THE AERODYNAMICS OF A RING AIRFOIL*

BY

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Abstract. The downwash required to produce a given vorticity distribution is computed for a ring airfoil and the results are compared with the corresponding two-dimensional case. From this it appears that if the curvature of the chord plane is small, as is the case with normal amounts of dihedral, the effect of this curvature on the chordwise lift distribution of a wing is extremely small. If the radius of curvature is small compared to the chord, as it is near the vertex of a cranked wing, it is seen that this curvature may cause comparatively large changes in the lift distribution.

1. Introduction. At the present time, the steady state two-dimensional airfoil theory is a highly developed subject; and, subject to the usual limitations of perfect fluid theory, solutions may be obtained with almost any desired degree of accuracy. The steady state three-dimensional airfoil theory is, however, in a much lower state of development. For most engineering problems the "lifting line" theory as developed by Prandtl and others is adequate to provide satisfactory results; however, for certain other problems, such as the flow near a wing tip, the effects of sweepback or of yaw, or the lift of a low aspect ratio wing, the lifting line theory cannot be used. At this time there have been only a fairly small number of solutions of finite wing problems in which lifting surfaces rather than lifting lines have been used and which may thus be used to throw light on these essentially more complicated problems. The best known of these lifting surface theories are those due to Blenk,[1] Kinner,[2] Krienes,[3] and Bollay.[4] As the number of such solutions is so limited almost any special solution involving a lifting surface is of interest.

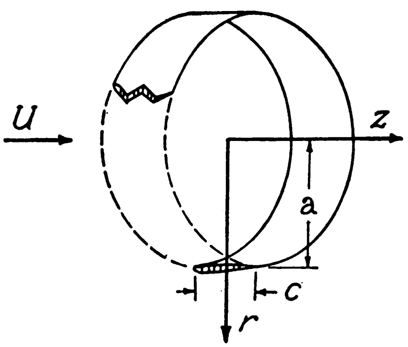

FIG. 1. Ring wing in a uniform flow.

From an analytical viewpoint, probably the simplest lifting surface problem which has not yet been investigated is that of the axially symmetric flow past a ring airfoil as shown in Fig. 1. This flow is especially simple as the vortex lines in the lifting surface are circular rings and there are thus no trailing vortices. The particular purpose of the present paper is to discuss the differences between this problem and the corresponding two dimensional problem. In addition to its intrinsic interest in the theory of the "anti-drag" cowl, the ring airfoil problem possesses a general interest insofar as it demonstrates, at least qualitatively, some of the effects of dihedral on the lift distribution of a wing.

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[1] Blenk, H., Zeit. f. angew. Math. u. Mechanik, **5**, 36 (1925).

[2] Kinner, W., Ingenieur Archiv, **8**, 47 (1937).

[3] Krienes, K., Zeit. f. angew. Math. u. Mechanik, **20**, 65 (1940).

[4] Bollay, W., Zeit. f. angew. Math. u. Mechanik, **19**, 21 (1939); also J. Aero. Sci., **4**, 294 (1937).

2. The vector potential. The mathematical analysis of this problem may be conveniently carried out by the method of the vector potential. Since the equation of continuity in an incompressible fluid is simply

$$\operatorname{div} \boldsymbol{q} = 0, \tag{1}$$

the velocity vector $\boldsymbol{q}$ may be written as the curl of a vector potential $\boldsymbol{A}$ or

$$\boldsymbol{q} = \operatorname{curl} \boldsymbol{A}. \tag{2}$$

By the Helmholtz decomposition theorem the vector potential may be subjected to the restriction that

$$\operatorname{div} \boldsymbol{A} = 0. \tag{3}$$

The differential equation for the determination of the vector potential is found by curling Eq. (2). This gives

$$\nabla^2 \boldsymbol{A} = -\operatorname{curl} \boldsymbol{q} = -\boldsymbol{\Omega}. \tag{4}$$

If the vorticity $\boldsymbol{\Omega}$ is a given function, this is a Poisson equation for the determination of the vector potential. The solution of this equation, which is well-known and may be obtained by the use of Green's theorem, is

$$\boldsymbol{A} = \frac{1}{4\pi} \int \boldsymbol{\Omega} \frac{dv}{r_1} \tag{5}$$

where the volume integral covers the entire region where the vorticity exists and r_1 is the distance from the point at which the vorticity exists to the point P at which the vector potential is being computed. If the vorticity is in the form of a single vortex filament of strength Γ then

$$\boldsymbol{A} = \frac{\Gamma}{4\pi} \int \frac{1}{r_1} d\boldsymbol{s}, \tag{6}$$

where $d\boldsymbol{s}$ is an infinitesimal distance vector along the vortex filament. If there are several vortex filaments the contribution from each one may be found by Eq. (6), and these results must then be summed to obtain the vector potential.

