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THE TRANSFORMATION OF PARTIAL DIFFERENTIAL EQUATIONS*

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1. Introduction. In the early stages of the use of partial differential equations for the solution of problems of mechanics and physics the separation of variables and construction of simple solutions was the primary aim. The introduction of the idea of an exact differential by Fontaine and Euler led to the idea of associated differential equations such as those for the velocity potential and stream function in hydrodynamics, the adjoint equations of Lagrange and Riemann, the contact transformations of Legendre and Ampère, the transformations of Euler and Laplace for the solution of differential equations by definite integrals and other transformations too numerous to mention. Another aim which led to the study of transformations was that of reducing an equation to a canonical form. Laplace's reduction of a linear partial differential equation of the second order to a form in which only one partial derivative of the second order occurs led to the study of transformations which preserve this form and of quantities which have a property of invariance. Conditions were then found that an equation may be reducible by means of a specified type of change of variables to some particular equations which had been fully studied. The conditions found by Campbell [1]† (constancy of his two invariants) that Laplace's canonical equation may be reducible to the equation of Euler and Poisson, may be cited as an example.

A classification of transformations may be made by including in group A all transformations which arise from the condition or conditions that a linear differential form may be of a specified type (for example an exact differential). Transformations arising from the study of a number of linear differential forms may be included in this group. Transformations associated with the Calculus of Variations are also included because the equations of Euler and Lagrange are closely associated with the conditions for an exact differential. The extension of Legendre's transformation found by Carathéodory [2] may be mentioned here. In this article attention will be devoted almost entirely to transformations of group A.

Transformations of group B include all those which arise from the conditions that a quadratic differential form may be of a specified type. The transformation of a

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† Numbers in square brackets refer to the list of references at the end of the paper.

linear differential equation to a form in which the variables are separated is thus a B-transformation. The transformations of group B are not necessarily point transformations, for instance, if $Q(a, b, c)$ is a non-negative quadratic form in the real variables a, b, c transformations from (x, y, z, t, u, v, w) to (X, Y, Z, T, U, V, W) may be considered in which $Q(dx-udt, dy-vdt, dz-wdt)$ goes over into $Q(dX-Udt, dY-Vdt, dZ-Wdt)$ the coefficients of Q in the first case being functions of x, y, z, t, u, v, w and in the second case functions of X, Y, Z, T, U, V, W. Since the equation $Q=0$ implies that $dx=udt$, $dy=vdt$, $dz=wdt$ it also implies that $dX=UdT$, $dY=VdT$, $dZ=WdT$. In other words if u, v, w can be regarded as the component velocities of a recognizable moving particle of fluid then U, V, W can be regarded as component velocities of a recognizable particle of a corresponding fluid. Such a transformation is of interest because the density of each fluid can be defined in such a way that the equation of continuity is invariant under the transformation.

Group C may be regarded as including all other transformations and some transformation of the other group which arise in the reduction of an equation to a canonical form.

2. Associated equations of the types of Monge and Legendre. In his work on partial differential equations of the second order in two variables x, y which can be regarded as independent, Monge [3] used z as dependent variable, p and q as the first derivatives z_x, z_y respectively and r, s, t as the second derivatives z_{xx}, z_{xy}, z_{yy}. As we shall have applications to fluid dynamics in mind, we shall deviate slightly from the notation of Monge and use u, v in place of p and q so that when z is the velocity potential u and v represent the component velocities as usual. This plan also allows us to use the symbol p to denote the pressure and q to denote the resultant velocity.

The equations of steady motion of a compressible fluid under no body forces when the flow is irrotational and the fluid barotropic (density a function of pressure only) can, as we know, be derived from a variational principle

$$\delta \iint p(u, v)dxdy = 0, \qquad u = z_x, v = z_y, \tag{1}$$

in which p is a specified function of q. For greater generality at the outset we shall suppose, however, that p is a specified function of u and v. Lagrange's partial differential equation for this variational problem is then derivable from Haar's condition [4] that $p_v dx - p_u dy$ should be an exact differential. When the differentiations are made, the equation has the form

$$p_{uu}r + 2p_{uv}s + p_{vv}t = 0. \tag{2}$$

When Legendre's transformation is applied to this differential equation the new dependent variable is

$$w = ux + vy - z \tag{3}$$

and since $dz = udx + vdy$,

$$dw = xdu + ydv. \tag{4}$$

When u and v can be regarded as independent this equation gives the relations

$$x = w_u, \qquad y = w_v, \tag{5}$$

and the equation for w is

$$p_{uu}w_{vv} - 2p_{uv}w_{uv} + p_{vv}w_{uu} = 0. \tag{6}$$

When, however, u and v are related so that they can be regarded as functions of a single variable τ the equation (4) indicates that w is then also a function of τ and we have the equations

$$w(\tau) = xu(\tau) + yv(\tau) - z, \qquad w'(\tau) = xu'(\tau) + yv'(\tau), \tag{7}$$

which furnish a solution of (2) if

$$p_{uu}u'^2(\tau) + 2p_{uv}u'(\tau)v'(\tau) + p_{vv}v'^2(\tau) = 0. \tag{8}$$

Let us now seek the conditions that 3 quantities

$$R = R(u, v), \qquad S = S(u, v), \qquad T = T(u, v), \tag{9}$$

may be the second derivatives Z_{xx}, Z_{xy}, Z_{zz} of a single function $Z(x, y)$. The required conditions $R_y = S_x$, $S_y = T_x$ may be written in the form

$$R_u s + R_v t = S_u r + S_v s, \qquad S_u s + S_v t = T_u r + T_v s. \tag{10}$$

We now seek the conditions that these two equations are both satisfied in virtue of equation (2). This will be the case when

$$\left.\begin{aligned} R_v &= wp_{vv}, & R_u &= wp_{uv} + h, & S_u &= -wp_{uu}, & S_v &= h - wp_{uv}, \\ T_u &= w'p_{uu}, & T_v &= w'p_{uv} + h', & S_u &= h' - w'p_{uv}, & S_v &= -w'p_{vv}. \end{aligned}\right\} \tag{11}$$

Equating the different expressions for R_{uv}, S_{uv}, T_{uv} we obtain the equations

$$\left.\begin{aligned} h_u &= w_u p_{uv} - w_v p_{uu}, & h_v &= w_u p_{vv} - w_v p_{uv}, \\ h_u' &= w_v' p_{uu} - w_u' p_{uv}, & h_v' &= w_v' p_{uv} - w_u' p_{vv}. \end{aligned}\right\} \tag{12}$$

The elimination of h and h' yields the two equations

$$\left.\begin{aligned} w_{uu}p_{vv} - 2w_{uv}p_{uv} + w_{vv}p_{uu} &= 0, \\ w_{uu}'p_{vv} - 2w_{uv}'p_{uv} + w_{vv}'p_{uu} &= 0 \end{aligned}\right\} \tag{13}$$

which show that w and w' are solutions of equation (6). The case of chief hydrodynamical interest is that in which the second derivatives of w and w' are all zero. We shall, however, look first at a possible alternative case.

