^{1/2}\int_0^h (h-x)^{1/2}f(x)dx=Q(h).$$

Its solution will involve a series of integral equations of the form

$$C_d(2g)^{1/2}\int_0^h (h-x)^{1/2}f_n(x)dx=a_n h^{n+\alpha}\qquad n=0,1,2,\cdots;\alpha>\tfrac{1}{2}$$

which can be solved by the use of (10), giving

$$f_n(x)=Ca_n\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-\frac{1}{2})}x^{n+\alpha-3/2}, \tag{18}$$

where

$$C=\frac{2}{C_d(2g\pi)^{1/2}}\cdot$$

Substitution of this in (17) gives the formal solution

$$f(x)=C\sum_{n=0}^{n=\infty}a_n\frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-\frac{1}{2})}x^{n+\alpha-3/2}. \tag{19}$$

The first term of this series

$$f_0(x) = Ca_0 \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - \frac{1}{2})} x^{\alpha - 3/2} \tag{20}$$

will be discontinuous at $x=0$ if $\frac{1}{2} < \alpha < \frac{3}{2}$ and continuous if $\alpha \geqq \frac{3}{2}$.

The series formed by all the terms after the first will converge and represent a continuous function of x. This may be proved as follows. By hypothesis the series $\sum_{n=1}^{n=\infty} a_n h^{n+\alpha}$ converges since it is the series for $Q(h)$, (16), with the first term omitted. Let

$$\begin{aligned} c_n &= \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + \alpha - \frac{1}{2})}, \qquad \alpha > \tfrac{1}{2} \\ &= \frac{(n + \alpha)(n + \alpha - 1)\Gamma(n + \alpha - 1)}{\Gamma(n + \alpha - \frac{1}{2})}, \qquad \Gamma(p + 1) = p\Gamma(p), \\ &= (n + \alpha)(n + \alpha - 1)c_n' \end{aligned}$$

where $c_n' = \Gamma(n+\alpha-1)/\Gamma(n+\alpha-\frac{1}{2})$ and $0 < c_n' < 1$ since $\Gamma(p)$ increases monotonically for $p > 1.46$. Now[8] if the series $\sum_{n=1}^{n=\infty} a_n h^{n+\alpha}$ converges, so also will the series

$$\sum_{n=1}^{n=\infty} c_n' a_n h^{n+\alpha}, \quad \sum_{n=1}^{n=\infty} n c_n' a_n h^{n+\alpha} \quad \text{and} \quad \sum_{n=1}^{n=\infty} n^2 c_n' a_n h^{n+\alpha}.$$

But the series $\sum_{n=1}^{n=\infty} c_n a_n h^{n+\alpha}$ is a simple combination of these three series, hence it also converges. In each case the function represented by the series is continuous.

We have then the form of weir given by

$$f(x) = f_0(x) + g(x), \tag{21}$$

where $f_0(x)$ is given by (20) and

$$g(x) = C \sum_{n=1}^{n=\infty} c_n a_n x^{n+\alpha-3/2},$$

the quantities α, C and c_n being as specified above. The solution $f(x)$ is discontinuous at $x=0$ if $\frac{1}{2} < \alpha < \frac{3}{2}$. It is continuous for $\alpha \geqq \frac{3}{2}$.

[8] F. S. Woods, *Advanced calculus*, Ginn & Co., 1926, p. 47.