MEMBRANE STRESSES IN SHELLS OF CONSTANT SLOPE*

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1. A surface S of constant slope may be generated by a straight line L sliding along a plane curve C_0 (say, in the xy plane), maintaining a right angle with the tangent to C_0 and a constant angle θ with its binormal (i.e., with the z axis). When a closed curve C_0 is chosen, the surface is an obvious generalization of a circular cone[2] (see Fig. 1). Since "near-conical" shells occur often in practice,[3] it may be of interest to discuss such effects as fall within the scope of the membrane theory of shells.

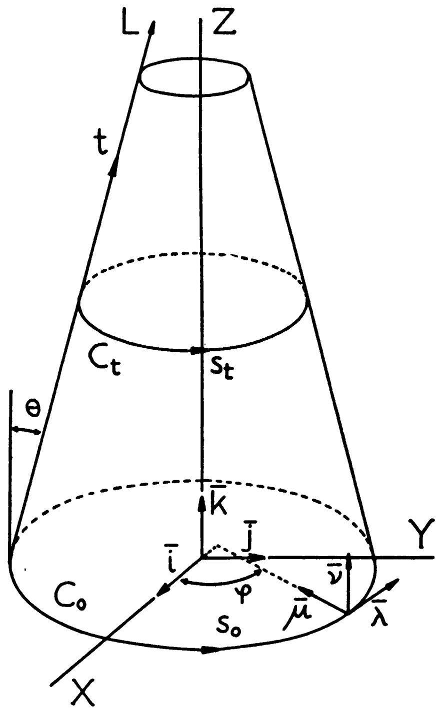

FIG. 1.

We introduce the following notations:

$\bar{\imath}, \bar{\jmath}, \bar{k}$, unit vectors in fixed rectangular directions x, y, z;

$\bar{\lambda}, \bar{\mu}, \bar{\nu}$, unit tangent, normal, and binormal of curve C_0;

t, length along generators L;

s_t, ρ_t, arc length and radius of curvature of a horizontal section C_t of the surface S; subscripts 0 and 1 will designate corresponding quantities in the end sections C_0 and C_1 of the shell;

$\bar{r} = \bar{r}(s_0)$, vector equation of curve C_0;

φ, angle between the positive x axis and the outward normal of C_0;

E, ν, G, Young's modulus, Poisson's ratio, and shear modulus;

h, thickness of shell having the surface S for middle surface;

N_s, N_t, normal forces per unit length of sections of the shell which are perpendicular to s- and t-directions respectively (Fig. 3);

N_{st}, shearing force in s-direction per unit length of shell section perpendicular to t-direction;

* Received Oct. 16, 1943.

[1] The author wishes to express his appreciation to Professor W. Prager for proposing the problem and for other valuable suggestions.

[2] Non-circular cones (for which the generators meet in one point while their "slope" varies) have been considered recently by A. Pflüger, Z. angew. Math. Mech. **22**, 99–116 (1942).

[3] The fuselages of some aeroplanes, for instance, can be approximated by one or several shells of different slopes connected by stiff bulkheads. The construction of models is relatively simple because each portion forms a developable surface.

e_{ss}, e_{tt}, e_{st}, strains corresponding to N_s, N_t, and N_{st}, respectively.

We note some simple relationships:

$$\frac{d\bar{r}}{ds_0} = \bar{\lambda}; \qquad \bar{\nu} = \bar{k}; \tag{1.1}$$

$$\bar{\lambda} = -\bar{\imath}\sin\varphi + \bar{\jmath}\cos\varphi; \qquad \bar{\mu} = -\bar{\imath}\cos\varphi - \bar{\jmath}\sin\varphi. \tag{1.2}$$

Since $\rho_0 = ds_0/d\varphi$, we obtain from (1.2) the Frenet-Serret formulae for a plane curve:

$$d\bar{\lambda}/ds_0 = \bar{\mu}/\rho_0; \qquad d\bar{\mu}/ds_0 = -\bar{\lambda}/\rho_0. \tag{1.3}$$

The vector equation of the surface of constant slope S has the form:

$$\bar{R}(s_0, t) = \bar{r}(s_0) + t(\bar{\mu}\sin\theta + \bar{\nu}\cos\theta). \tag{1.4}$$

For a constant value of t, (1.4) is the vector equation of the horizontal section C_t. Then, $\partial\bar{R}/\partial s_t$ is the unit vector tangent to C_t. Since $\partial\bar{R}/\partial s_t = \bar{\lambda}(\rho_0 - t\sin\theta)ds_0/\rho_0 ds_t$, C_t is parallel to C_0 at corresponding points (see Fig. 2), and

$$ds_t/ds_0 = (\rho_0 - t\sin\theta)/\rho_0. \tag{1.5}$$

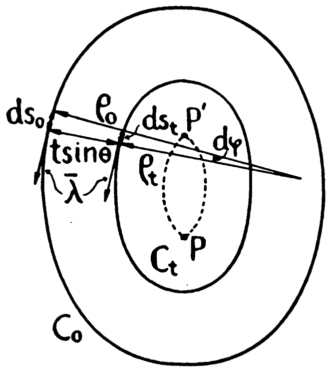

FIG. 2.

