THE TREATMENT OF DISCONTINUITIES IN BEAM DEFLECTION PROBLEMS*

By C. L. BROWN (*Purdue University*)

The multitude of methods of determining the deflections of beams all stem from the fundamental differential equation

$$\frac{d^2y}{dx^2} = \frac{M}{EI}, \tag{1}$$

where the abscissa x is measured along the axis of the beam, and y denotes the deflection, M the bending moment, and EI the bending stiffness. The most obvious method of determining y is direct integration of (1). However, in most cases the right hand side of (1) is but sectionally analytical. A differential equation of the form (1) is then written for each section of the beam. When these equations are integrated, two constants of integration appear for each section. The evaluation of these constants of integration, though elementary, is extremely cumbersome.

It is possible to avoid this sectionalizing treatment through the use of Heaviside's unit step function, well known from operational calculus. This function is defined as follows:

$$H_a(x) = \begin{cases} 0 & \text{for} \quad x < a, \\ 1 & \text{for} \quad x > a. \end{cases} \tag{2}$$

It can readily be seen that

$$H_aH_b = \begin{cases} H_a & \text{if} \quad a > b, \\ H_b & \text{if} \quad a < b. \end{cases} \tag{3}$$

Furthermore, if $u(x)$ and $v(x)$ are analytic, the continuous solution of

$$\frac{dy}{dx} = u(x) + H_a v(x - a)$$

is given by

$$y = \int_0^x u(\xi)d\xi + H_a \int_0^{x-a} v(\xi)d\xi + C, \tag{4}$$

where C denotes a constant of integration.

The use of the unit step function in the analysis of beams with concentrated loads is best shown by an example. The bending moment of the beam in Fig. 1 can be written as

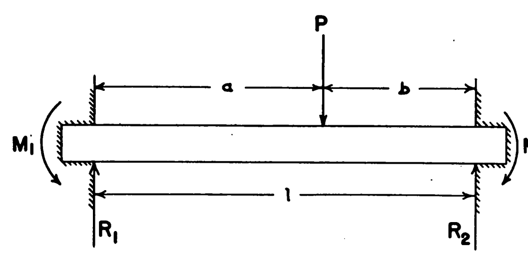

FIG. 1.

$$EI\frac{d^2y}{dx^2} = M = -M_1 + R_1x - H_aP(x - a).$$

* Received Aug. 26, 1943.

Integrating according to (4) and taking account of the fact that slope and deflection vanish for $x=0$, we find

$$EI\frac{dy}{dx} = -M_1x + \tfrac{1}{2}R_1x^2 - \tfrac{1}{2}H_aP(x-a)^2 \tag{5}$$

and

$$EIy = -\tfrac{1}{2}M_1x^2 + \tfrac{1}{6}R_1x^3 - \tfrac{1}{6}H_aP(x-a)^3. \tag{6}$$

Both slope and deflection are zero at $x=l$. Thus, from (5) and (6)

$$-M_1l + \tfrac{1}{2}R_1l^2 - \tfrac{1}{2}Pb^2 = 0, \qquad -\tfrac{1}{2}M_1l^2 + \tfrac{1}{6}R_1l^3 - \tfrac{1}{6}Pb^3 = 0.$$

Solving for M_1 and R_1,

$$M_1 = Pab^2/l^2, \qquad R_1 = Pb^2(3a+b)/l^3.$$

The deflection is then given by

$$EIy = -\frac{Pab^2}{2l^2}x^2 + \frac{Pb^2(3a+b)}{6l^3}x^3 - H_a\frac{P(x-a)^3}{6}.$$

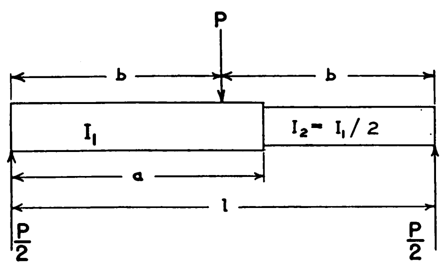

FIG. 2.

