THE INTRINSIC THEORY OF THIN SHELLS AND PLATES

PART III.—APPLICATION TO THIN SHELLS*

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10. Definitions and method of approximation. The method of approximation used below is essentially the same as in the case of thin plate theory. We define ϵ to be the average reduced thickness of a shell. (We may recall that the reduced thickness of a shell is the ratio of its thickness to a selected lateral dimension of its middle surface). Then for a thin shell, ϵ is a small quantity. This definition of a thin shell is in agreement with that of a thin plate given in Part II.

A thin shell is said to have *finite curvature* when the smallest radius of curvature of its middle surface and the selected lateral dimension are of the same order of magnitude. Furthermore, a thin shell is said to have *small curvature of order b* when the ratio of the selected lateral dimension to the smallest radius of curvature of its middle surface is of the same order of magnitude as ϵ^b, where $b \geqq 1$. Thus a thin plate may be regarded as a thin shell of small curvature of order ∞.

We consider a family of ∞^1 shells of the same material with diminishing reduced thickness, each in a state of stress under (i) external forces applied at the edge, (ii) surface forces and (iii) uniform body forces. We assign to each shell a value of a parameter ϵ $(0<\epsilon<\epsilon_1)$ denoting the average reduced thickness, so that the thickness is

$$2h = 2\epsilon\bar{h}(x^1, x^2). \tag{10.1}$$

The quantity ϵ_1 is supposed to be small, but the basic idea of the method is that we seek solutions valid for all ϵ in the range $0<\epsilon<\epsilon_1$. In this theory, ϵ is the only small quantity. All quantities occurring (except Poisson's ratio σ) are functions of ϵ. No quantity is *small* unless it tends to zero with ϵ.

For the greatest generality suppose all quantities to be power series in ϵ. Thus, supposing the middle surface itself to depend on ϵ, we have

$$\boldsymbol{a}_{\alpha\beta} = \sum_{s=0}^{\infty} \boldsymbol{a}_{(s)\alpha\beta}\epsilon^s, \qquad \boldsymbol{b}_{\alpha\beta} = \sum_{s=b}^{\infty} \boldsymbol{b}_{(s)\alpha\beta}\epsilon^s, \tag{10.2a, b}$$

where b is either zero or a positive integer. $\boldsymbol{a}_{(s)\alpha\beta}$ and $\boldsymbol{b}_{(s)\alpha\beta}$ are functions of $\boldsymbol{x}^\alpha$, independent of ϵ. For $b=0$, we are dealing with thin shells of finite curvature, while for $b\geqq 1$ we are dealing with thin shells of small curvature of order b.

Furthermore, we shall represent Q^i, P^i, $X^i_{[0]}$, $\tilde{T}^{\alpha\beta}$, $\tilde{T}^{\alpha 0}$, $\tilde{L}^{\alpha\beta}$, $p_{\alpha\beta}$, $q_{\alpha\beta}$ by power series as in Part II;

$$Q^0 = \sum_{s=k_0}^{\infty} Q^0_{(s)}\epsilon^s, \qquad Q^\alpha = \sum_{s=k}^{\infty} Q^\alpha_{(s)}\epsilon^s, \tag{10.3a}$$

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$$P^0 = \sum_{s=n_0}^{\infty} P^0_{(s)}\epsilon^s, \qquad P^\alpha = \sum_{s=k}^{\infty} P^\alpha_{(s)}\epsilon^s, \tag{10.3b}$$

$$X^0_{[0]} = \sum_{s=j_0}^{\infty} X^0_{(s)[0]}\epsilon^s, \qquad X^\alpha_{[0]} = \sum_{s=j}^{\infty} X^\alpha_{(s)[0]}\epsilon^s, \tag{10.3c}$$

$$\tilde{T}^{\alpha\beta} = \sum_{s=t}^{\infty} \tilde{T}^{\alpha\beta}_{(s)}\epsilon^s, \qquad \tilde{L}^{\alpha\beta} = \sum_{s=u}^{\infty} \tilde{L}^{\alpha\beta}_{(s)}\epsilon^s, \qquad \tilde{T}^{\alpha 0} = \sum_{s=l}^{\infty} \tilde{T}^{\alpha 0}_{(s)}\epsilon^s, \tag{10.4a, b, c}$$

$$p_{\alpha\beta} = \sum_{s=p}^{\infty} p_{(s)\alpha\beta}\epsilon^s, \qquad q_{\alpha\beta} = \sum_{s=q}^{\infty} q_{(s)\alpha\beta}\epsilon^s. \tag{10.5a, b}$$

Here k, k_0, n, n_0, j, j_0, t, u, l, p are integers greater than zero, and q is zero or a positive integer. The case $q=0$ corresponds to problems of finite deflection. The quantities $Q^0_{(s)}$, $Q^\alpha_{(s)}$, $P^0_{(s)}$ etc. are functions of x^α, independent of ϵ.

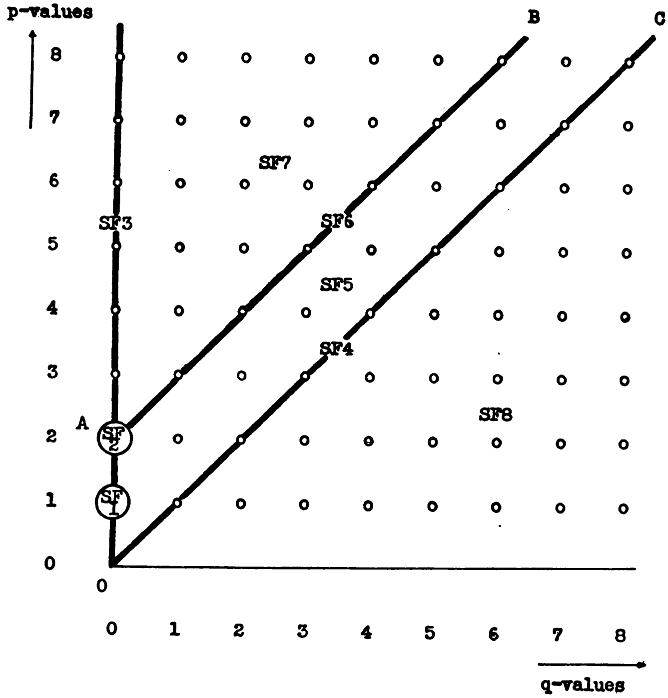

FIG. 4. Classification of problems of thin shells with finite curvature ($b=0$).
p = order of extension of middle surface.
q = order of change of curvature of middle surface.
b = order of initial curvature of middle surface.

Then the problems of thin shells can be classified by assigning integral values to p, q and b. With p, q, b given, the values of k_0, k, n_0, n, j_0, j in (10.3) are fixed by the condition that $X^0_{(j_0)[0]}$, $X^\alpha_{(j)[0]}$, $P^0_{(n_0)}$, $P^\alpha_{(n)}$, $Q^0_{(k_0)}$, $Q^\alpha_{(k)}$ should contribute to the principal parts of (6.34), (6.35), without dominating these equations to the exclusion of $p_{\alpha\beta}$ and $q_{\alpha\beta}$. The values of t, u, l of $T^{\alpha\beta}$, $L^{\alpha\beta}$, $T^{\alpha 0}$ are immediately fixed through the expressions (6.29), (6.30), (6.31). With p, q, b, k, k_0, j, j_0, n, n_0 fixed, the equations of equilibrium and compatibility in the first approximation are immediately obtained by substituting (10.1)–(10.5) into (6.34), (6.35), (6.43), (6.44), and picking out the principal terms in ϵ from the resulting equations. This gives us six differential equations in six unknowns $p_{(p)\alpha\beta}$ and $q_{(q)\alpha\beta}$. For the various combinations of values of p, q, b, the forms of these differential equations fall into several types. The classification of these types will be given below.

11. Classification of all thin shell problems. *The classification of the problems of thin shells with finite curvature* ($b=0$). The following is a complete classification of the problems of thin shells with finite curvature ($b=0$) based upon assigned values of p, q. The classification is shown graphically in Fig. 4.

It is found that the (p, q)-points in the diagram ($q\geqq 0$, $p\geqq 1$) are broken up into eight groups by the *division lines* AB, OC and the p-axis. For $q=0$, the principal part of (6.34) or (6.35) takes three different forms depending on the position of the point on the p-axis relative to the point A, while the principal parts of (6.43) and (6.44) are the same for all values of p. For $q\geqq 1$, the principal part of (6.34) or (6.35) takes three different forms depending on the position of the (p, q)-point relative to the line AB, and that of (6.43) or (6.44) takes three different forms depending on the position of the (p, q)-point relative to the line OC; each of these forms is different from that for $q=0$. It follows that the (p, q)-points are divided into eight groups and so the complete classification of all problems of thin shells of finite curvature involves consideration of eight types (Types $SF1$–$SF8$). (The letter 'S' denotes *shell*, while 'F' denotes *finite* curvature.)

In order to save space, we shall not discuss these types in detail. The results for these types are summarized together with those for thin shells with small curvature in the tables in the Appendices. The principal parts of the equations of equilibrium and compatibility are shown in Table III, and orders of magnitude of the external forces and the principal parts of the macroscopic tensors in Table IV.