Equating the different expressions for S_u and S_v we obtain the equations

$$\left.\begin{aligned} wp_{uu} &= w'p_{uv} + \int [w_u' d(p_v) - w_v' d(p_u)], \\ w'p_{vv} &= wp_{uv} + \int [w_u d(p_v) - w_v d(p_u)]. \end{aligned}\right\} \tag{14}$$

Hence

$$\left.\begin{aligned} w_v' p_{vv} + w'p_{uvv} &= 2w_v p_{uv} + wp_{uvv} - w_u p_{vv}, \\ 2w_u' p_{uv} - w_v' p_{uu} + w'p_{uuv} &= w_u p_{uu} + wp_{uuu}, \\ w_u' p_{vv} + w'p_{uvv} &= w_v p_{uu} + wp_{uuv}. \end{aligned}\right\} \tag{15}$$

These equations lead to the relation

$$w'D_v = wD_u, \quad \text{where} \quad D = p_{uu}p_{vv} - p_{uv}^2. \tag{16}$$

This equation is satisfied identically when D is constant but it may also be satisfied if $w=E_v$, $w'=E_u$ where D and E are related. An important case of this second type occurs when D is a function of q only and $w=v$, $w'=u$. In this case

$$h = -p_u, \quad h' = -p_v, \quad R = vp_v - p, \quad T = up_u - p, \quad S = -up_v = -vp_u, \tag{17}$$

and p is a function of q only. If $p_u = -u\rho$, $p_v = -v\rho$, where ρ is the density of the fluid we have

$$R = -p - \rho v^2, \qquad S = \rho uv, \qquad T = -p - \rho u^2, \tag{18}$$

and so Z is a kind of stress function which satisfies the equation

$$RT - S^2 = p(p + \rho u^2 + \rho v^2) = p(p + \rho q^2) = F(R + T) \tag{19}$$

since $R+T = -2p - \rho q^2$. The partial differential equation for Z is thus of Legendre's type [5]

$$\mathcal{H}(R, S, T) = 0. \tag{20}$$

In the special case in which $-F(R+T) = K^2 - \frac{1}{4}(R+T)^2$ the equation reduces to one which occurs in Saint Venant's theory of plastic bodies. This equation has been discussed by Hencky [6], Prandtl [7] and Carathéodory [8]. Oseen [9] uses the method of Legendre in which the equation is first solved for R, differentiated with respect to y and so reduced to an equation

$$(K^2 - V_x^2)^{1/2}(V_{xx} - V_{yy}) + 2V_xV_{xy} = 0 \tag{21}$$

in which

$$S = V_x, \qquad T = V_y.$$

It should be remarked that if $p_v = \bar{u} = \bar{z}_x$, $p_u = -\bar{v} = -\bar{z}_y$,

$$p + \rho u^2 + \rho v^2 = \bar{p}, \qquad 1/\rho = -\bar{\rho}, \tag{22}$$

we may write

$$R = -\bar{p} - \bar{\rho}\bar{v}^2, \qquad S = \bar{\rho}\bar{u}\bar{v}, \qquad T = -\bar{p} - \bar{\rho}\bar{u}^2, \tag{23}$$

and the equations $R_y = S_x$, $S_y = T_x$ lead to the partial differential equation for the stream-function $\bar{z}$.

In the theory of plane waves of finite amplitude equations of Legendre's type occur in at least two ways one of which is discussed by J. R. Wilton [10]. In the other way use is made of the equations

$$R = \rho, \qquad S = -\rho u, \qquad T = p + \rho u^2, \tag{24}$$

where now y denotes the time and x a co-ordinate in the direction in which the waves are travelling. The quantities u, v are again the derivatives of a velocity potential z, ρ is the density, p the pressure and u the velocity of the fluid. The quantities R, S, T are the second derivatives of a stress-function Z. The additional equations from which the relation between R, S and T may be derived, are

$$v + \tfrac{1}{2}u^2 = f'(\rho), \qquad p = f(\rho) - \rho f'(\rho). \tag{25}$$

The desired relation is thus

$$T - R^{-1}S^2 = f(R) - Rf'(R). \tag{26}$$

This equation, like that considered by Wilton, may be solved by the method of Legendre in which the equation is differentiated with respect to one of the independent variables (in this case x) so as to reduce it to an equation of the Monge-Ampère type. The transformation to the new equation can be regarded as a special Bäcklund transformation [11] as Oseen [9] observes. If $U=Z_x$, the new equation is

$$U_{yy} - 2(U_y/U_x)U_{xy} + (U_y^2/U_x^2)U_{xx} = - Rf''(R)U_{xx} = c^2U_{xx} \tag{27}$$

or

$$HU_{xx} + 2KU_{xy} + LU_{yy} = 0,$$

where $H=(U_y^2/U_x^2)-c^2$, $K=-(U_y/U_x)$, $L=1$. The invariant G is

$$G = K^2 - HL = c^2 \tag{28}$$

and the condition $G\neq 0$ is satisfied so long as $c^2\neq 0$.

In the present case

$$\begin{aligned} R_u &= - \rho u/c^2, \ R_v = - \rho/c^2, \ S_u = \rho(u^2/c^2 - 1)S_v = \rho u/c^2, \\ T_u &= \rho u(1 - u^2/c^2), \ \ T_v = - \rho(1 + u^2/c^2). \end{aligned} \tag{29}$$

The two equations (10) are both equivalent to

$$(u^2 - c^2)r + 2us + t = 0, \tag{30}$$

and it is readily seen that $w=-1$, $w'=-u$, $g=k^2-hl=c^2\neq 0$.