In order to illustrate the use of the unit step function in the analysis of beams with sectionally constant moment of inertia, we consider the beam shown in Fig. 2. The reciprocal of the bending stiffness can be written as

$$\frac{1}{EI} = \frac{1}{EI_1}[1 + H_a]$$

and the bending moment as

$$M = \frac{Px}{2} - PH_b(x-b).$$

Thus,

$$EI_1\frac{d^2y}{dx^2} = [1+H_a]\left[\frac{Px}{2} - PH_b(x-b)\right]$$

or, considering (3)

$$EI_1\frac{d^2v}{dx^2} = \frac{Px}{2} + \frac{P}{2}[H_a(x-a) + aH_a] - PH_b(x-b) - PH_a(x-b), \tag{7}$$

in which the second term on the right hand side has been written in this particular form in order to facilitate the application of (4). Integrating (7) we obtain

$$EI_1y = \frac{Px^3}{12} + \frac{P}{2}\left[H_a\frac{(x-a)^3}{6} + aH_a\frac{(x-a)^2}{2}\right]$$

$$- PH_b\frac{(x-b)^3}{6} - PH_a\frac{(x-b)^3}{6} + C_1x + C_2. \tag{8}$$

The deflection is zero at $x=0$ and $x=l$. The application of these conditions to (8) yields

$$C_1 = -\frac{P}{12b}(6b^3 - 3a^2b + a^3), \qquad C_2 = 0.$$

If, in addition to concentrated loads, the beam carries loads which are uniformly distributed over certain sections, the treatment is similar.

A VARIATIONAL PRINCIPLE FOR A STATE OF COMBINED PLASTIC STRESS*

By G. H. HANDELMAN† (*Brown University*)

In a recent paper[1] M. A. Sadowsky has stated a heuristic principle of maximum plastic resistance which he has applied to several states of combined plastic stress. The principle states that "among all statically possible stress distributions (satisfying all three equations of equilibrium, the condition of plasticity, and boundary conditions), the actual stress distribution in plastic flow requires a maximum value of the external effort necessary to maintain the flow." W. Prager, in a contribution to the discussion of this paper[2], has shown that the principle can be so interpreted as to lead to the correct differential equation for a beam under combined torsion and tension. This note is concerned with an accurate statement of the principle together with a proof of its validity for the case of a beam in a perfectly plastic state under combined torsion and bending by couples, the cross-section of the beam having an axis of symmetry. Specifically, we shall prove the following variational principle for such a system.

Among all statically possible stress distributions in a beam under a given torque (satisfying the equations of equilibrium, the condition of plasticity, and boundary conditions), the actual stress distribution when plastic flow occurs is the one for which the bending moment is stationary.

Let us choose the coordinate axes in the following fashion. y lies along the axis of symmetry of the cross-section, z passes through the center of gravity of the cross-section and is parallel to the generators of the cylindrical beam, and x is perpendicular to y and z. We assume that the strain velocities, v_x, v_y, v_z, are given by the same expressions as in the case of an incompressible elastic material; i.e.,

$$\begin{aligned} v_x &= -\omega yz + \tfrac{1}{2}\theta xy, \\ v_y &= \omega xz - \tfrac{1}{4}\theta(x^2 - y^2 - 2z^2), \\ v_z &= \omega w(x,y) - \theta yz. \end{aligned}$$

---

* Received Sept. 9, 1943.

† This note was prepared at the suggestion of Professor W. Prager while the author was a participant in the Program of Advanced Instruction and Research in Mechanics at Brown University and was presented to the American Mathematical Society on Sept. 12, 1943 under the title of *On a principle of M. A. Sadowsky.*

[1] M. A. Sadowsky, Journal of Applied Mechanics **10**, A-65 (1943).

[2] Journal of Applied Mechanics **10**, A-238 (1943).