The classification of the problems of thin shells with small curvature ($b\geqq 1$). The following is a complete classification of the problems of thin shells with small curvature based upon the assigned values of b, p, q. The classification is shown graphically in Fig. 5 (for $b=4$), Fig. 6 (for $b=2$), Fig. 7 (for $b=1$). The case $b=4$ is typical of the cases $3\leqq b<\infty$.

We shall now explain Fig. 5. We see that the (p, q)-points are broken up into 27 groups by the division lines and the p-axis. Of these division lines, the line $B'BB''$ (i.e., $q=b=4$) is the most important. It divides the (p, q)-plane into three main regions. For any point on $B'BB''$, the curvature in the unstrained state and the change of curvature during the strain are of the *same* order of magnitude ($q=b=4$). For any point on the left of $B'BB''$, the magnitude of the curvature in the unstrained state is *smaller* than the magnitude of the change of curvature ($q<b=4$), while for any point on the right of $B'B''$, the magnitude of the curvature in the unstrained state is *greater* than the magnitude of change of curvature ($q>b=4$).

For $q=0$ (i.e., on the p-axis) in Fig. 5, the principal parts of (6.34), (6.35) take three different forms depending on the position of the points on the p-axis relative to the point A, while the principal parts of (6.44), (6.43) are the same for all points on the p-axis. For $1\leqq q<b=4$ (i.e., in the region between the p-axis and $B'BB''$), the principal parts of (6.34) or (6.35) or (6.44) take three different forms depending on the position of the (p, q)-point relative to the division line AC or AB or OD respectively,

while the principal part of (6.43) is the same for all the (p, q)-points in this region. It follows that the (p, q)-points in the region on the left-hand side of $B'B''$ are divided into 11 groups (Types $SS1$–$SS11$). (The letters 'SS' denote the *shell* with *small* curvature.)

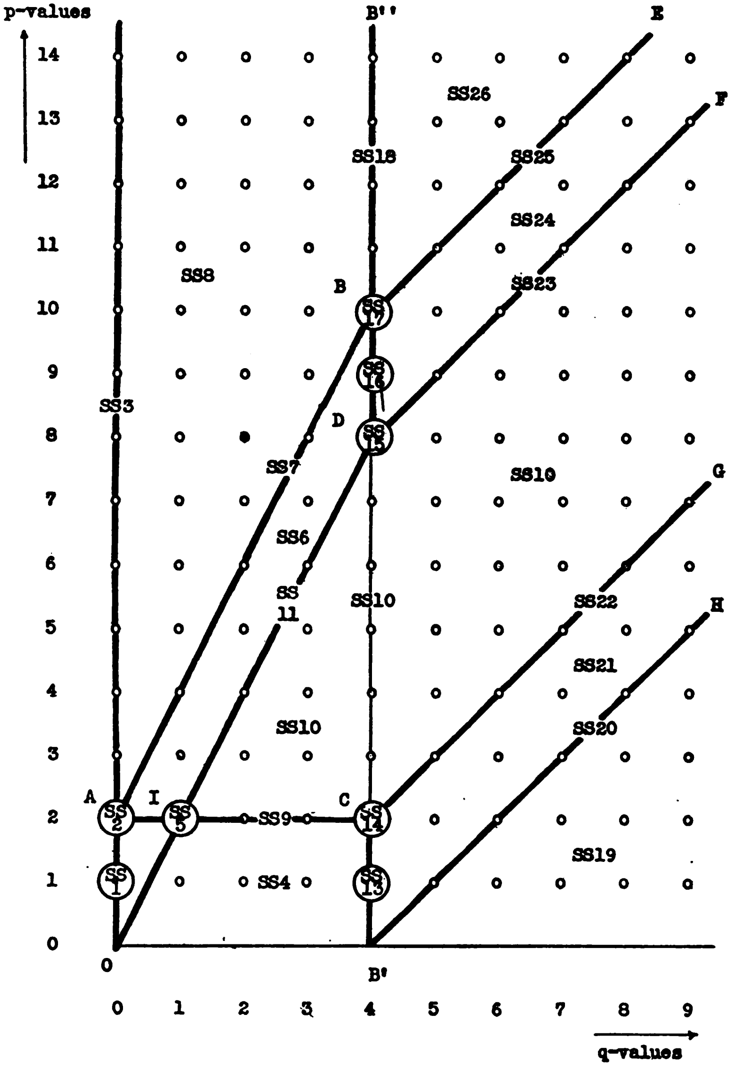

FIG. 5. Classification of problems of thin shells with small curvature $(b=4)$.
p = order of extension of middle surface.
q = order of change of curvature of middle surface.
b = order of initial curvature of middle surface.

For $q=b=4$ (i.e., on $B'B''$), the principal parts of (6.34) or (6.35) or (6.44) take three different forms depending on the position of the (p, q)-point relative to C or B or D respectively, while the principal part of (6.43) is the same for all points on this line. Furthermore, for $q>b=4$ (i.e., the region to the right of $B'B''$), the principal parts of (6.34) or (6.35) or (6.43) or (6.44) take three different forms depending on the position of the (p, q)-point relative to the division line CG or BE or $B'H$ or DF respectively. It follows that the (p, q)-points on the right-hand side of $B'B''$ are divided into 9 groups (Types $SS19$–$SS26$, $SS10$). It should be noted that, as far as the principal parts of (6.34), (6.35), (6.43), (6.44) are concerned, the (p, q)-points lying between the lines IDF and ICG are regarded as one group (Type $SS10$). Therefore,

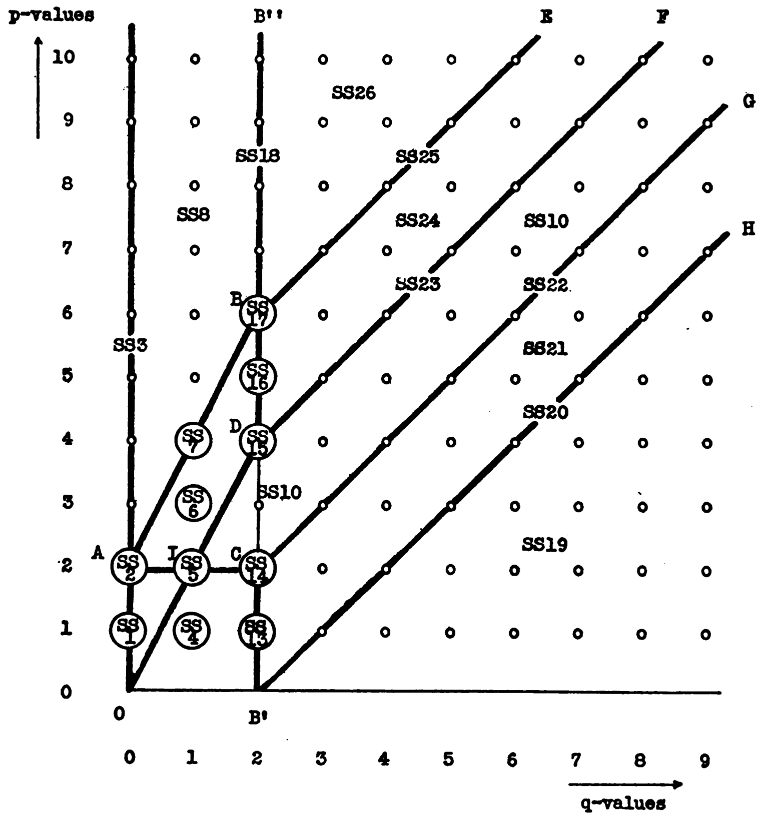

FIG. 6. Classification of problems of thin shells with small curvature ($b=2$).
p = order of extension of middle surface.
q = order of change of curvature of middle surface.
b = order of initial curvature of middle surface.

together with the groups on the left-hand side of $B'B''$, we have in all 25 groups of (p, q)-points in Fig. 5. And consequently the complete classification of all problems of thin shells with small curvature of order $b=4$ involves consideration of 25 types (Types $SS1$–$SS11$, $SS13$–$SS26$).

The general appearance of the classification diagrams for any b satisfying $3 \leqq b < \infty$ is the same as for $b=4$. An increase of b makes the line $B'B''$ shift to the right, while a decrease of b makes it shift to the left. On examining the various groups of (p, q)-points in these diagrams (for any integral value of b in the range of $3 \leqq b < \infty$), it is found that the corresponding groups occupying the same relative positions with respect to the division lines possess the same set of equations of equilibrium and compatibility in the first approximation, and so belong to the same type of problem.

Therefore the complete classification of all problems of thin shells with small curvature of order $3 \leqq b < \infty$ involves consideration of 25 types only.

For $b=2$ (Fig. 6), the situation is almost the same as in Fig. 5, but with the groups $SS9$, $SS11$ missing. The other groups are the same as those shown in Fig. 5 for $b=4$, and so no extra types arise.