It should be noticed that if we solve equations (9) for u and v in the form

$$u = F(R, S), \qquad v = G(S, T), \tag{31}$$

the equation $u_y=v_x$ is satisfied on account of $R_y=S_x$, $S_y=T_x$ if

$$F_R = G_S, \qquad F_S = G_T. \tag{32}$$

These two equations then are consequences of the single equation $\mathcal{H}(R, S, T)=0$. The expression of such an equation in the two forms (32) may be regarded as a problem of some interest.

In the case when D is a constant and p is a function of q only

$$D = p_q p_{qq}/q = \rho^2[1 - (q/c)^2], \tag{33}$$

and the flow is either entirely subsonic $(D>0)$ or entirely supersonic $(D<0)$. In many cases in which p is a function of q only, D can have either sign and so the flow is partly subsonic and partly supersonic. It is then of some interest to seek the condition satisfied by the function $\mathcal{H}(R, S, T)$ when $D>0$. For this purpose we write the equation in the form

$$0 = \mathcal{H}(R, S, T) \equiv [(R - T)^2 + 4S^2]^{1/2} - J(R + T), \tag{34}$$

where J is a function which is such that

$$2\rho f'(\rho) = - J[- 2f(\rho)]. \tag{35}$$

Here $p=f(\rho)-\rho f'(\rho)$ is the relation between the pressure p and density ρ. Now we find by differentiation that

$$\mathcal{H}_R + \mathcal{H}_T - 2 = -2 - 2J'(R+T) = -4 - 2\rho f''(\rho)/f'(\rho) = 4(c^2/q^2 - 1). \quad (36)$$

Hence $\mathcal{H}_R+\mathcal{H}_T-2>0$ when $c^2>q^2$ and $\mathcal{H}_R+\mathcal{H}_T-2<0$ when $c^2<q^2$. In the case of the plastic equation J is a constant and so

$$\mathcal{H}_R + \mathcal{H}_T = 0. \quad (37)$$

The corresponding flow is characterized by the relation $q^2=2c^2$ and is consequently supersonic.

A simple case in which D is constant is obtained by writing

$$p = au^2 + 2cuv + bv^2, \quad (38)$$

where a, b and c are constants. The functions w, w' both satisfy

$$bw_{uu} + aw_{vv} - 2cw_{uv} = 0, \quad (39)$$

and we may write $w=bV_u$, $w'=aV_v$, where V is a solution of this equation. If a function W is defined by the equations

$$W_u = cV_u - aV_v, \qquad W_v = bV_u - cV_v, \quad (40)$$

we may write $h=2bW_u$ and it is readily found that we can write

$$R = 2b(W + cV), \qquad S = -2abV, \qquad T = 2a(W + cV), \quad (41)$$

where V and W are connected by the foregoing equations. In this case the relation between R, S and T is simply

$$a\mathcal{H} = aR - bT = 0. \quad (42)$$

The quantity $\mathcal{H}_R+\mathcal{H}_T-2$ is now simply $-(a+b)/a$, a constant. There is no change in sign of the expression. It will be noticed that the equations $aR-bT=0$ and

$$ar + 2cs + bt = 0 \quad (43)$$

satisfy the condition of apolarity

$$Ab + Ba - 2Cc = 0, \quad (44)$$

when the first equation is written in the form $AR+2CS+BT=0$.

3. The transformation of the Monge-Ampère equation. If for the equation

$$hr + 2ks + lt + m + n(rt - s^2) = 0, \quad (45)$$

the expression

$$g = k^2 - hl + mn \quad (46)$$

is not zero and so the two systems in the methods of Monge and Boole are distinct, the equation is transformed by a contact transformation

$$\left.\begin{array}{l} X = X(x,y,z,u,v), \quad Y = Y(x,y,z,u,v), \quad Z = Z(x,y,z,u,v), \\ U = U(x,y,z,u,v), \ V = V(x,y,z,u,v), \ dZ = UdX - VdY = \sigma(dz - udx - vdy) \end{array}\right\} \quad (47)$$

into an equation

$$HR + 2KS + LT + M + N(RT - S^2) = 0 \quad (48)$$

for which the quantity $G=K^2-HL+MN=0$.

In a paper published in 1904 Sophus Lie [12] remarked that it would be desirable to have a direct proof of this theorem and Kürschák [13] gave one based upon a representation of the equation in the form of a Jacobian

$$d(a, b)/d(x, y) \tag{49}$$

where a and b are functions of x, y, z, u, v and $d/dx=\partial/\partial x+u(\partial/\partial z)+r(\partial/\partial u)+s(\partial/\partial v)$, $d/dy=\partial/\partial y+v(\partial/\partial z)+s(\partial/\partial u)+t(\partial/\partial v)$. When this representation is not used the proof is algebraically more difficult but the analysis is worth giving on account of the numerous relations to which it leads. Reference for this type of proof may be made to a paper by R. Garnier, *Sur la transformation des dérivées secondes dans les transformations de contact et les transformations ponctuelles*, Bull. des Sci. Math. (2), **64**, 13–32 (1940).

We shall suppose that $dz=udx+vdy$ and that consequently $dZ=UdX+VdY$. To make $dU=RdX+SdY$, $dV=SdX+TdY$ consequences of $du=rdx+sdy$, $dv=sdx+tdy$ we shall require that

$$\left.\begin{aligned} dU-RdX-SdY &= (U_u-RX_u-SY_u)(du-rdx-sdy)\\ &\quad+(U_v-RX_v-SY_u)(dv-sdx-tdy),\\ dV-SdX-TdY &= (V_u-SX_u-TY_u)(rdu-dx-sdy)\\ &\quad+(V_v-SX_v-TY_v)(dv-sdx-tdy). \end{aligned}\right\} \tag{50}$$

With the notation

$$\left.\begin{aligned} Z_1 &= Z_x+uZ_z, & Z_2 &= Z_y+vZ_z, \text{ etc.},\\ (uu) &= U_u-RX_u-SY_u, & (uv) &= U_v-RX_v-SY_v,\\ (vu) &= V_u-SX_u-TY_u, & (vv) &= V_v-SX_v-TY_v, \end{aligned}\right\} \tag{51}$$

the equations to be satisfied are

$$\left.\begin{aligned} r(uu)+s(uv)+U_1-RX_1-SY_1=0, \quad r(uv)+s(vv)+V_1-SX_1-TY_1=0,\\ s(uu)+t(uv)+U_2-RX_2-TY_2=0, \quad s(vu)+t(vv)+V_2-SX_2-TY_2=0. \end{aligned}\right\} \tag{52}$$