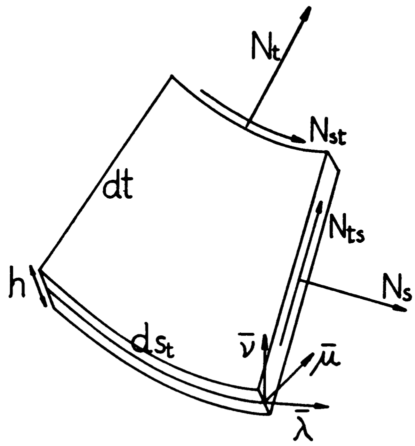

FIG. 3.

Hence, for corresponding points, the centers of curvature of C_0 and C_t coincide, and

$$\rho_t = \rho_0 - t\sin\theta. \tag{1.6}$$

If the shell is long, it may happen that at some point $\rho_t = 0$. At such a point the tangent to C_t ceases to turn continuously (see points P, P' in Fig. 2). We shall discuss only the portion of the shell where $t\sin\theta < \rho_0$, i.e., the open shell without the "tail edge."

2. An element of a shell of thickness h having the surface S for middle surface is shown in Fig. 3. According to the usual assumptions of the membrane theory of shells,[4] the bending stresses as well as effects of curvature of S are disregarded and

[4] See for instance S. P. Timoshenko, *Theory of plates and shells*, McGraw-Hill Co., New York, 1940, p. 356; also the first chapter of W. Flügge's *Statik und Dynamik der Schalen*, J. Springer, Berlin, 1934.

one has $N_{st}=N_{ts}$. The total forces acting on the faces hds_t and hdt of the element are respectively:

$$-\left\{N_t(\bar{\mu}\sin\theta+\bar{\nu}\cos\theta)+N_{st}\bar{\lambda}\right\}(\rho_0-t\sin\theta)d\varphi, \tag{2.1a}$$

$$-\left\{N_s\bar{\lambda}+N_{st}(\bar{\mu}\sin\theta+\bar{\nu}\cos\theta)\right\}dt. \tag{2.1b}$$

Let $\overline{P}=P_s\bar{\lambda}+P_t\,(\bar{\mu}\sin\theta+\bar{\nu}\cos\theta)+P_n(\bar{\mu}\cos\theta-\bar{\nu}\sin\theta)$ represent the load per unit area of the surface. Then the condition of equilibrium of the element of the shell is:

$$\frac{\partial}{\partial t}\left\{[N_t(\bar{\mu}\sin\theta+\bar{\nu}\cos\theta)+N_{st}\bar{\lambda}](\rho_0-t\sin\theta)\right\}dtd\varphi$$
$$+\frac{\partial}{\partial\varphi}\left\{N_s\bar{\lambda}+N_{st}(\bar{\mu}\sin\theta+\bar{\nu}\cos\theta)\right\}dtd\varphi=(\rho_0-t\sin\theta)\overline{P}dtd\varphi. \tag{2.2}$$

Equating the components of these forces in the n, s, and t directions, we obtain three equations for the determination of the three stress components:

$$\begin{aligned}
N_s &= (\rho_0-t\sin\theta)P_n\sec\theta,\\
\frac{\partial}{\partial t}\left\{N_{st}(\rho_0-t\sin\theta)\right\}-N_{st}\sin\theta &= (\rho_0-t\sin\theta)P_s-\frac{\partial N_s}{\partial\varphi},\\
\frac{\partial}{\partial t}\left\{N_t(\rho_0-t\sin\theta)\right\} &= -\frac{\partial N_{st}}{\partial\varphi}+(\rho_0-t\sin\theta)P_t-N_s\sin\theta.
\end{aligned} \tag{2.3}$$

We proceed to solve equations (2.3) with the simplifying assumption that the load $\overline{P}$ does not vary along the generators L, and obtain:[5]

$$\begin{aligned}
N_s &= (\rho_0-t\sin\theta)P_n\sec\theta,\\
N_{st} &= \frac{f(\varphi)\sin\theta}{(\rho_0-t\sin\theta)^2}-\tfrac{1}{3}\csc\theta(\rho_0-t\sin\theta)(P_s-P_n'\sec\theta)+\tfrac{1}{2}\rho_0'P_n\csc\theta\sec\theta,\\
N_t &= \frac{-1}{\rho_0-t\sin\theta}\left[\frac{f(\varphi)}{\rho_0-t\sin\theta}-g(\varphi)\right]'\\
&\quad+\frac{t\csc\theta}{\rho_0-t\sin\theta}\left[\tfrac{1}{3}\rho_0'P_s-\tfrac{5}{6}\rho_0'P_n'\sec\theta-\tfrac{1}{2}\rho_0''P_n\sec\theta\right]\\
&\quad-\tfrac{1}{2}\csc\theta(\rho_0-t\sin\theta)\left[P_t-P_n\tan\theta-\tfrac{1}{3}P_n''\csc\theta\sec\theta+\tfrac{1}{3}P_s'\csc\theta\right],
\end{aligned} \tag{2.4}$$

where $f(\varphi)$ and $g(\varphi)$ are arbitrary functions of φ and the prime denotes differentiation with respect to φ. If the curve C_0 is closed the continuity of stresses demands that f and g' have a period of 2π.