For $b=1$ (Fig. 7), the situation is only slightly different from those in Figs. 5 and 6. Instead of the two separate division lines IDF and ICG for Eqs. (6.34) and (6.43) in Figs. 5 and 6, we have one common division line $D'F'$ for both equations. Furthermore, the triangle formed by the division lines ID, DC, IC in Figs. 5, 6 collapses into

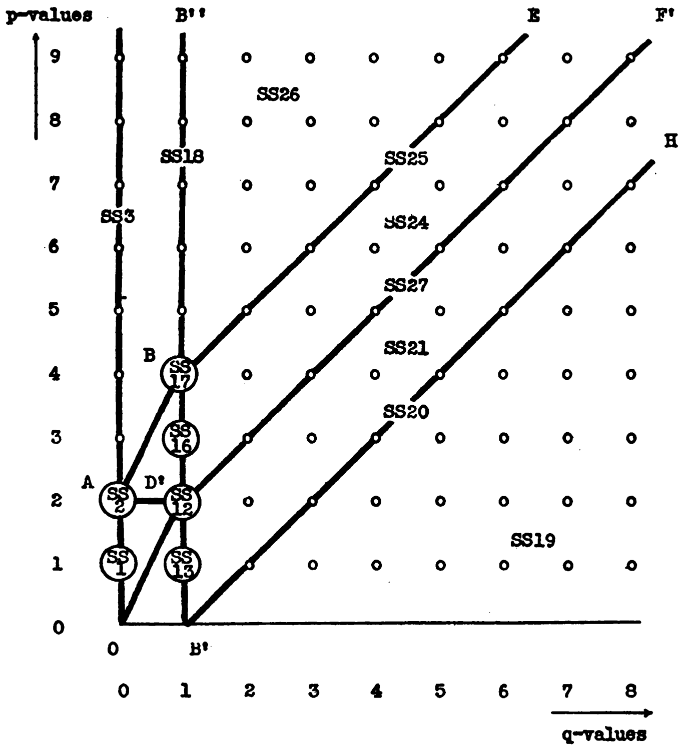

FIG. 7. Classification of problems of thin shells with small curvature ($b=1$).
p = order of extension of middle surface.
q = order of change of curvature of middle surface.
b = order of initial curvature of middle surface.

an isolated point D' in Fig. 7. Thus instead of 25 groups in Fig. 5, or 23 groups in Fig. 6, we have only 15 different groups. Among these groups, 13 belong to the types already mentioned in the case $3 \leqq b < \infty$ (Types $SS1$–$SS3$, $SS13$, $SS16$–$SS21$, $SS24$–$SS26$); the other two are Types $SS12$, $SS27$.

On comparing the classification of (p, q)-points on the left-hand side of $B'B''$ in Figs. 5, 6, 7 with that in the corresponding region of Fig. 3, it is found that they are identical with each other. In fact, for these types, the equations of equilibrium and compatibility in the first approximation are identical with those stated in Table I (Part II) for the corresponding types of thin plate problems. Therefore, we have the following important conclusion: *A problem of a thin shell with small curvature of order b is effectively equivalent to a problem of a thin plate in the first approximation, if* $q<b$, i.e., if the change of curvature is greater than the curvature of the shell in the unstrained state.

It should be noted that for $b=\infty$, Fig. 5 becomes exactly Fig. 3 for the thin plate problem.

The results are summed up as follows:

(i) The complete classification of the problems of thin shells with small curvature of order $b\geqq 1$ involves the consideration of 27 types (Types $SS1$–$SS27$).

(ii) Among these 27 types, 11 are equivalent to problems of thin plates; the characteristic of these types is $q<b$.

(iii) When $b=1$, these are two types (Types $SS12$, $SS27$) of particular interest.

We shall not discuss all these types in detail. The discussion of Type $SS12$ will serve as an example. The results for all types are summarized in tables in the Appendices. The principal parts of the equations of equilibrium and compatibility are shown in Table III, and the orders of magnitude of the external forces and the principal parts of the macroscopic tensors in Table IV.

Before entering on the detailed discussion of Type $SS12$, a useful result for small curvature ($b\geqq 1$) will be mentioned. On substituting $\boldsymbol{a}_{\alpha\beta}$, $\boldsymbol{b}_{\alpha\beta}$ from (10.2a, b) into (6.39b), it is found that the lowest power of ϵ in the resulting expression is ϵ^0. The corresponding coefficient gives rise to the equation

$$\begin{aligned}\boldsymbol{R}_{(0)\rho\alpha\beta\gamma} &= \tfrac{1}{2}(\boldsymbol{a}_{(0)\rho\gamma,\alpha\beta} + \boldsymbol{a}_{(0)\alpha\beta,\rho\gamma} - \boldsymbol{a}_{(0)\rho\beta,\alpha\gamma} - \boldsymbol{a}_{(0)\alpha\gamma,\beta\rho}) \\ &\quad + \boldsymbol{a}^{\pi\delta}_{(0)}\{[\rho\gamma,\pi]_{a_0}[\alpha\beta,\delta]_{a_0} - [\rho\beta,\pi]_{a_0}[\alpha\gamma,\delta]_{a_0}\} = 0, \qquad (11.1)\end{aligned}$$

where the Christoffel symbols are calculated for $\boldsymbol{a}_{(0)\alpha\beta}$. Eq. (11.1) expresses the fact that in the case of small curvature, the curvature tensor vanishes in the first approximation. Hence the order of the operations of covariant differentiation with respect to $\boldsymbol{a}_{(0)\alpha\beta}$ is immaterial; this result will be found very useful later.

12. Detailed discussion of type $SS12$ ($b=q=1$, $p=2$) and its applications. *General equations*. By the condition that in the first approximation, (6.34), (6.35) receive significant contributions from $P^0_{(n_0)}$, $P^\alpha_{(n)}$, $X^0_{(j_0)[0]}$, $X^\alpha_{(j)[0]}$, $Q^0_{(k_0)}$, $Q_{(k)}$, we must have

$$\begin{aligned} n_0 &= 4, & j_0 &= 3, & k_0 &= 2, \\ n &= 3, & j &= 2, & k &= 3. \end{aligned} \qquad (12.1)$$

By substituting the ϵ series into (6.34), (6.35), (6.43), (6.44), it is found that the lowest powers in ϵ occurring in the resulting equations are respectively ϵ^4, ϵ^3, ϵ^1, ϵ^2. The corresponding coefficients give rise to the following equations:

$$\begin{aligned} &- A^{\rho\gamma\pi\lambda}_{(01)}\boldsymbol{b}_{(1)\rho\gamma}\boldsymbol{p}_{(2)\pi\lambda}\bar{h} - 2A^{\rho\gamma\pi\lambda}_{(01)}\boldsymbol{q}_{(1)\rho\gamma}\boldsymbol{p}_{(2)\pi\lambda}\bar{h} + \tfrac{2}{3}A^{\rho\gamma\pi\lambda}_{(01)}(\boldsymbol{q}_{(1)\pi\lambda}\bar{h}^3)_{|\rho\gamma \atop a_0} \\ &+ P^0_{(4)} + 2X^0_{(3)[0]}\bar{h} + (Q^\lambda_{(3)}\bar{h})_{|\lambda \atop a_0} + \frac{2(1-2\sigma)}{1-\sigma}\boldsymbol{H}_{(1)}Q^0_{(2)}\bar{h} \\ &+ \frac{1-2\sigma}{1-\sigma}\boldsymbol{q}_{(1)\pi\lambda}\boldsymbol{a}^{\pi\lambda}_{(0)}Q^0_{(2)}\bar{h} = 0, \end{aligned} \qquad (12.2a)$$

$$2A^{\rho\alpha\pi\lambda}_{(01)}(\boldsymbol{p}_{(2)\pi\lambda}\bar{h})_{|\rho \atop a_0} + P^\alpha_{(3)} + 2X^\alpha_{(2)[0]}\bar{h} + \frac{\sigma}{1-\sigma}\boldsymbol{a}^{\alpha\rho}_{(0)}(Q^0_{(2)}\bar{h})_{|\rho \atop a_0} = 0, \qquad (12.2b)$$

$$\mathfrak{n}^{\beta\gamma}_{(0)} q_{(1)\alpha\beta\underset{a_0}{|}\gamma} = 0, \tag{12.2c}$$

$$2\mathfrak{n}^{\rho\alpha}_{(0)}\mathfrak{n}^{\beta\gamma}_{(0)} p_{(2)\rho\gamma\underset{a_0}{|}\alpha\beta} + \mathfrak{n}^{\rho\alpha}_{(0)}\mathfrak{n}^{\beta\gamma}_{(0)} q_{(1)\rho\gamma} q_{(1)\alpha\beta} + (b^{\alpha\beta}_{(1)} - 4H_{(1)}a^{\alpha\beta}_{(0)}) q_{(1)\alpha\beta} = 0, \tag{12.2d}$$

where a_0 under stroke indicates covariant differentiation with respect to the tensor $a_{(0)\alpha\beta}$ and x^α. The other symbols represent

$$A^{\alpha\beta\pi\lambda}_{(01)} = \frac{1}{1-\sigma^2}(\sigma a^{\alpha\beta}_{(0)} a^{\pi\lambda}_{(0)} + (1-\sigma) a^{\alpha\pi}_{(0)} a^{\beta\lambda}_{(0)}), \tag{12.3a}$$

$$\mathfrak{n}^{\alpha\beta}_{(0)} = \epsilon^{\alpha\beta}(a_0)^{-1/2}, \quad a_0 = \det.\ (a_{(0)\pi\lambda}), \quad \epsilon^{11} = \epsilon^{22} = 0, \quad \epsilon^{12} = -\epsilon^{21} = 1, \tag{12.3b}$$

$$b^{\alpha\beta}_{(1)} = a^{\alpha\pi}_{(0)} a^{\beta\lambda}_{(0)} b_{(1)\pi\lambda}, \quad H_{(1)} = \tfrac{1}{4} a^{\pi\lambda}_{(0)} b_{(1)\pi\lambda}. \tag{12.3c}$$