Hence

$$\left.\begin{aligned} r\Delta &= U_vV_1-U_1V_v+R(X_1V_v-V_1X_v)+S(Y_1V_v+U_1X_v-X_1U_v-V_1Y_v)\\ &\quad+T(U_1Y_v-Y_1U_v)+(RT-S^2)(Y_1X_v-X_1Y_v),\\ t\Delta &= U_2V_u-V_2U_u+R(X_uV_2-X_2V_u)+S(X_2U_u+Y_uV_2-X_uU_2-Y_2V_u)\\ &\quad+T(Y_2U_u-Y_uU_2)+(RT-S^2)(X_2Y_u-X_uY_2),\\ s\Delta &= U_1V_u-V_1U_u+R(X_uV_1-X_1V_u)+S(X_1U_u+Y_uV_1-X_uU_1-Y_1V_u)\\ &\quad+T(U_uY_1-U_1Y_u)+(RT-S^2)(X_1Y_u-X_uY_1),\\ s\Delta &= V_2U_v-U_2V_v+R(X_2V_v-X_vV_2)+S(U_2X_v+Y_2V_v-X_2U_v-V_2Y_v)\\ &\quad+T(U_2Y_v-Y_2U_v)+(RT-S^2)(Y_2X_v-X_2Y_v),\\ \Delta(rt-s^2) &= U_1V_2-U_2V_1+R(V_1X_2-V_2X_1)+T(U_2Y_1-U_1Y_2)\\ &\quad+S(U_2X_1-U_1X_2+V_1Y_2-V_2Y_1)+(RT-S^2)(X_1Y_2-X_2Y_1),\\ \Delta &= U_uV_v-U_vV_u+R(X_vV_u-X_uV_v)+T(U_vY_u-U_uY_v)\\ &\quad+S(Y_vV_u-Y_uV_v+U_vX_u-U_uX_v)+(RT-S^2)(X_uY_v-X_vY_u). \end{aligned}\right\} \tag{53}$$

The two expressions for s are equivalent on account of the relations

$$[UV] = [XV] = [YU] = [XY] = 0, \qquad [XU] = [YV] = \sigma \tag{54}$$

which, in addition to the relations $[YZ]=[ZX]=0$, $[UZ]=\sigma U[VZ]=\sigma V$ are satisfied because the transformation is a contact transformation. In these relations $[AB]$ is the Poisson bracket

$$[AB] = A_uB_1 - A_1B_u + A_vB_2 - A_2B_v. \tag{55}$$

The relations are derived by Lie [12] by a clever device. In the book of Cerf [14] the relations are derived from the equation

$$\sigma[AB] = [ab] \tag{56}$$

where A, B are the expressions for $a(x, y, z, p, q)$, $b(x, y, z, p, q)$ in the new co-ordinates $[X, Y, Z, P, Q]$, while F. Engel [15] obtained them with the aid of the bilinear covariant by a development of a method used by G. Darboux.
It is readily seen that the equation $hr+2ks+lt+m+n(rt-s^2)=0$ becomes $HR+2KS+LT+M+N(RT-S)=0$, where

$$\left.\begin{aligned} H &= hP_p + k(P_b + P_q) + lP_a + mP_r + nP_c, \\ 2K &= h(R_p + C_p) + k(C_b + R_b + C_q + R_q) + l(R_a + C_a) \\ &\qquad + m(R_r + C_r) + n(R_c + C_c), \\ L &= hA_p + k(A_b + A_q) + lA_a + mA_r + nA_c, \\ M &= hQ_p + k(Q_b + Q_q) + lQ_a + mQ_r + nQ_c, \\ N &= hB_p + k(B_b + B_q) + lB_a + mB_r + nB_c, \end{aligned}\right\} \tag{57}$$

where

$$\begin{array}{lll}
A_a = U_uY_2 - U_2Y_u, & A_b = U_uY_1 - U_1Y_u, & A_c = U_2Y_1 - U_1Y_2, \\
A_p = U_1Y_v - U_vY_1, & A_q = U_2Y_v - U_vY_2, & A_r = U_vY_u - U_uY_v, \\
B_a = X_2Y_u - X_uY_2, & B_b = X_1Y_u - X_uY_1, & B_c = X_1Y_2 - X_2Y_1, \\
B_p = X_vY_1 - X_1Y_v, & B_q = X_vY_2 - X_2Y_v, & B_r = X_uY_v - X_vY_u, \\
C_a = X_2U_u - X_uU_2, & C_b = X_1U_u - X_uU_1, & C_c = X_1U_2 - X_2U_1, \\
C_p = X_vU_1 - X_1U_v, & C_q = X_vU_2 - X_2U_v, & C_r = X_uU_v - X_vU_u, \\
P_a = X_uV_2 - X_2V_u, & P_b = X_uV_1 - X_1V_u, & P_c = X_2V_1 - X_1V_2, \\
P_p = X_1V_v - X_vV_1, & P_q = X_2V_v - X_vV_2, & P_r = X_vV_u - X_uV_v, \\
Q_a = U_2V_u - U_uV_2, & Q_b = U_1V_u - U_uV_1, & Q_c = U_1V_2 - U_2V_1, \\
Q_p = U_vV_1 - U_1V_v, & Q_q = U_vV_2 - U_2V_v, & Q_r = U_uV_v - U_vV_u, \\
R_a = Y_uV_2 - Y_2V_u, & R_b = Y_uV_1 - Y_1V_u, & R_c = Y_2V_1 - Y_1V_2, \\
R_p = Y_1V_v - Y_vV_1, & R_q = Y_2V_v - Y_vV_2, & R_r = Y_vV_u - Y_uV_v.
\end{array}$$