3. The vector potential for a vortex ring. As the vortex filaments are all circles for the axially symmetric flow past a ring wing, the complete vector potential can easily be obtained if the vector potential of a single filament is known. For such a filament of strength Γ and lying in the plane $z=0$ (see Fig. 2), it is obvious that the vector potential is not a function of the meridian angle θ, and it may be calculated at points in the plane $\theta=0$. Since $d\boldsymbol{s}$ is in the plane $z=0$, the vector potential can have no z-component. Furthermore, by considering two vortex elements, one having the negative of the other's θ coordinate, it is evident that the vector potential can have no radial component. The vector potential has thus only the component A_θ which is perpendicular to the meridian planes. By Eq. (6), this is

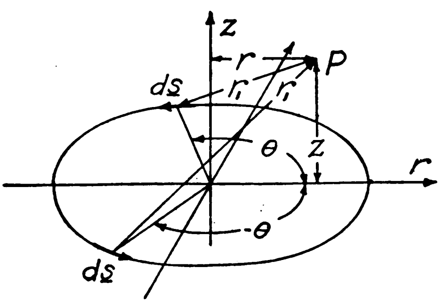

FIG. 2. Vortex ring.

$$A_\theta = \frac{a\Gamma}{2\pi}\int_0^\pi \frac{\cos\theta\, d\theta}{[a^2 + r^2 + z^2 - 2ar\cos\theta]^{1/2}}. \tag{7}$$

With the vector potential expressed in this form as an elliptic integral, it is rather difficult to superimpose the vector potentials for a band of vorticity of radius a and of chord c in order to represent the ring wing. A much more convenient form can be obtained by the use of the Fourier integral. For an even function $f(z)$, the Fourier integral theorem states that

$$f(z) = \frac{2}{\pi}\int_0^\infty \cos kz\left\{\int_0^\infty f(t)\cos kt\, dt\right\}dk. \tag{8}$$

Since A_θ is an even function of z, it follows that

$$A_\theta = \frac{a\Gamma}{\pi^2}\int_0^\infty \cos kz\left[\int_0^\infty \cos kt\left\{\int_0^\pi \frac{\cos\theta\, d\theta}{[a^2 + r^2 - 2ar\cos\theta + t^2]^{1/2}}\right\}dt\right]dk. \tag{9}$$

Since[5]

$$\int_0^\infty \frac{\cos(kt)}{[x^2 + t^2]^{1/2}}\,dt = K_0(kx), \tag{10}$$

the inner two integrals of Eq. (9) become, after inversion of the order of integration,

$$I = \int_0^\pi K_0\left[k\sqrt{a^2 + r^2 - 2ar\cos\theta}\right]\cos\theta\, d\theta. \tag{11}$$

The addition theorem for the modified Bessel functions of the second kind (see Ref. 5, p. 74) states that

$$K_0\left[k\sqrt{a^2 + r^2 - 2ar\cos\theta}\right] = \begin{cases} I_0(ka)K_0(kr) + 2\sum_{n=1}^{\infty} I_n(ka)K_n(kr)\cos n\theta \text{ if } r > a. \\ I_0(kr)K_0(ka) + 2\sum_{n=1}^{\infty} I_n(kr)K_n(ka)\cos n\theta \text{ if } r < a. \end{cases} \tag{12}$$

Since the trigonometrical functions are orthogonal over the range $0 \leq \theta \leq \pi$,

$$I = \begin{cases} \pi I_1(ka)K_1(kr) & \text{if } \quad r > a. \\ \pi I_1(kr)K_1(ka) & \text{if } \quad r < a. \end{cases} \tag{13}$$

The vector potential for the vortex ring in the outer range where $r > a$ can thus be written as

$$A_\theta = \frac{a\Gamma}{\pi}\int_0^\infty I_1(ka)K_1(kr)\cos(kz)dk \qquad (r > a). \tag{14}$$

For the inner range it is necessary to interchange the arguments of the two Bessel functions.