The macroscopic tensors (6.29)–(6.31) can be written as

$$T^{\alpha\beta} = \left\{ 2A^{\alpha\beta\pi\lambda}_{(01)} p_{(2)\pi\lambda}\bar{h} + \frac{\sigma}{1-\sigma} a^{\alpha\beta}_{(0)} Q^0_{(2)}\bar{h} \right\}\epsilon^3 + O(\epsilon^4), \tag{12.4a}$$

$$L^{\alpha\beta} = \tfrac{2}{3}\mathfrak{n}^{\gamma\beta}_{(0)} a_{(0)\pi\gamma} A^{\alpha\pi\lambda\delta}_{(01)} q_{(1)\lambda\delta}\bar{h}^3\epsilon^4 + O(\epsilon^5), \tag{12.4b}$$

$$T^{\alpha 0} = \left\{ \tfrac{2}{3} A^{\pi\alpha\lambda\delta}_{(01)} (q_{(1)\lambda\delta}\bar{h}^3)_{\underset{a_0}{|}\pi} + Q^\alpha_{(3)}\bar{h} \right\}\epsilon^4 + O(\epsilon^5). \tag{12.4c}$$

Equations (12.2a, b, c, d) are six equations for the six unknowns $p_{(2)\pi\lambda}$ and $q_{(1)\pi\lambda}$.

Since by (11.1) the order of the operations of covariant differentiation with respect to $a_{(0)\pi\lambda}$ is immaterial, (12.2c) implies the existence of $w_{(1)}$ such that

$$q_{(1)\alpha\beta} = w_{(1)\underset{a_0}{|}\alpha\beta}. \tag{12.5}$$

Thus the determination of $q_{(1)\alpha\beta}$ is reduced to the determination of the single function $w_{(1)}$. Furthermore, instead of using $p_{(2)\alpha\beta}$ as the rest of the unknowns, we may use $T^{\alpha\beta}_{(3)}$. By definition, $T^{\alpha\beta}_{(3)}\epsilon^3$ is the principal part of the macroscopic tensor $T^{\alpha\beta}$, namely, by (12.4a),

$$T^{\alpha\beta}_{(3)} = 2A^{\alpha\beta\pi\lambda}_{(01)} p_{(2)\pi\lambda}\bar{h} + \frac{\sigma}{1-\sigma} a^{\alpha\beta}_{(0)} Q^0_{(2)}\bar{h}. \tag{12.6}$$

This is a symmetrical tensor; so it has only three independent components. Substituting (12.5), (12.6) into (12.2a, b, d), we have

$$\begin{aligned} &- T^{\pi\lambda}_{(3)} w_{(1)\underset{a_0}{|}\pi\lambda} - \tfrac{1}{2} b_{(1)\pi\lambda} T^{\pi\lambda}_{(3)} + \tfrac{2}{3} A^{\rho\gamma\pi\lambda}_{(01)} (w_{(1)\underset{a_0}{|}\pi\lambda}\bar{h}^3)_{\underset{a_0}{|}\rho\gamma} + P^0_{(4)} \\ &\qquad + 2X^0_{(3)[0]}\bar{h} + (Q^\pi_{(3)}\bar{h})_{\underset{a_0}{|}\pi} + 2H_{(1)} Q^0_{(2)}\bar{h} + w_{(1)\underset{a_0}{|}\pi\lambda} a^{\pi\lambda}_{(0)} Q^0_{(2)}\bar{h} = 0, \end{aligned} \tag{12.7a}$$

$$T^{\pi\alpha}_{(3)\underset{a_0}{|}\pi} + P^\alpha_{(3)} + 2X^\alpha_{(2)[0]}\bar{h} = 0, \tag{12.7b}$$

$$\begin{aligned} &\mathfrak{n}^{\rho\alpha}_{(0)}\mathfrak{n}^{\beta\gamma}_{(0)} \{(1+\sigma) a_{(0)\pi\rho} a_{(0)\gamma\delta} - \sigma a_{(0)\rho\gamma} a_{(0)\pi\delta}\} \left\{ \frac{1}{\bar{h}} T^{\pi\delta}_{(3)} \right\}_{\underset{a_0}{|}\alpha\beta} + \sigma a^{\pi\lambda}_{(0)} Q^0_{(2)\underset{a_0}{|}\pi\lambda} \\ &\qquad + \mathfrak{n}^{\beta\alpha}_{(0)}\mathfrak{n}^{\rho\gamma}_{(0)} w_{(1)\underset{a_0}{|}\beta\gamma} w_{(1)\underset{a_0}{|}\alpha\rho} + (b^{\alpha\beta}_{(1)} - 4H_{(1)} a^{\alpha\beta}_{(0)}) w_{(1)\underset{a_0}{|}\alpha\beta} = 0. \end{aligned} \tag{12.7c}$$

Equations (12.7a, b, c) form a set of four equations for the four unknowns $w_{(1)}$ and $T^{\alpha\beta}_{(3)}$.

Special case. The following special case is interesting. If

$$P^{\alpha}_{(3)} = X^{\alpha}_{(2)[0]} = 0, \tag{12.8}$$

then by (12.7b) there exists a stress function $\chi_{(3)}$ such that

$$T^{\alpha\beta}_{(3)} = n^{\alpha\pi}_{(0)} n^{\beta\lambda}_{(0)} \chi_{(3)\underset{a_0}{|}\pi\lambda}. \tag{12.9}$$

Here $\chi_{(3)}$ is a function of x^α, having properties similar to those of the Airy function in the thin plate theory. Substituting (12.8), (12.9) into (12.7a, c), we have

$$- \tfrac{1}{2} n^{\pi\lambda}_{(0)} n^{\delta\rho}_{(0)} (2w_{(1)\underset{a_0}{|}\pi\delta} + b_{(1)\pi\delta}) \chi_{(3)\underset{a_0}{|}\lambda\rho} + \tfrac{2}{3} A^{\rho\gamma\pi\lambda}_{(01)} (w_{(1)\underset{a_0}{|}\pi\lambda} \bar{h}^3)_{\underset{a_0}{|}\rho\gamma}$$
$$+ P^0_{(4)} + 2X^0_{(3)[0]}\bar{h} + 2H_{(1)}Q^0_{(2)}\bar{h} + a^{\pi\lambda}_{(0)} w_{(1)\underset{a_0}{|}\pi\lambda} Q^0_{(2)}\bar{h} = 0, \tag{12.10a}$$

$$\{\sigma a^{\pi\rho}_{(0)} a^{\delta\lambda}_{(0)} - (1+\sigma) a^{\pi\delta}_{(0)} a^{\rho\lambda}_{(0)}\} \left\{\frac{1}{\bar{h}} \chi_{(3)\underset{a_0}{|}\delta\lambda}\right\}_{\underset{a_0}{|}\pi\rho} + \sigma a^{\pi\lambda}_{(0)} Q^0_{(2)\underset{a_0}{|}\pi\lambda}$$
$$+ n^{\rho\pi}_{(0)} n^{\beta\gamma}_{(0)} w_{(1)\underset{a_0}{|}\rho\gamma} w_{(1)\underset{a_0}{|}\pi\beta} + (b^{\pi\lambda}_{(1)} - 4H_{(1)} a^{\pi\lambda}_{(0)}) w_{(1)\underset{a_0}{|}\pi\lambda} = 0. \tag{12.10b}$$

Equations (13.10a, b) are two equations for the two unknowns $\chi_{(3)}$ and $w_{(1)}$. These equations are valid in general for a shell of non-uniform thickness. For the case of uniform thickness, (12.10a, b) are immediately simplified to the forms

$$- \tfrac{1}{2} n^{\pi\lambda}_{(0)} n^{\delta\rho}_{(0)} (2w_{(1)\underset{a_0}{|}\pi\delta} + b_{(1)\pi\delta}) \chi_{(3)\underset{a_0}{|}\lambda\rho} + D a^{\pi\gamma}_{(0)} a^{\lambda\delta}_{(0)} w_{(1)\underset{a_0}{|}\pi\gamma\lambda\delta}$$
$$+ P^0_{(4)} + 2X^0_{(3)[0]}\bar{h} + 2H_{(1)}Q^0_{(2)}\bar{h} + a^{\pi\lambda}_{(0)} w_{(1)\underset{a_0}{|}\pi\lambda} Q^0_{(2)}\bar{h} = 0, \tag{12.11a}$$

$$a^{\pi\gamma}_{(0)} a^{\lambda\delta}_{(0)} \chi_{(3)\underset{a_0}{|}\pi\gamma\lambda\delta} - \sigma \bar{h} a^{\pi\lambda}_{(0)} Q_{(2)\underset{a_0}{|}\pi\lambda} + \bar{h} n^{\pi\rho}_{(0)} n^{\lambda\gamma}_{(0)} w_{(1)\underset{a_0}{|}\rho\gamma} w_{(1)\underset{a_0}{|}\pi\lambda}$$
$$+ \bar{h}(4H_{(1)} a^{\pi\lambda}_{(0)} - b^{\pi\lambda}_{(1)}) w_{(1)\underset{a_0}{|}\pi\lambda} = 0, \tag{12.11b}$$

where D is the reduced flexural rigidity, as given in (9.14). Applications of these two equations will be discussed below.