When the load on the shell is applied only through the end sections C_0 and C_1 the stress system becomes:

$$N_s=0;\quad N_{st}=\frac{f\sin\theta}{(\rho_0-t\sin\theta)^2};\quad N_t=\frac{-1}{(\rho_0-t\sin\theta)}\left[\frac{f}{\rho_0-t\sin\theta}-g\right]'. \tag{2.5}$$

Substituting (2.5) into (2.1a) and integrating between 0 and 2π, we obtain the resultant force $\overline{F}_t$ acting on the section C_t; the expression for $\overline{F}_t$ simplifies readily by virtue of (1.2):

[5] In the case of cylindrical surfaces, $\theta=0$ and integration of (2.3) leads to the special solution: $N_s=\rho P_n$; $N_{st}=f(s)+t(P_s-dN_s/ds)$; $N_t=g(s)-tdf/ds+tP_t+t^2/2(d^2N_s/ds^2-dP_s/ds)$; where $f(s)$ and $g(s)$ are arbitrary functions of the arc length s. In this connection see pp. 66–76 of Flügge's book.

$$\bar{F}_t = \int_0^{2\pi} \left\{ -\left[\frac{f}{\rho_0 - t\sin\theta} (\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) \right]' + g'(\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) \right\} d\varphi$$

$$= -\sin\theta \left\{ \bar{\imath} \int_0^{2\pi} g'\cos\varphi\, d\varphi + \bar{\jmath} \int_0^{2\pi} g'\sin\varphi\, d\varphi \right\} + \bar{k}\cos\theta \{ g(2\pi) - g(0) \}. \quad (2.6)$$

The resultant moment $\bar{M}_t$ about the origin due to the forces on the section C_t is found similarly:

$$\bar{M}_t = \int_0^{2\pi} \bar{R} \times \left\{ -\left[\frac{f}{\rho_0 - t\sin\theta} (\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) \right]' + g'(\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) \right\} d\varphi.$$

It follows by integration by parts that

$$\bar{M}_t = \bar{R}(0) \times \bar{F}_t + \bar{k}\sin\theta \int_0^{2\pi} f d\varphi + \cos\theta \int_0^{2\pi} (\bar{\imath}\cos\varphi + \bar{\jmath}\sin\varphi) f\, d\varphi$$

$$- \int_0^{2\pi} \bar{\lambda} \times \left\{ \int_0^{\varphi} g'(\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) d\varphi \right\} (\rho_0 - t\sin\theta) d\varphi. \quad (2.7)$$

The results (2.6) and (2.7) will form the basis of analysis in later sections.

3. Let the vector of infinitesimal displacement be

$$\bar{D} = u\bar{\lambda} + v(\bar{\mu}\sin\theta + \bar{\nu}\cos\theta) + w(\bar{\mu}\cos\theta - \bar{\nu}\sin\theta). \quad (3.1)$$

The strains in the surface are given by the following scalar products between the rates of change of the displacement $\bar{D}$ and the unit vectors in the t and s directions:

$$e_{tt} = \frac{\partial \bar{R}}{\partial t} \cdot \frac{\partial \bar{D}}{\partial t}, \qquad e_{ss} = \frac{\partial \bar{R}}{\partial s_t} \cdot \frac{\partial \bar{D}}{\partial s_t}, \qquad e_{st} = \left\{ \frac{\partial \bar{R}}{\partial s_t} \cdot \frac{\partial \bar{D}}{\partial t} + \frac{\partial \bar{R}}{\partial t} \cdot \frac{\partial \bar{D}}{\partial s_t} \right\}. \quad (3.2)$$

We evaluate (3.2) and substitute the results into Hooke's Law:

$$\frac{1}{Eh}\{N_t - \nu N_s\} = \frac{\partial v}{\partial t},$$

$$\frac{1}{Eh}\{N_s - \nu N_t\} = \frac{1}{\rho_0 - t\sin\theta}\{u' - (v\sin\theta + w\cos\theta)\}, \quad (3.3)$$

$$\frac{2(1+\nu)}{Eh} N_{st} = \frac{1}{\rho_0 - t\sin\theta}\left\{ (\rho_0 - t\sin\theta)\frac{\partial u}{\partial t} + u\sin\theta + v' \right\}.$$