4. The vector potential for a ring airfoil. A ring airfoil may be considered to be a system of ring vortices of radius a and distributed over the chord c from $z = -c/2$ to $z = c/2$. If the strength of this vortex sheet is $\gamma(z_0)$, then the vector potential for $r > a$ is

[5] Grey, Mathews and MacRobert, *Bessel functions*, Macmillan and Co., London, 1931, p. 52.

$$A_\theta = \frac{a}{\pi}\int_{-c/2}^{c/2}\gamma(z_0)\left\{\int_0^\infty I_1(ka)K_1(kr)\cos k(z-z_0)dk\right\}dz_0. \tag{15}$$

If the vortex strength is known, Eq. (15), after inversion of the order of integration, can conveniently be used to compute the vector potential or the radial or axial velocity components, u_r and u_z respectively. From Eq. (2),

$$u_r = -\frac{\partial A_\theta}{\partial z}, \qquad u_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta). \tag{16}$$

The radial velocity is of the most interest as it corresponds to the downwash velocity in the ordinary two-dimensional airfoil theory. The downwash at the ring airfoil, $r=a$, is

$$u_r = \frac{a}{\pi}\int_0^\infty kI_1(ka)K_1(ka)\left\{\int_{-c/2}^{c/2}\gamma(z_0)\sin k(z-z_0)dz_0\right\}dk. \tag{17}$$

5. Comparison with two-dimensional flat plate airfoil. If the airfoil shape is given, the downwash u_r is known, and Eq. (17) may be considered as an integral equation for the determination of the vortex strength $\gamma(z_0)$. It is, however, an integral equation of a difficult type. The importance of the curvature of the chord plane may be estimated by comparing the downwash for some given vortex distribution with the corresponding two dimensional result. This process will be carried out for the vortex distribution

$$\gamma(z_0) = A\sqrt{\frac{c-2z_0}{c+2z_0}}\,. \tag{18}$$

In the two-dimensional case, this vorticity distribution corresponds to a flat plate airfoil with its leading edge at $z_0=-c/2$. The downwash is then constant over the airfoil and equal to $A/2$. For this vorticity distribution it can easily be seen by use of the transformation $2z_0=c\sin\theta$ that

$$\int_{-c/2}^{c/2}\gamma(z_0)\sin k(z-z_0)dz_0 = \frac{\pi}{2}Ac\left[J_0(\tfrac{1}{2}kc)\sin kz + J_1(\tfrac{1}{2}kc)\cos kz\right]. \tag{19}$$

The downwash velocity is thus

$$u_r = \tfrac{1}{2}Aca\int_0^\infty kI_1(ka)K_1(ka)\left[J_0(\tfrac{1}{2}kc)\sin kz + J_1(\tfrac{1}{2}kc)\cos kz\right]dk. \tag{20}$$

It is of interest to note that the two-dimensional result can be obtained directly from this by considering the limiting form as the radius of the ring becomes infinitely large; for

$$\lim_{x\to\infty}\left\{xI_1(x)K_1(x)\right\} = \tfrac{1}{2}; \tag{21}$$

so the downwash in the two-dimensional case is given by

$$u_r = \tfrac{1}{4}Ac\int_0^\infty\left[J_0(\tfrac{1}{2}kc)\sin(kz) + J_1(\tfrac{1}{2}kc)\cos(kz)\right]dk. \tag{22}$$

TABLE I
Comparison of $F(x)$ and $F_1(x)$

x	$F(x)$	$F_1(x)$	$F(x)-F_1(x)$
0.1	0.0493	0.0132	0.0361
0.5	0.2132	0.2000	0.0132
1	0.3402	0.3637	−0.0235
2	0.4450	0.4571	−0.0121
3	0.4762	0.4800	−0.0038
4	0.4873	0.4885	−0.0012
5	0.4921	0.4926	−0.0005

The first integral vanishes on the airfoil where $z^2 \leqq c^2/4$ and the second is equal to $2/c$ on the airfoil (see Ref. 5, p. 65); so in the two-dimensional case, for this vorticity distribution

$$u_r = \tfrac{1}{2}A \qquad (z^2 \leqq c^2/4). \tag{23}$$

An exact evaluation of the integral of Eq. (20) is rather difficult; however, an approximate evaluation, valid for large values of a/c, may be obtained quite easily. If

$$F(x) = xI_1(x)K_1(x), \tag{24}$$

a very close approximation to $F(x)$ is given by

$$F_1(x) = \frac{x^2/2}{3/8 + x^2}. \tag{25}$$

It may be noted that the asymptotic expansions for $F(x)$ and $F_1(x)$ are the same up through terms of order (x^{-2}). It is shown in Table I and Fig. 3 that $F_1(x)$ is a good approximation to $F(x)$ even for small values of x.