These equations give the relation

$$
\left.\begin{aligned}
&K^2-HL+MN=J(k^2-hl+mn), \quad \text{if}\\
&\tfrac{1}{2}(C_b+R_b+C_q+R_q)^2-(P_b+P_q)(A_b+A_q)+(Q_b+Q_q)(B_b+B_q)=J,\\
&\tfrac{1}{2}(R_r+C_r)(R_c+C_c)+(Q_rB_c+Q_cB_r)-(P_rA_c+P_cA_r)=J,\\
&\tfrac{1}{2}(R_p+C_p)(R_a+C_a)+(Q_pB_a+Q_aB_p)-(P_pA_a+P_aA_p)=-J,\\
&\tfrac{1}{4}(R_p+C_p)^2+Q_pB_p-P_pA_p=0, \qquad \tfrac{1}{4}(R_a+C_a)^2+Q_aB_a-P_aA_a=0,\\
&\tfrac{1}{4}(R_r+C_r)^2+Q_rB_r-P_rA_r=0, \qquad \tfrac{1}{4}(R_c+C_c)^2+Q_cB_c-P_cA_c=0,\\
&\tfrac{1}{2}(R_p+C_p)(C_b+R_b+C_q+R_q)+Q_p(B_b+B_q)+B_p(Q_b+Q_q)\\
&\qquad\qquad=P_p(A_b+A_q)+A_p(P_b+P_q),\\
&\tfrac{1}{2}(R_a+C_a)(C_b+R_b+C_q+R_q)+Q_a(B_b+B_q)+B_a(Q_b+Q_q)\\
&\qquad\qquad=P_a(A_b+A_q)+A_a(P_b+P_q),\\
&\tfrac{1}{2}(R_r+C_r)(C_b+R_b+C_q+R_q)+Q_r(B_b+B_q)+B_r(Q_b+Q_q)\\
&\qquad\qquad=P_r(A_b+A_q)+A_r(P_b+P_q),\\
&\tfrac{1}{2}(R_c+C_c)(C_b+R_b+C_q+R_q)+Q_c(B_b+B_q)+B_c(Q_b+Q_q)\\
&\qquad\qquad=P_c(A_b+A_q)+A_c(P_b+P_q),\\
&\tfrac{1}{2}(R_p+C_p)(R_r+C_r)+Q_pB_r+Q_rB_p-P_pA_r-P_rA_p=0,\\
&\tfrac{1}{2}(R_p+C_p)(R_c+C_c)+Q_pB_c+Q_cB_p-P_pA_c-P_cA_p=0,\\
&\tfrac{1}{2}(R_a+C_a)(R_r+C_r)+Q_aB_r+Q_rB_a-P_aA_r-P_rA_a=0,\\
&\tfrac{1}{2}(R_a+C_a)(R_c+C_c)+Q_aB_c+Q_cB_a-P_aA_c-P_cA_a=0.
\end{aligned}\right\} \tag{58}
$$

These relations may be established by using a parametric representation of the quantities satisfying Lie's conditions for a contact transformation, we therefore write

$$
\left.\begin{aligned}
X_1 &= a_1e + a_2e', & X_2 &= b_1e + b_2e', & X_u &= c_1e + c_2e', & X_v &= d_1e + d_2e',\\
U_1 &= a_1f + a_2f', & U_2 &= b_1f + b_2f', & U_u &= c_1f + c_2f', & U_v &= d_1f + d_2f',\\
-Y_u &= a_3p + a_4p', & -Y_v &= b_3p + b_4p', & Y_1 &= c_3p + c_4p', & Y_2 &= d_3p + d_4p',\\
-V_u &= a_3q + a_4q', & -V_v &= b_3q + b_4q', & V_1 &= c_3q + c_4q', & V_2 &= d_3q + d_4q',
\end{aligned}\right\} \tag{59}
$$

where the quantities

$$
\begin{matrix}
a_1 & a_2 & a_3 & a_4\\
b_1 & b_2 & b_3 & b_4\\
c_1 & c_2 & c_3 & c_4\\
d_1 & d_2 & d_3 & d_4
\end{matrix}
$$

form an orthogonal matrix and the quantities $e, e', f, f', p, p', q, q'$ are such that

$$(c_1a_2 - c_2a_1 + d_1b_2 - d_2b_1)(ef' + e'f) = (a_3a_4 - a_4c_3 + b_3d_4 - b_4d_3)(pq' - p'q) = \sigma. \tag{60}$$

It is then found that

$$
\left.\begin{aligned}
-H &= eq(1, 3) + e'q(2, 3) + eq'(1, 4) + e'q'(2, 4),\\
-L &= fp(1, 3) + f'p(2, 3) + fp'(1, 4) + f'p'(2, 4),\\
2K &= (ef' - e'f)(1, 2) + (pq' - p'q)(3, 4),\\
M &= fq(1, 3) + f'q(1, 3) + fq'(1, 4) + f'q'(2, 4),\\
N &= ep(1, 3) + e'p(2, 3) + ep'(1, 4) + e'p'(2, 4),
\end{aligned}\right\} \tag{61}
$$

where

$$\left.\begin{aligned}
(1, 3) &= h(d_1c_3 + a_1b_3) - l(b_1a_3 + c_1d_3) + k(d_1d_3 + b_1b_3 - a_1a_3 - c_1c_3)\\
&\quad + n(a_1d_3 - b_1c_3) + m(d_1a_3 - c_1b_3),\\
(2, 3) &= h(d_2c_3 + a_2b_3) - l(b_2a_3 + c_2d_3) + k(d_2d_3 + b_2b_3 - a_2a_3 - c_2c_3)\\
&\quad + n(a_2d_3 - b_2c_3) + m(d_2a_3 - c_2b_3),\\
(1, 4) &= h(d_1c_4 + a_1b_4) - l(b_1a_4 + c_1d_4) + k(d_1d_4 - b_1b_4 - a_1a_4 - c_1c_4)\\
&\quad + n(a_1d_4 - b_1c_4) + m(d_1a_4 - c_1b_4),\\
(2, 4) &= h(d_2a_4 + a_2b_4) - l(b_2a_4 + c_2d_4) + k(d_2d_4 + b_2b_4 - a_2a_4 - c_2c_4)\\
&\quad + n(a_2d_4 - b_2c_4) + m(d_2a_4 - c_2b_4),\\
(1, 2) &= h(d_1a_2 - d_2a_1) + l(b_1c_2 - b_2c_1) + k(a_1c_2 - a_2c_1 + d_1b_2 - d_2b_1)\\
&\quad + n(a_1b_2 + a_2b_1) + m(c_1d_2 - d_1c_2),\\
(3, 4) &= h(b_3c_4 - b_4c_3) + l(a_4d_3 - a_3d_4) + k(c_3a_4 - c_4a_3 + b_3d_4 - b_4d_3)\\
&\quad + n(d_3c_4 - d_4c_3) + m(b_3a_4 - b_4a_3).
\end{aligned}\right\} \quad (62)$$

It is readily seen that

$$MN - HL = [(1, 3)(2, 4) - (2, 3)(1, 4)](e'f - ef')(p'q - pq') \quad (63)$$

and that, on account of the properties of an orthogonal matrix

$$(e'f - ef')^2 = (p'q - pq')^2. \quad (64)$$

The expression for K also simplifies considerably and the proof may be readily completed. The quantity J as in Kürschák's analysis, is equal to σ^2 and so is not zero.