Equations (3.3) are easily integrated to yield expressions for the displacements:

$$v = \frac{1}{Eh}\int^t (N_t - \nu N_s) dt + A(\varphi),$$

$$u = \frac{2(1+\nu)}{Eh}(\rho_0 - t\sin\theta)\int^t \frac{N_{st}}{\rho_0 - t\sin\theta} dt$$

$$- \frac{(\rho_0 - t\sin\theta)}{Eh}\int^t \frac{\int^t (N_t' - \nu N_s') dt}{(\rho_0 - t\sin\theta)^2} dt - A'(\varphi)\csc\theta + (\rho_0 - t\sin\theta)B(\varphi), \quad (3.4)$$

$$w = u'\sec\theta + v\tan\theta - \frac{1}{Eh}(\rho_0 - t\sin\theta)(N_s - \nu N_t)\sec\theta,$$

where $A(\varphi)$ and $B(\varphi)$ are arbitrary functions. When the stresses have the form (2.5), the displacements can be expressed directly in terms of the functions f and g:

$$
\begin{aligned}
v &= \frac{\csc\theta}{Eh}\left\{-f'\rho_t^{-1} + \tfrac{1}{2}f\rho'\rho_t^{-2} - g'\ln\rho_t\right\} + A,\\
u &= \frac{\csc^2\theta}{Eh}\left\{(1+\nu)f\,\rho_t^{-1}\sin^2\theta + \tfrac{1}{2}f''\rho_t^{-1} - \tfrac{1}{6}(f\rho'' + 3f'\rho')\rho_t^{-2} + \tfrac{1}{4}f\rho'^2\rho_t^{-3}\right.\\
&\quad\left. + \tfrac{1}{2}g'\rho'\rho_t^{-1} + g''(\ln\rho_t + 1)\right\} - A'\csc\theta + \rho_t B,\\
w &= \frac{\sec\theta\csc^2\theta}{Eh}\left\{\sin^2\theta\left[-\tfrac{3}{2}f\rho'\rho_t^{-2} + 2f'\rho_t^{-1} + g'(\ln\rho_t + \nu)\right] + \tfrac{1}{2}f'''\rho_t^{-1}\right.\\
&\quad - \tfrac{1}{6}(f\rho''' + 4f'\rho'' + 6f''\rho')\rho_t^{-2} + \tfrac{1}{12}(15f'\rho'^2 + 10f\rho'\rho'')\rho_t^{-3}\\
&\quad\left. - \tfrac{3}{4}f\rho'^3\rho_t^{-4} + \tfrac{1}{2}(g'\rho'' + 3g''\rho')\rho_t^{-1} - \tfrac{1}{2}g'\rho'^2\rho_t^{-2} + g'''(\ln\rho_t + 1)\right\}\\
&\quad - \tan\theta(A + A''\csc^2\theta) + B'\rho_t\sec\theta + B\rho'\sec\theta.
\end{aligned}
\tag{3.5}
$$

Expressions for displacements D_x, D_y, D_z in the x, y, z (or any other) directions are best derived by taking a scalar product between a unit vector in the given direction and $\bar{D}$ of (3.1). For instance,

$$D_z = \bar{k}\cdot\bar{D} = v\cos\theta - w\sin\theta. \tag{3.6}$$

4. The current literature on shells contains very little on the boundary conditions in the membrane theory of shells. We recall that local bending of the shell was disregarded according to the simplifying assumptions of the theory. Thus we cannot expect to satisfy all of the usual boundary conditions. For instance, we cannot ask that the heavy end bulkhead be considered rigid; in bending of the shell as a whole this would entail $e_{ss} = N_t = 0$ in the end section which could consequently transmit no bending moment. By allowing deformations in the plane of the end sections we remove the restriction on N_t and the problem of bending has a solution (see section 5). One has to decide in every particular problem which boundary conditions correspond more nearly to the assumption of no local bending.

A casual reader might be tempted to interpret the contribution of A and B to the displacements in (3.4) as that of rigid body motion since it is present when the stresses vanish. However, it is conceivable that a given state of stress induces inextensional displacements other than those of a rigid body as necessitated by the shape of the shell. Thus, in the case of a non-circular cylindrical shell under torsion, A accounts for the warping of the cross-sections.[6]

In general, these inextensional deformations are accompanied by local bending stresses which must be small to be neglected in accordance with our assumptions. One would expect that no energy is expended in the inextensional deformations. The strain energy in shells loaded through the end-sections is

$$V = \frac{1}{2h}\int_0^{t_1}\int_0^{2\pi}\left(\frac{N_t^2}{E} + \frac{N_{st}^2}{G}\right)\rho_t\,d\varphi\,dt, \tag{4.1}$$

[6] Specifically, $A = (T/2A_0Gh)\left\{\int_0^{\varphi}\rho\,d\varphi - (1/2A_0)\int_0^{\varphi}(x\rho\cos\varphi + y\rho\sin\varphi)\,d\varphi\right\}$. This expression is found by the method indicated in section **8**.

or

$$V = \frac{1}{2}\int_0^{2\pi} (N_{st}u + N_t v)\rho_t d\varphi \Big|_{t=0}^{t=t_1}. \tag{4.2}$$

Substituting (2.5) and the contribution due to A and B into (4.2) and integrating by parts, we find that our expectation is verified. The accompanying local bending, however, absorbs energy and, therefore, places limitations on the inextensional displacements according to the principle of minimum strain energy. The minimum expenditure of energy in bending occurs when the inextensional displacements reduce to rigid body displacements.