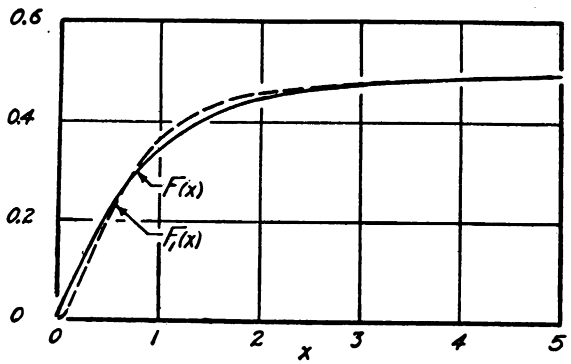

FIG. 3. Comparison of $F(x)$ and $F_1(x)$.

Since

$$F_1(x) - \tfrac{1}{2} = -\frac{3/16}{3/8 + x^2}, \tag{26}$$

an approximate expression for Δu, the difference between the ring airfoil downwash of Eq. (20) and the corresponding two-dimensional case is given by

$$\Delta u = -(3/32)Ac\int_0^\infty \left[J_0(\tfrac{1}{2}kc)\sin kz + J_1(\tfrac{1}{2}kc)\cos kz\right]\frac{dk}{3/8 + a^2k^2}\cdot \qquad (27)$$

Let $\lambda = kc/2$, $\alpha = \sqrt{3/32}\,c/a$ and $\beta = 2z/c$. Then

$$\Delta u = -\tfrac{1}{2}A\alpha^2\int_0^\infty \left[J_0(\lambda)\sin\beta\lambda + J_1(\lambda)\cos\beta\lambda\right]\frac{d\lambda}{\lambda^2+\alpha^2}\cdot \qquad (28)$$

On the airfoil where $\beta^2 \leq 1$, this gives (see Ref. 5, p. 78)

$$\Delta u = \tfrac{1}{2}A\left[\alpha\cosh(\alpha\beta)K_1(\alpha) - \alpha\sinh(\alpha\beta)K_0(\alpha) - 1\right]. \qquad (29)$$

The ratio of the change in downwash to the two dimensional downwash is $2(\Delta u)/A$. For $\alpha = 0.02$ and 0.20 corresponding to $a/c = 15.3$ and 1.53 respectively, this ratio is given in Table 2 for the leading edge ($\beta = -1$), the center of the airfoil ($\beta = 0$), and for the trailing edge ($\beta = 1$).

TABLE 2

Values of $2(\Delta u)/A$ for the ring airfoil

α	0.02	0.20
a/c	15.3	1.53
$\beta = -1$	0.0009	0 .0451
$\beta = 0$	−0.0009	−0.0448
$\beta = 1$	−0.0023	−0.0963

As the downwash velocity is determined by the slope of the camber line, the airfoil camber required to produce the lift distribution of Eq. (18) may be computed by integrating the downwash velocity. The camber lines for $a/c = 1.53$ and for the two dimensional case are shown for comparison in Fig. 4.

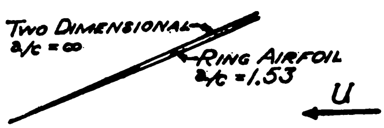

FIG. 4. Airfoil profiles having the same vorticity. See Eq. (18).

6. Conclusions. From Table 2, it is apparent that the effects of the curvature of the chord plane of the ring airfoil are negligibly small if $a/c = 15.3$ while they are fairly large for $a/c = 1.53$. From Fig. 4 it appears that the lift of a ring airfoil having a constant angle of attack across the chord would be somewhat more than that of the corresponding two dimensional airfoil and the lift is shifted away from the leading edge toward the center of the airfoil.

It seems reasonable to suppose that the changes at any given section of the ring wing are caused primarily by the vortex elements near that section. These results may thus be applied in estimating the effects of the dihedral of a wing on the lift distribution over the wing's surface. This indicates that if the curvature of the chord plane is small, as is normally the case, no appreciable changes in the lift distribution need be expected; however, if the radius of curvature of the chord plane is of the same order as the chord, fairly large effects may be expected. This should be particularly noticeable near the vertex of a cranked wing.