Contact transformations are not the only ones in which the condition $g \neq 0$ is invariant. In the theory of the steady two-dimensional motion of an inviscid elastic fluid the equations satisfied by the velocity potential z and stream-function $\bar{z}$ are respectively

$$p_{uu}r + 2p_{uv}s + p_{vv}t = 0, \qquad p_{uu}\bar{r} + 2p_{uv}\bar{s} + p_{vv}\bar{t} = 0. \quad (65)$$

In this case $dz = udx + vdy$, $d\bar{z} = \bar{u}\, dx + \bar{v}\, dy = p_v dx - p_u dy$ and so

$$\bar{u} = p_v, \quad \bar{v} = -p_u, \quad g = \bar{g} = p_{uv}^2 - p_{uu}p_{vv} = c^2(u^2 + v^2 - c^2). \quad (66)$$

Thus $g = 0$ either when $c = 0$ or when $q = c$. The supersonic region is characterized by the condition $g > 0$ and the subsonic region by the condition $g < 0$. The curve for which $c = 0$ is a boundary for the flow just as in the case of the Prandtl-Meyer flow round a corner. The transformation under consideration is a special Bäcklund transformation and is included in the group of Bäcklund transformations

$$\bar{x} = X(x, y, u, v), \quad \bar{y} = Y(x, y, u, v), \quad \bar{u} = U(x, y, u, v), \quad \bar{v} = V(x, y, u, v), \quad (67)$$

for which the Jacobian $\partial(X, Y, U, V)/\partial(x, y, u, v)$ is not zero. These transformations have been studied carefully by Goursat [16]. The requirement that

$$\bar{u}\, d\bar{x} + \bar{v}\, d\bar{y}$$

should be exact leads to an equation of the Monge-Ampère type in which z does not occur explicitly. It is shown, however, that the general Monge-Ampère equation of this type cannot be obtained in this way and a similar result has been found by J. Clairin [17] in his studies of more general Bäcklund transformations. Clairin has studied in particular transformations of type

$$\left.\begin{aligned} x' &= f_1(x, y, z, u, v, z'), \quad y' = f_2(x, y, z, u, v, z'), \\ u' &= f_3(x, y, z, u, v, z'), \quad v' = f_4(x, y, z, u, v, z'). \end{aligned}\right\} \tag{68}$$

Some of Clairin's work is summarized in the book of Forsyth [18] and illustrated by means of examples.

Another transformation of type (67) which preserves the condition $g \neq 0$ is obtained by writing

$$dU = Rdx + Sdy, \qquad dV = Sdx + Tdy,$$

where R, S, T are the functions of u and v used in section 2. Making use of the equations

$$(up_u + vp_v)dx = p_u dz + vd\bar{z}, \qquad (up_u + vp_v)dy = p_v dz - ud\bar{z}, \tag{69}$$

we find that

$$\left.\begin{aligned} (Tu - Sv)dU + (Rv - Su)dV &= (RT - S^2)dz, \\ (Sp_u + Tp_v)dU - (Rp_u + Sp_v)dV &= (RT - S^2)d\bar{z}. \end{aligned}\right\} \tag{70}$$

Hence, if

$$x' = U, \quad y' = V, \quad u' = \partial z/\partial U, \quad v' = \partial z/\partial V, \quad \bar{u}' = \partial \bar{z}/\partial U, \quad \bar{v}' = \partial \bar{z}/\partial V, \tag{71}$$

we have the relations

$$\left.\begin{aligned} u' &= (Tu - Sv)/(RT - S^2), & v' &= (Rv - Su)/(RT - S^2), \\ \bar{u}' &= (Sp_u + Tp_v)/(RT - S^2), & \bar{v}' &= -(Rp_u + Sp_v)/(RT - S^2) \end{aligned}\right\} \tag{72}$$

which, in conjunction with the preceding relations define two Bäcklund transformations. The transformations considered in my paper on the lift and drag functions are of this type [19] and are not generally contact transformations as is apparently implied by a statement relating to the correspondence of the supersonic regions in the two associated types of flow.

In the case in which p is a function of q only the relations between u', v', u, v are

$$u' = -u/p, \qquad v' = -v/p, \qquad q' = q/p$$

and, if $p' = -1/p$, $p' + \rho' q'^2 = -1/(p + \rho q^2)$ we have

$$\rho' = \frac{\rho p}{p + \rho q^2},$$

$$dp'/dq' = -q\rho/(p + \rho q^2) = -\rho' q',$$

$$1 - q'^2/c'^2 = (1 - q^2/c^2)[p/(p + \rho q^2)]^2, \quad \text{where} \quad c'^2 = dp'/d\rho'.$$

This transformation may be compared with that obtained by means of Haar's adjoint variation problems [20]. In this case

$$u^* = p_u/(p - up_u - vp_v), \qquad v^* = p_v/(p - up_u - vp_v), \qquad p^* = 1/(p - up_u - vp_v),$$

and, when p depends only on q,

$$u^* = -\rho u/(p + \rho q^2), \qquad v^* = -\rho v/(p + \rho q^2),$$
$$p^* = +1/(p + q^2), \qquad q^* = q/(p + q^2).$$