One can easily verify that the most general functions A and B corresponding to rigid body displacements have the form

$$\begin{aligned} A &= -a_x \sin\theta\cos\varphi - a_y \sin\theta\sin\varphi + a_z\cos\theta + \alpha_x y_0 \cos\theta \\ &\quad - \alpha_y x_0 \cos\theta + \alpha_z \sin\theta(y_0\cos\varphi - x_0\sin\varphi), \\ B &= \alpha_x \cot\theta\cos\varphi + \alpha_y\cot\theta\sin\varphi + \alpha_z, \end{aligned} \tag{4.3}$$

where a_x, a_y, a_z represent the infinitesimal translations in the x, y, z directions; α_x, α_y, α_z the infinitesimal rotations about the x, y, z axes; and x_0, y_0 the coordinates in the base section C_0.

Instead of imposing conditions on the displacements, one may prescribe a sensible distribution of stresses at the boundary. We note that by (2.5) the state of stress in the whole shell is determined as soon as the stresses N_t and N_{st} are given at *one* end-section. Thus two different stress distributions which are statically equivalent over an end-section will determine distinctly different stress distributions in the rest of the shell.[7]

5. We shall study first the effects of taper[8] as exhibited in a conical shell of circular cross-section; later, we shall discuss the influence of a variable radius of curvature ρ_t of the section C_t.

Let M represent the bending moment (causing tension for $x_t > 0$) applied to the shell through the end-sections C_0 and C_1. We shall try to satisfy the conditions that the end-sections (bulkheads) remain plane, i.e.,

$$D_z = 0 \text{ for } t = 0; \qquad D_z = \beta(x_0 - t_1\sin\theta) \text{ for } t = t_1; \tag{5.1}$$

where β is the (undetermined) angle of bending, and that the displacements due to A and B reduce to rigid body displacements (4.3). By virtue of (3.6), (3.5), and (4.3), we obtain for the first of conditions (5.1)

$$\begin{aligned} -\frac{\sec\theta\csc\theta}{Eh}\Big\{ &\frac{1}{2r}(2f'(1+\sin^2\theta) + f''') \\ &+ (g' + g''')\ln r + \nu g'\sin^2\theta + g'''\Big\} - \alpha_y r\cos\varphi = 0. \end{aligned} \tag{5.2}$$

[7] This is the price that has to be paid for the simplifications due to the assumptions of the membrane theory. A "disturbance" of the state of stress on one end-section (the difference between the equivalent stress distributions) "propagates" itself along the generators without "dying out." The general theory of thin shells would lead to differential equations of higher order; for these one can find solutions representing disturbances that die out with the distance from the end-section.

[8] All the results of sections **5–9** simplify to the corresponding expressions for a cylinder as the taper approaches zero.

By symmetry, the functions g' and f' are odd in x_0; let their Fourier expansions read

$$g' = \sum_0^\infty (2n+1)a_{2n+1} \cos (2n+1)\varphi, \quad f' = \sum_0^\infty (2n+1)b_{2n+1} \cos (2n+1)\varphi. \tag{5.3}$$

Since the resultant force $\overline{F}_0$ on C_0 must vanish, one concludes from (2.6) that $a_1=0$. Equation (2.7) yields $\overline{M}_t=\bar{j}\pi b_1 \cos\theta$ or $b_1=-(1/\pi)M \sec\theta$. It follows from the coefficient of $\cos\varphi$ in (5.2) that

$$\alpha_y = (1/2\pi Ehr^2) \sec^2\theta \csc\theta(1 + 2\sin^2\theta)M. \tag{5.4}$$

Substituting (5.4) and (5.3) into the second of conditions (5.1) and equating coefficients of $\cos\varphi$ in the two members, we obtain

$$\beta = \frac{M \sin\theta(2+\csc^2\theta)}{2\pi Eh\cos^2\theta}\left\{\frac{1}{(r-t_1\sin\theta)^2} - \frac{1}{r^2}\right\}. \tag{5.5}$$

For the coefficients a_{2n+1} and b_{2n+1}, $n>0$, one obtains a system of two homogeneous equations with a non-vanishing determinant. Therefore $a_{2n+1}=b_{2n+1}=0$, $n>0$, and

$$N_t = \frac{M\sec\theta\cos\varphi}{\pi(r-t\sin\theta)^2}, \qquad N_{st} = -\frac{M\tan\theta\sin\varphi}{\pi(r-t\sin\theta)^2}, \tag{5.6}$$

$$D_z = \frac{M \sin\theta(2+\csc^2\theta)}{2\pi Eh\cos^2\theta}\left\{\frac{1}{(r-t\sin\theta)^2} - \frac{1}{r^2}\right\} x_t. \tag{5.7}$$