A circular cylindrical thin shell with small curvature and uniform thickness under end thrust and normal pressure. We shall assume that the external forces and the edge loading are such that the problem is of Type $SS12$. Furthermore let us assume that

$$X^0_{(3)[0]} = Q^0_{(2)} = 0. \tag{12.12}$$

We have in mind the case where body force is negligible and where the shell is loaded normally on one side only. A number of terms disappear from the equations of equilibrium and compatibility (12.11a, b) for Type $SS12$. Thus if we write these equations in terms of the small principal parts instead of in terms of the finite coefficients of the lowest power in ϵ, we have

$$- \tfrac{1}{2} n^{\pi\lambda}_{[0]} n^{\delta\rho}_{[0]} (2w_{\underset{a}{|}\pi\delta} + b_{\pi\delta}) \chi_{\underset{a}{|}\lambda\rho} + D a^{\pi\gamma} a^{\lambda\delta} w_{\underset{a}{|}\pi\gamma\lambda\delta} + P^0 = 0, \tag{12.13a}$$

$$a^{\pi\gamma} a^{\lambda\delta} \chi_{\underset{a}{|}\pi\gamma\lambda\delta} + h n^{\pi\rho}_{[0]} n^{\delta\gamma}_{[0]} w_{\underset{a}{|}\rho\gamma} w_{\underset{a}{|}\pi\delta} + h(4Ha^{\pi\lambda} - b^{\pi\lambda}) w_{\underset{a}{|}\pi\lambda} = 0. \tag{12.13b}$$

Here a under a stroke indicates covariant differentiation with respect to the tensor $a_{\alpha\beta}$ and x^α; also

$$D = \frac{2h^3}{3(1-\sigma^2)}, \qquad 4H = a_{\alpha\beta} b^{\alpha\beta}. \tag{12.14}$$

Let us choose the set of intrinsic rectangular Cartesian coordinates on the middle surface so that $x = x^1$ is the distance measured along the generators of the cylinder and $y = x^2$ is the distance measured perpendicular to the generators. Then we have

$$\begin{aligned} a_{11} &= a^{11} = a_{22} = a^{22} = 1, & a_{12} &= a^{12} = 0, \\ b_{11} &= b^{11} = b_{12} = b^{12} = 0, & b_{22} &= b^{22} = 2/R, \end{aligned} \tag{12.15}$$

where R is the radius of curvature of the cylindrical middle surface. In these coordinates, Eqs. (12.13a, b) become

$$D\Delta\Delta w + (2w_{,xy}\chi_{,xy} - w_{,xx}\chi_{,yy} - w_{,yy}\chi_{,xx}) - \frac{1}{R}\chi_{,xx} + P^0 = 0, \tag{12.16a}$$

$$\Delta\Delta\chi + 2h(w_{,xx}w_{,yy} - w_{,xy}w_{,xy}) + 2h\frac{1}{R}w_{,xx} = 0, \tag{12.16b}$$

where subscripts preceded by a comma denote partial differentiation. If we let R tend to infinity, we get the von Kármán equations for a flat plate. The equation (12.16b) was recently obtained by von Kármán and Tsien [1] in their treatment of buckling of a thin-walled circular cylindrical shell under compression on the two ends. If we apply the operators $\Delta\Delta$ to (12.16a) and $(1/R)\partial^2/\partial x^2$ to (12.16b) and add the resulting equations, we obtain

$$\begin{aligned} D\Delta\Delta\Delta\Delta w + \frac{2h}{R^2}w_{,xxxx} + \frac{2h}{R}(w_{,xx}w_{,yy} - w_{,xy}w_{,xy})_{,xx} \\ = \Delta\Delta(P^0 + 2w_{,xy}\chi_{,xy} - w_{,xx}\chi_{,yy} - w_{,yy}\chi_{,xx}). \end{aligned} \tag{12.17}$$

This is the equation of equilibrium used by von Kármán and Tsien, except that they omit the term

$$\frac{2h}{R}(w_{,xx}w_{,yy} - w_{,xy}w_{,xy})_{,xx}. \tag{12.18}$$

This term is important when the deflection is comparable with thickness. However, it seems simpler to treat the problem directly by means of (12.16a, b) instead of using the higher-order equation (12.17). Equation (12.16a) appears to be new.

A small segment of a thin spherical shell under external pressure. We shall assume that the solid angle of the segment is small, so that the curvature is small; we shall assume it to be of the same order as the thickness, so that $b = 1$ (cf. section 10). We shall use spherical polar coordinates as in Fig. 8, so that on the middle surface in the unstrained state we have

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta\, d\varphi^2. \tag{12.19}$$

Since θ is small, we write

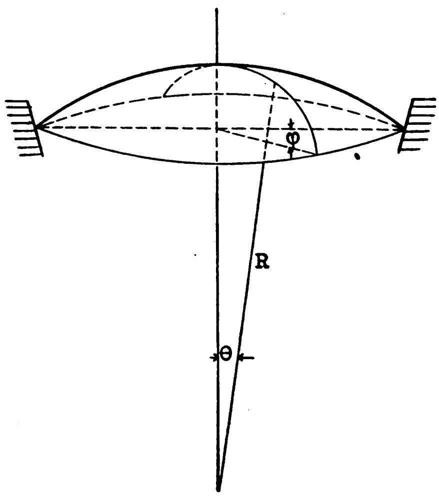

FIG. 8.

$$ds^2 = R^2 d\theta^2 + R^2\theta^2 d\varphi^2. \tag{12.20}$$

If we put

$$x^1 = \theta, \qquad x^2 = \varphi, \tag{12.21}$$

the components of the first and second fundamental tensors are given by

$$a_{11} = R^2, \quad a_{22} = R^2\theta^2, \quad a_{12} = 0, \quad a^{11} = 1/R^2, \quad a^{22} = 1/R^2\theta^2, \quad a^{12} = 0, \tag{12.22}$$

$$b_{11} = 2R, \quad b_{22} = 2R\theta^2, \quad b^{11} = 2/R^3, \quad b^{22} = 2/R^3\theta^2, \quad b_{12} = b^{12} = 0. \tag{12.23}$$

Futhermore, we have from (12.22), (12.23)

$$H = 1/R, \qquad a = R^4\theta^2. \tag{12.24}$$

All the Christoffel symbols are equal to zero, except

$$\left\{ {1 \atop 2\ 2} \right\} = -\theta, \qquad \left\{ {2 \atop 1\ 2} \right\} = 1/\theta. \tag{12.25}$$

We shall assume that the problem is of Type $SS12$. Substituting (12.21)–(12.25) into (12.13a, b), we have

$$-\frac{1}{R^4\theta^2}\left\{(w_{,\theta\theta} + R)(\chi_{,\varphi\varphi} + \theta\chi_{,\theta}) - 2\left(w_{,\theta\varphi} - \frac{1}{\theta}w_{,\varphi}\right)\left(\chi_{,\theta\varphi} - \frac{1}{\theta}\chi_{,\varphi}\right)\right.$$
$$\left. + (w_{,\varphi\varphi} + R\theta^2 + \theta w_{,\theta})\chi_{,\theta\theta}\right\} + D\Delta\Delta w + P^0 = 0, \tag{12.26a}$$

$$\Delta\Delta\chi + \frac{2h}{R^4\theta^2}\left\{w_{,\theta\theta}(w_{,\varphi\varphi} + \theta w_{,\theta}) - \left(w_{,\theta\varphi} - \frac{1}{\theta}w_{,\varphi}\right)^2\right\}$$
$$+ 2h\left\{\frac{1}{R^3}w_{,\theta\theta} + \frac{1}{R^3\theta^3}(w_{,\varphi\varphi} + \theta w_{,\theta})\right\} = 0. \tag{12.26b}$$

Here Δ is the Laplace operator

$$\Delta = \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{R^2\theta^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{R^2\theta}\frac{\partial}{\partial\theta} = \frac{1}{R^2\theta}\frac{\partial}{\partial\theta}\theta\frac{\partial}{\partial\theta} + \frac{1}{R^2\theta^2}\frac{\partial^2}{\partial\varphi^2}\cdot \tag{12.27}$$

Equations (12.26a, b) are two nonlinear partial differential equations for two unknowns χ, w.