Defining ρ^* by the equation $dp^* = -q^* dq^*$, we have

$$\rho^* = (p + \rho q^2)/p\rho, \qquad p^* + q^{*2} = 1/p,$$
$$c^{*2} = \rho^2 p^2 q^2 (q^2 - c^2)/(p + \rho q^2)^2 [c^2(p + \rho q^2)^2 + p^2(q^2 - c^2)],$$
$$c^{*2} - q^{*2} = -\rho^2 q^2 c^2/[c^2(p + \rho q^2)^2 + p^2(q^2 - c^2)].$$

Hence $c^{*2} = q^{*2}$ when $c^2 = 0$ and $c^{*2} = 0$ when $q^2 = c^2$. It should be noticed, however, that

$$c^{*2}(c^{*2} - q^2) = c^2(c^2 - q^2)p^2\rho^4 q^4/(p + \rho q^2)^2[c^2(p + \rho q^2)^2 + p^2(q^2 - c^2)]^2.$$

This may be compared with Haar's general relation*

$$(p^*_{u^*u^*} p^*_{v^*v^*} - p^{*2}_{u^*v^*})(p_{uu} p_{vv} - p^2_{uv}) = p^{-4} p^{*-4}.$$

Transformations more general than those of Bäcklund have been considered by Gau [21] but so far no hydrodynamical applications have been found for these so far as I know. Mention should be made, however, of the equiareal transformations from the Eulerian to Lagrangian co-ordinates in the two-dimensional flow of an incompressible fluid. These transformations have been much used in mapping but the hydrodynamical applications are beset with formidable difficulties.

No mention has been made of the use of transformations in the theory of surfaces, congruences, etc. This is a subject which is well treated in the books of Darboux [22], Forsyth [18], Goursat [16], Bianchi [23] and Eisenhart [24].

4. Transformation of the linear equation. In the special case in which h, k and l are functions of x and y only, $n = 0$ and m is a linear homogeneous function of u, v and z with coefficients depending only on x and y, the Monge-Ampère equation reduces to a linear equation. The behavior of this equation in transformations of type

$$X = X(x, y), \qquad Y = Y(x, y), \qquad Z = zF(x, y) \tag{73}$$

has been studied by Darboux [22], Cotton [25], Rivereau [26], J. E. Campbell after the case $F = 1$ had been discussed by Laplace [27], Chini [28] and others [16]. Campbell uses the equation in a form in which $g = 1$, a form to which the general equation can be reduced by multiplying it by a suitable factor. He then shows that there are two invariants I, J and an absolute invariant J/I where if suffixes denote partial derivatives

$$\left.\begin{aligned} I &= ha_x + k(a_y + b_x) + lb_y + (h_x + k_y)a + (k_x + l_y)b \\ &\quad + ha^2 + 2kab + lb^2 - m_z, \qquad J = a_y - b_x, \\ 2a &= l(h_x + k_y - m_u) - k(k_x + l_y - m_v), \\ 2b &= h(k_x + l_y - m_v) - k(h_x + k_y - m_u). \end{aligned}\right\} \tag{74}$$

* This is a consequence of the relations

$$\frac{\partial(u^*, v^*)}{\partial(u, v)} = pp^{*3}(p_{uu}p_{vv} - p_{uv}), \qquad \frac{\partial(u, v)}{\partial(u^*, v^*)} = p^* p^3 (p^*_{u^*u^*} p^*_{v^*v^*} - p^{*2}_{u^*v^*}).$$

In the correspondence between the two hodograph planes complications arise on account of the relation of $p_{uu}p_{vv} - p^2_{uv}$ to these Jacobians.

Laplace's invariants are $\frac{1}{2}(I-J)$, $\frac{1}{2}(I+J)$. These are used with a different notation, in Darboux's Théorie des Surfaces, t.2. Campbell shows that in the case of the equation of Euler and Poisson the invariants I and J are constant. This may be compared with Cotton's result. The harmonic equations belong to the group characterized by the relation $J=0$. The equations considered by Burgatti [29] are such that $I=0$.

In mathematical physics the simple solutions of linear equations play an important part and the primary problem is that of separability. Even in the case of the equation with two independent variables there are some unsolved problems. A good idea of the progress which has been made may be derived from Darboux's book [22]. The method of Laplace provides an important way of reducing equations by a cascade process which is particularly useful in the treatment of equations arising in the theory of plane waves of finite amplitude. Reference may be made to a paper of Love and Pidduck [30], an article by Platrier [31], some papers by Bechert [32] and to two papers by Oseen [9] in which the transformation and reduction is given for equations occurring in the theory of earth pressure and in the theory of plasticity.

In the theory of the steady motion of an inviscid compressible fluid the equations in the hodograph plane are linear. These equations are

$$\left.\begin{aligned} &p_{vv}w_{uu} - 2p_{uv}w_{uv} + p_{uu}w_{vv} = 0, \qquad &&p_{vv}\bar{w}_{\bar{u}\bar{u}} - 2p_{uv}\bar{w}_{\bar{u}\bar{v}} + p_{uu}\bar{w}_{\bar{v}\bar{v}} = 0,\\ &z = uw_u + vw_v - w = qw_q - w, &&\bar{z} = \bar{u}\bar{w}_{\bar{u}} + \bar{v}\bar{w}_{\bar{v}} - \bar{w} = \bar{q}\bar{w}_{\bar{q}} - \bar{w}.\end{aligned}\right\} \tag{75}$$

When p is a function of q only the equations $\bar{u}=p_v$, $v=-p_u$ take the form

$$\bar{u} = -\rho v = -\bar{q}\sin\tau, \qquad \bar{v} = \rho u = \bar{q}\cos\tau, \qquad \bar{q} = \rho q, \tag{76}$$

and the equations become

$$w_{\tau\tau} + q(\bar{q}w_q)_{\bar{q}} = 0, \qquad \bar{w}_{\tau\tau} + q(q\bar{w}_{\bar{q}})_q = 0 \qquad (\bar{q} \text{ a function of } q). \tag{77}$$

These are consequences of simple relations between w and $\bar{w}$

$$\bar{w}_\tau + \bar{q}w_{\bar{q}} = 0, \qquad w_\tau - q\bar{w}_{\bar{q}} = 0. \tag{78}$$

The corresponding relations between z and $\bar{z}$ are found to be

$$q^2\bar{z}_q = \bar{q}z_\tau, \qquad \bar{q}^2 z_q = -q\bar{z}_\tau \tag{79}$$

and so the equations for z and $\bar{z}$ are

$$\left.\begin{aligned}(q^2/\bar{q})](\bar{q}^2/q)z_{\bar{q}}]_q + z_{\tau\tau} = 0,\\ (\bar{q}^2/q)[(q^2/\bar{q})\bar{z}_q]_{\bar{q}} + z_{\tau\tau} = 0.\end{aligned}\right\} \tag{80}$$

These are equivalent to the equations obtained by Molenbroek [33] and Tschaplygin [34] for the case in which the relation between p and ρ is of the polytopic or adiabatic type. The symmetrical forms of the equations are easy to remember.