If we designate by I_t the moment of inertia, $\pi(r-t\sin\theta)^3$, of C_t about the neutral axis, we can write $N_t=(1/I_t)Mx_t\sec\theta$. Essentially the stresses in the z direction follow the classical beam formula; the influence of taper is manifested by the presence of the x components of stresses N_t which have to be balanced by N_{st}. From (5.7) we see that all sections C_t remain plane. The rate of change of the angle of bending increases as the shell grows narrower:

$$\frac{d\beta}{dz} = \frac{M(1+2\sin^2\theta)}{\pi Eh\cos^3\theta(r-t\sin\theta)^3} = \frac{M(1+2\sin^2\theta)}{EhI_t\cos^3\theta}. \tag{5.8}$$

Further effects of taper are apparent in the other displacements:

$$v = \frac{M\tan\theta}{2\pi Eh}\cos\varphi\left\{\frac{2\csc^2\theta}{r-t\sin\theta} - \frac{1}{r}(2\csc^2\theta - \nu)\right\}, \tag{5.9}$$

$$u = \frac{M\sec\theta}{2\pi Eh}\sin\varphi\left\{\frac{\csc^2\theta - 2 - 2\nu}{r-t\sin\theta} - \frac{1}{r}(2\csc^2\theta-\nu) + \frac{1}{r^2}(r-t\sin\theta)(\csc^2\theta+2)\right\}, \tag{5.10}$$

$$w = \frac{M\sec^2\theta}{2\pi Eh}\cos\varphi\left\{\frac{\csc^2\theta - 4}{r-t\sin\theta} - \frac{1}{r}\cos^2\theta(2\csc^2\theta-\nu) + \frac{1}{r^2}(r-t\sin\theta)(\csc^2\theta+2)\right\}, \tag{5.11}$$

$$D_x = \frac{M\sec\theta}{2\pi Eh}\left\{\frac{2-\csc^2\theta+2\nu\sin^2\varphi}{r-t\sin\theta} + \frac{1}{r}(2\csc^2\theta-\nu) - \frac{1}{r^2}(r-t\sin\theta)(\csc^2\theta+2)\right\}. \tag{5.12}$$

If we take for X, the fictitious displacement of the axis of the cone, the average of D_x over C_t (by analogy with a cylinder or prism), we obtain for the slope of the deformed axis

$$\frac{dX}{dz} = \frac{M \sin \theta}{2\pi Eh \cos^2 \theta}\left\{\frac{2 + \nu - \csc^2 \theta}{(r - t \sin \theta)^2} + \frac{1}{r^2}(2 + \csc^2 \theta)\right\}. \tag{5.13}$$

Comparison with equation (5.7) shows that the axis is not perpendicular to the sections C_t as one might expect. Nor is the increment in slope equal to β, the angle between the end sections. In fact, for $\csc^2 \theta = 2+\nu$, a large taper, the axis remains altogether straight despite the angle between C_0 and C_1. This is due to a slipping effect caused by an interplay of the shearing forces N_{st} and the x components of N_t. Finally, let us check (5.5) by the customary[9] application of Castigliano's Principle, $\partial V/\partial M = \beta$. Substituting (5.6), (5.9), and (5.10) into (4.2), we have

$$V\Big|_{t=0} = \frac{M^2 \sin \theta}{4\pi Eh \cos^2 \theta}\,\frac{2\nu}{r^2}, \tag{5.14}$$

$$V = \frac{M^2 \sin \theta(2 + \csc^2 \theta + 2\nu)}{4\pi Eh \cos^2 \theta}\left\{\frac{1}{(r - t_1 \sin \theta)^2} - \frac{1}{r^2}\right\}, \tag{5.15}$$

$$\beta_V = \frac{M \sin \theta(2 + \csc^2 \theta + 2\nu)}{2\pi Eh \cos^2 \theta}\left\{\frac{1}{(r - t_1 \sin \theta)^2} - \frac{1}{r^2}\right\}. \tag{5.16}$$

The discrepancy between (5.5) and (5.16) is negligible in practical applications, but is interesting theoretically. It springs from a loose interpretation of Castigliano's Principle above, which is strictly true only for a concentrated couple M. Since M is distributed over the end sections, it does work not only in bending the shell but also in deforming the end-sections within their planes, as seen from (5.14). When the end-sections are alike as in a cylinder or prism, as much energy is spent in the deformation of one end as is gained at the other end; then, Castigliano's Principle holds even for a distributed moment. But to obtain the correct angle of bending in the case of a cone, one must deduct from the total strain energy (5.15) the net energy absorbed in the plane deformation of C_0 and C_1, namely

$$\frac{M^2\nu \sin \theta}{2\pi Eh \cos^2 \theta}\left\{\frac{1}{(r - t_1 \sin \theta)^2} - \frac{1}{r^2}\right\}.$$

6. We derive easily the expressions for the stresses in a cone twisted by a torque T by making either $D_z = 0$ or $N_t = 0$ at C_0 and C_1 and using (2.6) and (2.7),

$$N_{st} = \frac{T}{2\pi(r - t \sin \theta)^2}, \qquad N_t = 0. \tag{6.1}$$

From the displacements or the strain energy we obtain the total angle of twist γ and the angle of twist per unit length of the cone

[9] See for instance Timoshenko, *Strength of materials*, vol. 1, D. Van Nostrand, New York, 1940, p. 312.

$$\gamma = \frac{T \csc \theta}{4\pi G h}\left\{\frac{1}{(r - t_1 \sin \theta)^2} - \frac{1}{r^2}\right\}, \qquad \frac{d\gamma}{dz} = \frac{T \sec \theta}{2\pi G h (r - t \sin \theta)^3}. \tag{6.2}$$

Here, the effects of taper as manifested in (6.1) and (6.2) are not unexpected.