We suppose that the problem has rotational symmetry. Then w, χ are independent of φ, and (12.26a, b) reduce to the form

$$\frac{d}{d\theta}\theta\frac{d}{d\theta}\frac{1}{\theta}\frac{d}{d\theta}\theta\frac{dw}{d\theta} - \frac{1}{D}\frac{d}{d\theta}\left(\frac{dw}{d\theta}\frac{d\chi}{d\theta}\right) - \frac{R}{D}\frac{d}{d\theta}\left(\theta\frac{d\chi}{d\theta}\right) + \frac{P^0\theta R^4}{D} = 0, \tag{12.28a}$$

$$\frac{d}{d\theta}\theta\frac{d}{d\theta}\frac{1}{\theta}\frac{d}{d\theta}\theta\frac{d\chi}{d\theta} + h\frac{d}{d\theta}\left(\frac{dw}{d\theta}\right)^2 + 2hR\frac{d}{d\theta}\left(\theta\frac{dw}{d\theta}\right) = 0. \tag{12.28b}$$

The equations can be integrated once giving

$$\theta \frac{d}{d\theta} \frac{1}{\theta} \frac{d}{d\theta} \theta \frac{dw}{d\theta} - \frac{1}{D} \frac{dw}{d\theta} \frac{d\chi}{d\theta} - \frac{R}{D} \theta \frac{d\chi}{d\theta} + \frac{P^0 \theta^2 R^4}{2D} = \text{constant}, \tag{12.29a}$$

$$\theta \frac{d}{d\theta} \frac{1}{\theta} \frac{d}{d\theta} \theta \frac{d\chi}{d\theta} + \left(\frac{dw}{d\theta}\right)^2 h + 2hR\theta \frac{dw}{d\theta} = \text{constant}. \tag{12.29b}$$

Since $dw/d\theta$ vanishes for $\theta = 0$, the constants are zero. If we introduce the quantities

$$\alpha = \frac{1}{R} \frac{dw}{d\theta} + \theta, \qquad \beta = \frac{1}{R^2} \frac{d\chi}{d\theta}, \tag{12.30}$$

the equations can be further simplified to the form

$$\theta \frac{d^2\alpha}{d\theta^2} + \frac{d\alpha}{d\theta} - \frac{\alpha}{\theta} - \frac{R^2}{D} \alpha\beta + \frac{P^0 R^3}{2D} \theta^2 = 0, \tag{12.31a}$$

$$\theta \frac{d^2\beta}{d\theta^2} + \frac{d\beta}{d\theta} - \frac{\beta}{\theta} + h(\alpha^2 - \theta^2) = 0. \tag{12.31b}$$

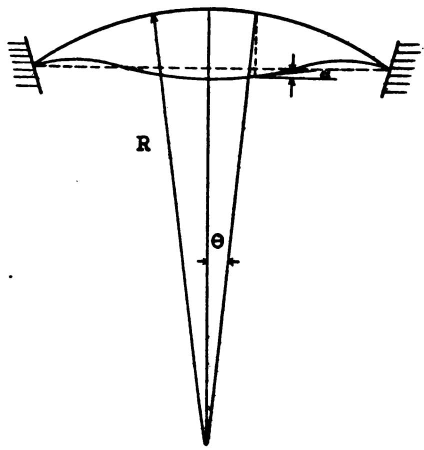

FIG. 9.

The quantity α is the slope of the meridian line in the strained state (Fig. 9). The significance of the quantity β is that β/θ is the radial membrane stress (tension). Equations (12.31a, b) are the fundamental equations for the determination of the buckling pressure of a small segment of spherical shell.

If we assume that the first and second terms in (12.31b) are negligible in comparison with the other terms, then we can solve (12.31b) immediately for β. Substituting the resulting expression for β into (12.31a), we have

$$\theta \frac{d^2\alpha}{d\theta^2} + \frac{d\alpha}{d\theta} - \frac{\alpha}{\theta} = \frac{hR^2}{D} \theta\alpha(\alpha^2 - \theta^2) - \frac{P^0 \theta^2 R^3}{2D}. \tag{12.32}$$

This is the equation used by von Kármán and Tsien [2] in their treatment of buckling of spherical shells by external pressure. It should be noted that the neglect of the first two terms in (12.31b) is a rough approximation. Actually the first three terms in (12.31b) are of the same order of magnitude.

Furthermore, if we introduce

$$r = R\theta, \tag{12.33}$$

Eqs. (12.28a, b) can be written in the form

$$\frac{d}{dr} r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} r \frac{dw}{dr} - \frac{1}{D} \frac{d}{dr}\left(\frac{dw}{dr} \frac{d\chi}{dr}\right) - \frac{1}{RD} \frac{d}{dr}\left(r \frac{d\chi}{dr}\right) + \frac{P^0 r}{D} = 0, \tag{12.34a}$$

$$\frac{d}{dr} r \frac{d}{dr} \frac{1}{r} \frac{d}{dr} r \frac{d\chi}{dr} + h \frac{d}{dr}\left(\frac{dw}{dr}\right)^2 + \frac{2h}{R} \frac{d}{dr}\left(r \frac{dw}{dr}\right) = 0. \tag{12.34b}$$

The quantity r is the radial distance measured along the meridian line from the center of the shell. We see that these two equations are the same as the corresponding von Kármán equations for the circular plate under symmetrical loading [3], with the exception of the terms proportional to $1/R$; this is evident if we make R infinite in (12.34a, b).

A summary of the whole paper was given at the end of the first section (Part I).

APPENDICES

(iii) TABLE III.—Table of the equations of equilibrium and compatibility of thin shell problems.

Types	b	q	p	(6.34)									(6.35)						(6.44)				(6.43)	
				I_1^0	I_2^0	I_3^0	I_4^0	I_5^0	I_6^0	I_7^0	I_8^0	I_9^0	I_1^α	I_2^α	I_3^α	I_4^α	I_5^α	I_6^α	J_1^0	J_2^0	J_3^0	J_4^0	$J_{\alpha 1}$	$J_{\alpha 2}$
*SS*1	≧1	0	1	x			x	x					x		x	x				x			x	
*SS*2	≧1	0	2	x	x	x	x	x					x	x	x	x				x			x	
*SS*3*	≧1	0	>2		x	x	x	x						x	x	x				x			x	
*SS*4	≧2	$1 \leqq q < b$	1	x			x	x					x		x				x				x	
*SS*5	≧2	1	2	x	x		x	x					x		x				x	x			x	
*SS*6*	≧2	$1 \leqq q < b$	$2q+1$		x		x						x		x					x			x	
*SS*7*	≧2	$1 \leqq q < b$	$2q+2$		x		x						x	x	x	x				x			x	
*SS*8*	≧2	$1 \leqq q < b$	$>2q+2$		x		x							x	x	x				x			x	
*SS*9	≧3	$2 \leqq q < b$	2	x	x		x	x					x		x				x				x	
*SS*10	≧3	$2 \leqq q < b$	$2 < p < 2q$																					
*SS*10	≧2	$\geqq b$	$2+q-b < q < q+b$		x		x						x		x				x				x	
*SS*11	≧3	$2 \leqq q < b$	$2q$		x		x						x		x				x	x			x	
*SS*12	1	1	2	x	x		x	x	x			x	x		x				x	x		x	x	
*SS*13	≧1	b	1	x			x	x	x			x	x		x				x				x	
*SS*14	≧2	b	2	x	x		x	x	x			x	x		x				x				x	
*SS*15	≧2	b	$2b$		x		x						x		x				x	x		x	x	
*SS*16*	≧1	b	$2b+1$		x		x						x		x					x		x	x	
*SS*17*	≧1	b	$2b+2$		x		x						x	x	x	x	x	x		x		x	x	
*SS*18*	≧1	b	$>2b+2$		x		x							x	x	x	x	x		x		x	x	
*SS*19*	≧1	$>b$	$<q-b$				x		x			x	x		x				x					x
*SS*20*	≧1	$>b$	$q-b$				x		x			x	x		x				x				x	x
*SS*21*	≧1	$>b$	$q-b+1$				x		x			x	x		x				x				x	
*SS*22	≧2	$>b$	$q-b+2$		x		x		x			x	x		x				x				x	
*SS*23	≧2	$>b$	$q+b$		x		x						x		x				x			x	x	
*SS*24*	≧1	$>b$	$q+b+1$		x		x						x		x							x	x	
*SS*25*	≧1	$>b$	$q+b+2$		x		x						x		x		x	x				x	x	
*SS*26*	≧1	$>b$	$>q+b+2$		x		x								x		x	x				x	x	
*SS*27	1	$>b$	$q+1$		x		x		x			x	x		x				x			x	x	
*SF*1	0	0	1	x			x	x	x			x	x		x	x		x		x		x	x	
*SF*2	0	0	2	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x		x		x	x	
*SF*3*	0	0	>2		x	x	x	x		x	x	x		x	x	x	x	x		x		x	x	
*SF*4	0	≧1	q				x		x			x	x		x			x	x		x	x	x	x
*SF*5	0	≧1	$q+1$				x		x			x	x		x			x				x	x	
*SF*6	0	≧1	$q+2$		x		x		x	x		x	x		x		x	x				x	x	
*SF*7*	0	≧1	$>q+2$		x		x			x		x			x		x	x				x	x	
*SF*8	0	≧2	$<q$				x		x			x	x		x			x	x		x		x	