It is sometimes useful to introduce other quantities which satisfy linear relations. Thus we may obtain the desired relations between z, $\bar{z}$, w, $\bar{w}$ by writing

$$\begin{aligned} &w = \bar{e}_\tau = qe_{\bar{q}}, \qquad &&\bar{w} = e_\tau = -\bar{q}e_q,\\ &z = -\bar{e}_\tau - (q/\bar{q})e_{\tau\tau}, &&\bar{z} = (\bar{q}/q)e_{\tau\tau} - e_\tau,\end{aligned}$$

where

$$e_{\tau\tau} + \bar{q}(qe_{\bar{q}})_q = 0, \qquad \bar{e}_{\tau\tau} + q(\bar{q}\bar{e}_q)_q = 0.$$

The literature dealing with the transformation of linear equations in several variables is very extensive and only a brief summary can be attempted here. Beltrami's work on differential parameters [35] was extended by Ricci and Levi-Civita [36], Cotton [25], Levi-Civita [37] and many other writers. The development of general relativity, electrodynamics and the theory of elasticity has made this work more or less known. The work of Lamé on simple solutions of the potential equation [38] was much developed by later writers and a good summary of results up to 1893 is given in the book of Bôcher [39]. The use of a variational principle for obtaining the transformation of the equation was recommended by Larmor [40], Volterra and others [41]. Since the advent of the new quantum theory the interest in separable equations and separable systems has much increased. Mention may be made of the work of Staeckel [42], Eisenhart [43] and Robertson [44].

In addition to the simple solutions of partial differential equations there are solutions having the form of products in which one or more or the factors satisfies a partial differential equation instead of an ordinary differential equation. Comparatively little work has been done on this problem. In the case of Laplace's equation $V_{xx}+V_{yy}+V_{zz}=0$, the aim is to find a solution of form [45]

$$V = ZF(X, Y), \qquad \text{(generalized binary potential)}$$

where F satisfies a partial differential equation of the second order in the variables X and Y. The problem seems to depend on the formation of a relation of type

$$(p^2 + q^2 + r^2)(dx^2 + dy^2 + dz^2) - (pdx + qdy + rdz)^2 = adX^2 + 2hdXdY + bdY^2$$

in which a, b and h are functions of X and Y only. There is a similar relation for the corresponding problem in any number of variables.

5. The transformation of Legendre's equation. Legendre's equation

$$\mathcal{H}(R, S, T) = 0$$

is unaltered in form by a Legendre contact transformation

$$X' = U, \qquad Y' = V, \qquad U' = X, \qquad V = Y', \qquad Z' = UX + VY - Z,$$

which makes

$$R' = T/(RT - S^2), \qquad S' = - S/(RT - S^2), \qquad T' = R/(RT - S^2).$$

In particular, the equation

$$R'T' - S'^2 = F(R' + T')$$

becomes

$$1/(RT - S^2) = F\left(\frac{R + T}{RT - S}\right),$$

an equation of the same general type. Again, if a, b and h are constants the contact transformation

$$Z' = \tfrac{1}{2}(aX^2 + 2hXY + bY^2) + Z, \qquad X' = X, \qquad Y' = Y,$$
$$U' = aX + hY + U, \qquad V' = hX + bY + V$$

makes $R'=a+R$, $S'=h+S$, $T'=b+T$ and so transforms an equation of Legendre's type into another equation of the same type. Equations of the preceding type usually go into Legendre equations of a slightly different type.

Other transformations may be found by first transforming the equation to the Monge-Ampère form by Legendre's device. If, for instance, the equation is

$$T = F(R, S)$$

and we differentiate with respect to x using then the new notation $z = U$, $R = U_x = u$, $S = U_y = v$, $T_x = S_y = t$, $R_x = r$, $S_x = s$, the new equation is

$$t = F_u(u, v)r + F_v(u, v)s.$$

Comparing this with the equation $p_{uu}r + 2p_{uv}s + p_{vv}t = 0$ we find that

$$F_u = - p_{uu}/p_{vv}, \qquad F_v = - 2p_{uv}/p_{vv}.$$

Eliminating F we find that the equation for z is not a general equation of the type considered in §2 because the function $p(u, v)$ satisfies the condition

$$\partial D/\partial v = 0 \quad \text{where} \quad D = p_{uu}p_{vv} - p_{uv}^2.$$

An equation for which D is constant satisfies this condition and the equation of type

$$\overline{\mathcal{H}}(R, S, T) = 0$$

associated with it by the analysis of section 2 may be regarded as a transform of the original equation $\mathcal{H}(R, S, T) = 0$. In the case when $D = 1$, we may write

$$p_{uu} = e^a \sec b, \qquad p_{vv} = e^{-a} \sec b, \qquad p_{uv} = \tan b,$$

where a and b are functions of u and v which must be chosen so that

$$\sec^2 bb_u = e^2 \sec b(a_v) + e^a \sec b \tan b(b_v),$$
$$\sec^2 bb_v = - e^{-a} \sec b(a_u) + e^{-a} \sec b \tan b(b_u).$$

Also, since $F_u = -e^{2a}$, $F_v = -2e^a \sin v$, we must have the additional equation

$$2e^a \cos b(b_u) - 2e^{2a}(a_v) + 2e^a \sin v(a_u) = 0$$

which is seen, however, to be a consequence of the other two. Elimination of the derivatives of a gives the equation

$$e^a b_{vv} + e^{-a} b_{uu} - 2 \sin bb_{uv} = 0 \quad \text{or} \quad p_{uu}b_{vv} + p_{vv}b_{uu} - 2p_{uv}b_{uv} = 0$$

and it is readily seen that a satisfies the same equation. The equation $D = 1$ is given as an example in Forsyth's book, p. 220, Ex. 11.

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