More interesting is the case of a cone supported at C_0 and bent by a force R (in the x direction) distributed over C_1. We learn from (2.6) that the function $f(\varphi)$ and hence the shear stress N_{st} do not actually contribute to the resultant R acting on any section C_t. Expressions (2.6) and (2.7) show that the term in $\cos \varphi$ of g' alone influences the resultant force as well as moment on C_t. We superpose a state of stress given by (5.6) with $M = R \cot \theta (r - t_1 \sin \theta)$ in order to bring the moment across C_1 to zero, and obtain the final result

$$N_t = \frac{-R(t_1 - t) \cos \varphi}{\pi (r - t \sin \theta)^2}, \qquad N_{st} = \frac{-R(r - t_1 \sin \theta) \sin \varphi}{\pi (r - t \sin \theta)^2}. \tag{6.3}$$

7. Let us now consider shells with non-circular cross-sections C_t. The coordinates of points on C_t are expressed in terms of ρ_t and φ

$$x_t = x_t(0) - \int_0^{\varphi} \rho_t \sin \varphi \, d\varphi, \qquad y_t = \int_0^{\varphi} \rho_t \cos \varphi \, d\varphi. \tag{7.1}$$

It is clear from (7.1) that ρ_t cannot contain any terms in $\cos \varphi$ or $\sin \varphi$ if the shell is closed. If only cosine terms appear in the Fourier expansion

$$\rho_t = r_t - \sum_2^{\infty} r_n \cos n\varphi, \tag{7.2}$$

the section C_t is symmetric with respect to the x axis. The simple section, for which $r_n = 0$ if $n \neq 3$, approximates the cross-section of many a fuselage:

$$\begin{aligned} x_t &= r_t \cos \varphi + \tfrac{1}{4} r_3 \cos 2\varphi - \tfrac{1}{8} r_3 \cos 4\varphi; \\ x_t(0) &= r_t + \frac{r_3}{8}, \qquad x_t(\pi) = -r_t + \frac{r_3}{8}; \\ y_t &= r_t \sin \varphi - \tfrac{1}{4} r_3 \sin 2\varphi - \tfrac{1}{8} r_3 \sin 4\varphi; \qquad y_t(\pi/2) = r_t. \end{aligned} \tag{7.3}$$

The neutral axis of the sections coincides with the y axis (i.e., is independent of t) if x_t contains no constant term and if

$$\int_0^{2\pi} x_0 \rho_0 d\varphi = \frac{\pi}{2} \sum_2^{\infty} \frac{r_n (r_{n+1} - r_{n-1})}{n} = 0. \tag{7.4}$$

In bending, only sections satisfying (7.4) will be considered.

8. The stresses in a shell of constant slope under torsion are determined from the conditions that the load is applied in such a manner that only shearing stresses are generated at the end-sections. The conditions $N_t = 0$ at $t = 0$ and $t = t_1$ yield $f = k\rho_0(\rho_0 - t_1 \sin \theta)$ and $g = k(\rho_0 - t_1 \sin \theta)$. Substituting the expression for f into N_{st}, we find the torque T on C_0

$$T = \int_0^{2\pi} \overline{R} \times N_{st} \overline{\lambda} \rho_0 d\varphi = k \sin \theta \left\{ \oint \overline{R} \times \frac{d\overline{R}}{ds_0} ds_0 - t_1 \sin \theta \int_0^{2\pi} \overline{R} \times \overline{\lambda} d\varphi \right\},$$

which reduces to

$$T = k \sin \theta \{ 2A_0 - t_1 L_0 \sin \theta \} = k \sin \theta \{ A_0 + A_1 - \pi t_1^2 \sin^2 \theta \}. \tag{8.1}$$

Here the A's represent the areas of the sections and L_0 is the length of C_0, all quantities easily measurable. Then,

$$N_{st} = \frac{T}{(A_0 + A_1 - \pi t_1^2 \sin^2 \theta)} \cdot \frac{\rho_0 \rho_1}{\rho_t^2}, \tag{8.2}$$

$$N_t = \frac{-T}{(A_0 + A_1 - \pi t_1^2 \sin^2 \theta)} \cdot \frac{t}{\rho_t} \left(\frac{\rho_1}{\rho_t} \right)' = \frac{-T \sin \theta}{(A_0 + A_1 - \pi t_1^2 \sin^2 \theta)} \cdot \frac{t(t_1 - t)\rho'}{\rho_t^3}. \tag{8.3}$$

The effect of the variable radius of curvature of C_t is observed in the expression for N_t; tensile stresses increase directly with ρ' and inversely with ρ_t^3.