In this table, the following notation is used:

The terms occurring in the first equation of equilibrium (6.34) are

$$I_1^0 = -2A_{(1)}^{\rho\gamma\pi\lambda} q_{\rho\gamma} p_{\pi\lambda} h, \qquad I_2^0 = \tfrac{2}{3}A_{(1)}^{\rho\gamma\pi\lambda}(q_{\pi\lambda}h^3)_{|\rho\gamma}, \qquad I_3^0 = A_{(3)}^{\rho\gamma\pi\omega\lambda\delta} q_{\pi\omega} q_{\rho\gamma} q_{\lambda\delta} h^3,$$

$$I_4^0 = P^0 + 2X_{[0]}^0 h + (Q^\lambda h)_{\underset{a}{|}\lambda}, \qquad I_5^0 = \frac{1-2\sigma}{1-\sigma} a_{\pi\lambda} q^{\pi\lambda} Q^0 h, \qquad I_6^0 = -A_{(1)}^{\rho\gamma\pi\lambda} b_{\rho\gamma} p_{\pi\lambda} h,$$

$$I_7^0 = -\tfrac{1}{2} A_{(4)}^{\rho\gamma\pi\omega\lambda\delta} b_{\rho\gamma} b_{\lambda\delta} q_{\pi\omega} h^3, \qquad I_8^0 = A_{(6)}^{\rho\gamma\pi\omega\lambda\delta} q_{\pi\omega} b_{\rho\gamma} q_{\lambda\delta} h^3, \qquad I_9^0 = \frac{1-2\sigma}{1-\sigma} 2HQ^0 h.$$

The terms occurring in the second and third equations of equilibrium (6.35) are

$$I_1^\alpha = 2A_{(1)}^{\rho\alpha\pi\lambda}(p_{\pi\lambda}h)_{\underset{a}{|}\rho}, \qquad I_2^\alpha = \tfrac{2}{3} a^{\alpha\tau} q_{\tau\gamma} A_{(1)}^{\gamma\rho\lambda\delta}(q_{\lambda\delta}h^3)_{\underset{a}{|}\rho} - A_{(3)}^{\rho\alpha\pi\omega\delta\lambda}(q_{\pi\omega}q_{\delta\lambda}h^3)_{\underset{a}{|}\rho},$$

$$I_3^\alpha = P^\alpha + 2X_{[0]}^\alpha h + \frac{\sigma}{1-\sigma} a^{\alpha\rho}(Q^0 h)_{\underset{a}{|}\rho}, \qquad I_4^\alpha = (a^{\pi\lambda} q_{\pi\lambda} a_\gamma^\alpha + 2a^{\alpha\pi} q_{\pi\gamma}) Q^\gamma h,$$

$$I_5^\alpha = A_{(4)}^{\rho\alpha\pi\omega\lambda\delta}(b_{\lambda\delta} q_{\pi\omega} h^3)_{\underset{a}{|}\rho} + \tfrac{1}{3} A_{(1)}^{\gamma\rho\lambda\delta} a^{\alpha\pi} b_{\pi\gamma}(q_{\lambda\delta}h^3)_{\underset{a}{|}\rho}, \qquad I_6^\alpha = (2Ha_\gamma^\alpha + b_\gamma^\alpha) Q^\gamma h.$$

The terms occurring in the first equation of compatibility (6.44) are

$$J_1^0 = 2\overset{\rho\alpha}{n}_{[0]}\overset{\beta\gamma}{n}_{[0]} p_{\rho\gamma\underset{a}{|}\alpha\beta}, \qquad J_2^0 = \overset{\rho\alpha}{n}_{[0]}\overset{\beta\gamma}{n}_{[0]} q_{\rho\gamma} q_{\alpha\beta},$$

$$J_3^0 = 2a^{\alpha\beta} p_{\alpha\beta} K, \qquad J_4^0 = -(4Ha^{\alpha\beta} - b^{\alpha\beta}) q_{\alpha\beta}.$$

The terms occurring in the second and third equations of compatibility (6.43) are

$$J_{\alpha 1} = 2\overset{\beta\gamma}{n}_{[0]} q_{\alpha\beta\underset{a}{|}\gamma}, \qquad J_{\alpha 2} = \overset{\beta\gamma}{n}_{[0]} b_{\beta\pi} a^{\pi\lambda}(p_{\alpha\lambda\underset{a}{|}\gamma} + p_{\gamma\lambda\underset{a}{|}\alpha} - p_{\alpha\gamma\underset{a}{|}\lambda}).$$

On account of the conditions which hold in the various types of problems, some of these terms may be negligible in comparison with others. Table III shows by the symbol 'x' those terms which are to be retained in the first approximation for the various types. (The over-determined problems are denoted by *.) Thus for example, for problems of type $SS1$, we have the following equations of equilibrium and compatibility in the first approximation:

$$I_1^0 + I_4^0 + I_5^0 = 0, \qquad I_1^\alpha + I_3^\alpha + I_4^\alpha = 0, \qquad J_2^0 = 0, \qquad J_{\alpha 1} = 0.$$

These equations are written in terms of the small principal parts instead of in terms of the finite coefficients of the lowest power in ϵ.

(iv) In Table IV, the following notation is used:

The terms occurring in the expresion (6.29) for the membrane stress tensor $T^{\alpha\beta}$ are denoted by

$$T_1^{\alpha\beta} = 2A_{(1)}^{\alpha\beta\pi\lambda} p_{\pi\lambda} h, \qquad T_2^{\alpha\beta} = -A_{(3)}^{\alpha\beta\pi\omega\lambda\delta} q_{\pi\omega} q_{\lambda\delta} h^3,$$

$$T_3^{\alpha\beta} = \frac{\sigma}{1-\sigma} a^{\alpha\beta} Q^0 h, \qquad T_4^{\alpha\beta} = A_{(4)}^{\alpha\beta\pi\omega\lambda\delta} b_{\lambda\delta} q_{\pi\omega} h^3.$$

The terms occurring in the expression (6.30) for the bending moment tensor $L^{\alpha\beta}$ are denoted by

$$L_1^{\alpha\beta} = \tfrac{2}{3}\overset{\rho\beta}{n}_{[0]} a_{\pi\rho} A_{(1)}^{\alpha\pi\lambda\delta} q_{\lambda\delta} h^3,$$

$$L_2^{\alpha\beta} = 2\overset{\rho\beta}{n}_{[0]} a_{\pi\rho} A_{(5)}^{\alpha\pi\lambda\delta\rho\gamma} b_{\lambda\delta} p_{\rho\gamma} h^3$$

$$+ \frac{\sigma}{6(1-\sigma)} \overset{\lambda\beta}{n}_{[0]} \left\{ a_\lambda^\alpha \left(\frac{4\sigma}{1-\sigma} HQ^0 - 4X_{[0]}^0 - 2Q_{\underset{a}{|}\gamma}^\gamma \right) - b_\lambda^\alpha Q^0 \right\} h^3.$$

TABLE IV.—Table of the external force system and the macroscopic tensors for various types of thin shell problems.