The expression (4.2) for strain energy takes the form

$$V = \frac{k^2 \csc \theta}{2Eh} \left\{ t_1(1 + \nu) \sin^3 \theta (L_0 + L_1) + \frac{t_1^2 \sin^2 \theta}{12} \int_0^{2\pi} \left(\frac{\rho'}{\rho_t} \right)^2 d\varphi \Bigg|_0^{t_1} \right.$$
$$\left. - \frac{t_1 \sin \theta}{2} \int_0^{2\pi} \rho'^2 \left(\frac{1}{\rho_1} + \frac{1}{\rho_0} \right) d\varphi - \int_0^{2\pi} \rho'^2 \ln \left(\frac{\rho_1}{\rho_0} \right) d\varphi \right\}, \tag{8.4}$$

and the angle of twist is

$$\gamma = \frac{T}{Eh(A_0 + A_1 - \pi t_1^2 \sin^2 \theta)^2} \left[t_1(1+\nu)(L_0 + L_1) + \csc^3 \theta \int_0^{2\pi} \left\{ \frac{t_1^2 \sin^2 \theta}{12} \rho'^2 \left(\frac{1}{\rho_1^2} - \frac{1}{\rho_0^2} \right) \right. \right.$$
$$\left. \left. - \frac{t_1 \sin \theta}{2} \rho'^2 \left(\frac{1}{\rho_1} + \frac{1}{\rho_0} \right) - \rho'^2 \ln \left(\frac{\rho_1}{\rho_0} \right) \right\} d\varphi \right]. \tag{8.5}$$

The quantity in the braces is of the order of $\sin^5 \theta$. Also, each of its terms contains the factor ρ'^2. In the common case of small taper and nearly circular shell we may use as a good approximation

$$\gamma_{app} = \frac{T t_1 (L_0 + L_1)}{2Gh(A_0 + A_1 - \pi t_1^2 \sin^2 \theta)^2}. \tag{8.6}$$

Neglecting the terms in the braces of (8.5) is equivalent to disregarding the effect of the stress N_t; see (4.1).

The inextensional displacements given by A and B in (3.5) can be determined from the twist of the end-sections (centers of twist along z axis)

$$u = 0, \quad t = 0; \qquad u = \gamma(x_1 \cos \varphi + y_1 \sin \varphi), \quad t = t_1. \tag{8.7}$$

These displacements include warping.[10] The actual process of solving (8.7) is quite tedious even when a definite section is given.

9. We conclude with a short discussion of stresses in a general shell of constant slope bent by couples M as in section 5. We assume that the moments at the end-sec-

[10] For a treatment of warping along similar lines see R. V. Southwell, *On the torsion of conical shells*, Proc. Royal Soc. London, (A)**163**, 337–355 (1937).

tions C_0 and C_1 are applied in such a manner that the stress N_t at these sections is proportional to the distance from the neutral axis:

$$N_t = e_0 x_0 \text{ for } t = 0; \qquad N_t = e_1 x_1 \text{ for } t = t_1. \tag{9.1}$$

Conditions (9.1) and the fact that the moments across C_0 and C_1 are alike lead us to the following expressions:

$$f = \frac{M\rho_0\rho_1}{t_1 \sin\theta \cos\theta}\left\{\frac{Q_0}{I_0} - \frac{Q_1}{I_1}\right\}; \qquad g = \frac{M}{t_1 \sin\theta \cos\theta}\left\{\frac{\rho_0 Q_0}{I_0} - \frac{\rho_1 Q_1}{I_1}\right\}, \tag{9.2}$$

where the I's are the moments of inertia about the neutral axis of the full respective sections and $Q_t = \int_0^\varphi x_t \rho_t d\varphi$ the variable first moment (about the same axis) of the section included between 0 and φ. The expressions for the stresses themselves read:

$$N_{st} = \frac{M\rho_0\rho_1}{t_1\rho_t^2 \cos\theta}\left\{\frac{Q_0}{I_0} - \frac{Q_1}{I_1}\right\}, \tag{9.3}$$

$$N_t = -\frac{M(t_1 - t)t\sin\theta}{t_1\cos\theta}\cdot\frac{\rho'}{\rho_t^3}\left\{\frac{Q_0}{I_0} - \frac{Q_1}{I_1}\right\} + \frac{M(t_1 - t)x_0}{t_1 I_0 \cos\theta}\cdot\frac{\rho_0^2}{\rho_t^2} + \frac{Mt}{t_1\cos\theta}\cdot\frac{x_1\rho_1^2}{I_1\rho_t^2}. \tag{9.4}$$

The corresponding expressions for strain energy and displacements are very cumbersome and can hardly be useful in practical applications.