Types	n_0	n	j_0	j	k_0	k	$T^{\alpha\beta}$ t	$T_1^{\alpha\beta}$	$T_2^{\alpha\beta}$	$T_3^{\alpha\beta}$	$T^{\alpha\beta}$	$L^{\alpha\beta}$ u	$L_1^{\alpha\beta}$	$L_2^{\alpha\beta}$	$T^{\alpha 0}$ l	T_1^α	T_2^α	T_3^α
*SS*1	2	2	1	1	1	1	2	x		x		3	x		2	x		
	2	2	1	1	1	2	2	x		x		3	x		3	x	x	
*SS*2	3	3	2	2	2	2	3	x	x	x		3	x		3	x	x	
*SS*3	3	3	2	2	2	2	3		x	x		3	x		3	x	x	
*SS*4	$q+2$	2	$q+1$	1	1	$q+1$	2	x		x		$q+3$	x		$q+2$	x		
	$q+2$	2	$q+1$	1	1	$q+2$	2	x		x		$q+3$	x		$q+3$	x	x	
*SS*5	4	3	3	2	2	3	3	x		x		4	x		4	x	x	
*SS*6	$q+3$	$2q+2$	$q+2$	$2q+1$	$2q+1$	$q+2$	$2q+2$	x		x		$q+3$	x		$q+3$	x	x	
*SS*7	$q+3$	$2q+3$	$2q+2$	$q+2$	$q+2$	$2q+2$	$2q+3$	x	x	x		$q+3$	x		$q+3$	x	x	
*SS*8	$q+3$	$2q+3$	$2q+2$	$q+2$	$q+2$	$2q+2$	$2q+3$		x	x		$q+3$	x		$q+3$	x	x	
*SS*9	$q+3$	3	$q+2$	2	2	$q+2$	3	x		x		$q+3$	x		$q+3$	x	x	
*SS*10	$q+3$	$p+1$	$q+2$	p	p	$q+2$	$p+1$	x		x		$q+3$	x		$q+3$	x	x	
*SS*11	$q+3$	$2q+1$	$q+2$	$2q$	$2q$	$q+2$	$2q+1$	x		x		$q+3$	x		$q+3$	x	x	
*SS*12	4	3	3	2	2	3	3	x		x		4	x		4	x	x	
*SS*13	$b+2$	2	$b+1$	1	1	$b+1$	2	x		x		$b+3$	x		$b+2$	x		
	$b+2$	2	$b+1$	1	1	$b+2$	2	x		x		$b+3$	x		$b+3$	x	x	
*SS*14	$b+3$	3	$b+2$	2	2	$b+2$	3	x		x		$b+3$	x		$b+3$	x	x	
*SS*15	$b+3$	$2b+1$	$b+2$	$2b$	$2b$	$b+2$	$2b+1$	x		x		$b+3$	x		$b+3$	x	x	
*SS*16	$b+3$	$2b+2$	$b+2$	$2b+1$	$2b+1$	$b+2$	$2b+2$	x		x		$b+3$	x		$b+3$	x	x	
*SS*17	$b+3$	$2b+3$	$b+2$	$2b+2$	$2b+2$	$b+2$	$2b+3$	x	x	x	x	$b+3$	x		$b+3$	x	x	
*SS*18	$b+3$	$2b+3$	$b+2$	$2b+2$	$2b+2$	$b+2$	$2b+3$		x	x	x	$b+3$	x		$b+3$	x	x	
	$b+p+1$	$p+1$	$b+p$	p	p	$b+p$	$p+1$	x		x		$b+p+3$		x	$b+p+1$	x		
*SS*19	$b+p+1$	$p+1$	$b+p$	p	p	$b+p+1$	$p+1$	x		x		$b+p+3$		x	$b+p+2$	x		
	$b+p+1$	$p+1$	$b+p$	p	p	$b+p+2$	$p+1$	x		x		$b+p+3$		x	$b+p+3$	x		x
	$b+p+1$	$p+1$	$b+p$	p	p	$b+p$	$p+1$	x		x		$b+p+3$	x	x	$b+p+1$	x		
*SS*20	$b+p+1$	$p+1$	$b+p$	p	p	$b+p+1$	$p+1$	x		x		$b+p+3$	x	x	$b+p+2$	x		
	$b+p+1$	$p+1$	$b+p$	p	p	$b+p+2$	$p+1$	x		x		$b+p+3$	x	x	$b+p+3$	x	x	x
*SS*21	$b+p+1$	$p+1$	$b+p$	p	p	$b+p$	$p+1$	x		x		$b+p+2$	x		$b+p+1$	x		
	$b+p+1$	$p+1$	$b+p$	p	p	$b+p+1$	$p+1$	x		x		$b+p+2$	x		$b+p+2$	x	x	
*SS*22	$q+2$	$q-b+3$	$q+2$	$q-b+2$	$q-b+2$	$q+2$	$q-b+3$	x		x		$q+3$	x		$q+3$	x	x	
*SS*23	$q+3$	$q+b+1$	$q+2$	$q+b$	$q+b$	$q+2$	$q+b+1$	x		x		$q+3$	x		$q+3$	x	x	
*SS*24	$q+3$	$q+b+2$	$q+2$	$q+b+1$	$q+b+1$	$q+2$	$q+b+2$	x		x		$q+3$	x		$q+3$	x	x	
*SS*25	$q+3$	$q+b+3$	$q+2$	$q+b+2$	$q+b+2$	$q+2$	$q+b+3$	x		x	x	$q+3$	x		$q+3$	x	x	
*SS*26	$q+3$	$q+b+3$	$q+2$	$q+b+2$	$q+b+2$	$q+2$	$q+b+3$			x	x	$q+3$	x		$q+3$	x	x	
*SS*27	$q+3$	$q+2$	$q+2$	$q+1$	$q+1$	$q+2$	$q+2$	x		x		$q+3$	x		$q+3$	x	x	
*SF*1	2	2	1	1	1	1	2	x		x		3	x		2	x		
	2	2	1	1	1	2	2	x		x		3	x		3	x	x	
*SF*2	3	3	2	2	2	2	3	x	x	x	x	3	x		3	x	x	
*SF*3	3	3	2	2	2	2	3		x	x	x	3	x		3	x	x	
	$q+1$	$q+1$	q	q	q	q	$q+1$	x		x		$q+3$	x	x	$q+1$	x		
*SF*4	$q+1$	$q+1$	q	q	q	$q+1$	$q+1$	x		x		$q+3$	x	x	$q+2$	x		
	$q+1$	$q+1$	q	q	q	$q+2$	$q+1$	x		x		$q+3$	x	x	$q+3$	x	x	x
*SF*5	$q+2$	$q+2$	$q+1$	$q+1$	$q+1$	$q+1$	$q+2$	x		x		$q+3$	x		$q+2$	x		
	$q+2$	$q+2$	$q+1$	$q+1$	$q+1$	$q+2$	$q+2$	x		x		$q+3$	x		$q+3$	x	x	
*SF*6	$q+3$	$q+3$	$q+2$	$q+2$	$q+2$	$q+2$	$q+3$	x		x	x	$q+3$	x		$q+3$	x	x	
*SF*7	$q+3$	$q+3$	$q+2$	$q+2$	$q+2$	$q+2$	$q+3$			x	x	$q+3$	x		$q+3$	x	x	
	$p+1$	$p+1$	p	p	p	p	$p+1$	x		x		$p+3$		x	$p+1$	x		
*SF*8	$p+1$	$p+1$	p	p	p	$p+1$	$p+1$	x		x		$p+3$		x	$p+2$	x		
	$p+1$	$p+1$	p	p	p	$p+2$	$p+1$	x		x		$p+3$		x	$p+3$	x		x

The terms occurring in the expression (6.31) for the shearing stress tensor $T^{\alpha 0}$ are denoted by

$$T_1^\alpha = Q^\alpha h, \qquad T_2^\alpha = \tfrac{2}{3}A_{(1)}^{\pi\alpha\lambda\delta}(q_{\lambda\delta}h^3)_{\underset{a}{|}\pi},$$

$$T_3^\alpha = 2A_{(5)}^{\tau\alpha\lambda\delta\rho\gamma}(b_{\lambda\delta}p_{\rho\gamma}h^3)_{\underset{a}{|\tau}} + \tfrac{1}{2}(4HP^\alpha + b_\tau^\alpha P^\tau)h^2 + \tfrac{4}{3}HX_{[0]}^\alpha h^3$$
$$+ \frac{\sigma}{6(1-\sigma)}\left\{\left[a^{\alpha\tau}\left(\frac{2\sigma}{1-\sigma}HQ^0 - 4X_{[0]}^0\right) - b^{\alpha\tau}Q^0\right]h^3\right\}_{\underset{a}{|\tau}}.$$

Furthermore,

n_0 = order of sum of the normal forces acting on the upper and lower boundary surfaces, or order of P^0,

n = order of sum of the tangential forces acting on the upper and lower boundary surfaces, or order or P^α,

j_0 = order of normal component of body force, or order of $X_{[0]}^0$,

j = order of tangential component of body force, or order of $X_{[0]}^\alpha$,

k_0 = order of difference of normal forces acting on the upper and lower surfaces, or order of Q^0,

k = order of difference of tangential components of forces acting on the upper and lower boundary surfaces, or order of Q^α,

t = order of membrane stress tensor $T^{\alpha\beta}$,

u = order of bending moment tensor $L^{\alpha\beta}$,

l = order of shearing stress tensor $T^{\alpha 0}$.

This table gives (a) the values of n_0, n, j_0, j, k_0, k, t, u, l, (b) the principal terms in the expressions for $T^{\alpha\beta}$, $L^{\alpha\beta}$, $T^{\alpha 0}$ (denoted by 'x'). The terms not marked with 'x' are negligible in comparison those principal terms. It will be noted that there are two lines in the table for $SS1$, $SS4$, $SS13$, $SS21$, $SF1$, $SF5$, and three lines for $SS19$, $SS20$, $SF4$, $SF8$. This is because, in each case, k may have two or three values.

For example, in the case of Type $SS1$, we have for $T^{\alpha\beta}$, $L^{\alpha\beta}$,

$$T^{\alpha\beta} = T_1^{\alpha\beta} + T_3^{\alpha\beta}, \qquad L^{\alpha\beta} = L_1^{\alpha\beta},$$

while for $T^{\alpha 0}$,

$$T^{\alpha 0} = T_1^\alpha \qquad (\text{if } k = 1),$$
$$T^{\alpha 0} = T_1^\alpha + T_2^\alpha \qquad (\text{if } k = 2).$$

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