—NOTES—

USE OF SINE TRANSFORM FOR NON-SIMPLY SUPPORTED BEAMS*

BY A. G. STRANDHAGEN (*Carnegie Institute of Technology*)

The problem of non-simply supported beams is approached by various mathematical procedures. In certain applications several of the common methods are long and tedious. By employing the sine transform a certain ease can be claimed for most cases.

The definition of the sine transform of a function $y(x)$ in the interval $(0, l)$ is

$$S[y(x)] = \int_0^l y(x) \sin (n\pi x/l)dx = v(n). \qquad (0 < x < l;\, n = 1, 2, \cdots) \tag{1}$$

Recalling that the expression of a function $y(x)$ in a Fourier sine series is

$$y(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x/l, \tag{2}$$

where

$$b_n = (2/l) \int_0^l y(x) \sin (n\pi x/l)dx, \qquad (0 < x < l;\, n = 1, 2, \cdots) \tag{3}$$

it becomes evident that the connection between the sine transform and the coefficients of the Fourier sine series is

$$S[y(x)] = (l/2)b_n. \tag{4}$$

Forms given by Eq. (2) and Eq. (3) are altered for the sake of convenience as follows:

$$y(x) = (2/l) \sum_{n=1}^{\infty} v(n) \sin (n\pi x/l), \tag{5}$$

where

$$v(n) = S[y(x)] = \int_0^l y(x) \sin (n\pi x/l)dx. \tag{6}$$

For example, consider the sine transform of (d^2y/dx^2) in the interval $(0, l)$; by definition

$$S[d^2y/dx^2] = \int_0^l (d^2y/dx^2) \sin (n\pi x/l)dx. \quad (n = 1, 2, \cdots)$$

Integrating formally by parts gives

$$S[d^2y/dx^2] = -\frac{n\pi}{l}[(-1)^n y(l) - y(0)] - \left(\frac{n\pi}{l}\right)^2 v(n). \qquad (n = 1, 2, \cdots) \tag{7}$$

Likewise the sine transform of (d^4y/dx^4) in $(0, l)$ is:

* Received August 9, 1943.

$$S[d^4y/dx^4] = -\frac{n\pi}{l}[(-1)^n y''(l) - y''(0)]$$
$$+\left(\frac{n\pi}{l}\right)^3[(-1)^n y(l) - y(0)] + \left(\frac{n\pi}{l}\right)^4 v(n), \quad (n = 1, 2, \cdots) \tag{8}$$

where $v(n)$ in (7) and (8) is defined by Eq. (1).

Consider a beam fixed at $x=0$ with axial loads P. The intensity of transverse loading is $q(x)$, Fig. 1. The differential equation and boundary conditions are as follows:

1. $d^4y/dx^4 + k^2(d^2y/dx^2) = q(x)/EI, \qquad (0 < x < l)$
2. $y(0) = y(l) = 0,$
3. $y''(l) = 0, \qquad y'(0) = 0,$

where

$$\begin{aligned} q(x) &= 0 && \text{when} \quad 0 < x < b, \\ &= \theta(x) && \text{when} \quad b < x < c, \\ &= 0 && \text{when} \quad c < x < l, \end{aligned}$$

and $c > b$. Let $k^2 = P/EI$, and primes indicate differentiation with respect to x. Let $S[y(x)] = v(n)$. Transforming d^4y/dx^4 and d^2y/dx^2 and $q(x)$ and substituting $y(0) = y(l) = y''(l) = 0$, there results

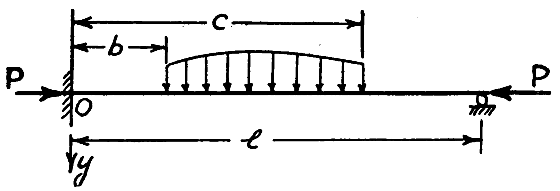

FIG. 1.

$$(n\pi/l)y''(0) + (n\pi/l)^4 v(n) - k^2(n\pi/l)^2 v(n) = (1/EI)\int_b^c \theta(x)\sin(n\pi x/l)dx.$$

Solving tor $v(n)$, where $\alpha^2 = (kl/\pi)^2$,

$$v(n) = -(l/\pi)^3 y''(0)\frac{1}{n(n^2-\alpha^2)} + \frac{l^4}{\pi^4 EI}\frac{1}{n^2(n^2-\alpha^2)}\int_b^c \theta(x)\sin(n\pi x/l)dx. \tag{9}$$

Since $y(x) = (2/l)\sum_{n=1}^{\infty} v(n)\sin n\pi x/l$, then

$$y(x) = -(2l^2/\pi^3)y''(0)\sum_{n=1}^{\infty}\frac{1}{n(n^2-\alpha^2)}\sin(n\pi x/l)$$
$$+ 2(l^3/\pi^4 EI)\sum_{n=1}^{\infty}\frac{\sin(n\pi x/l)}{n^2(n^2-\alpha^2)}\int_b^c \theta(x')\sin(n\pi x'/l)dx'. \quad (n \neq \alpha) \tag{10}$$

The remaining boundary condition $y'(0) = 0$ gives the following:

$$y''(0)\sum_{n=1}^{\infty}\frac{1}{(n^2-\alpha^2)} = \frac{l}{\pi EI}\sum_{n=1}^{\infty}\frac{1}{n(n^2-\alpha^2)}\int_b^c \theta(x')\sin(n\pi x'/l)dx'. \tag{11}$$

Since $\sum_{n=1}^{\infty} 1/(n^2-\alpha^2) = (1/2\alpha^2)(1-\pi\alpha\cot\pi\alpha)$, then $y''(0)$ becomes

$$y''(0) = \frac{(2\alpha^2 l/\pi EI)}{(1-\pi\alpha\cot\pi\alpha)}\sum_{n=1}^{\infty}\frac{1}{n(n^2-\alpha^2)}\int_b^c \theta(x')\sin(n\pi x'/l)dx'. \quad (n \neq \alpha) \tag{12}$$

Further simplifications are possible in Eq. (12). Interchanging formally the integral and summation sign and summing, the following is obtained for $y''(0)$:

$$y''(0) = \frac{(l/EI)}{(1 - \pi\alpha \cot\pi\alpha)} \int_b^c \theta(x) \left\{ \frac{\sin k(l - x)}{\sin kl} - \frac{l - x}{l} \right\} dx,$$

where

$$\sum_{n=1}^{\infty} \frac{\sin (n\pi x/l)}{n(n^2 - \alpha^2)} = \frac{\pi}{2\alpha^2} \left\{ \frac{\sin k(l - x)}{\sin kl} - \frac{l - x}{l} \right\} = \phi(x). \tag{13}$$

Thus by substitution of (13) in (10),

$$y(x) = -\frac{4\alpha^2 l^3}{\pi^4 EI} \frac{\phi(x)}{(1 - \pi\alpha \cot \pi\alpha)} \int_b^c \theta(x')\phi(x')dx'$$
$$+ \frac{2l^3}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{\sin (n\pi x/l)}{n^2(n^2 - \alpha^2)} \int_b^c \theta(x') \sin (n\pi x'/l)dx'. \quad (n \neq \alpha,\ 0 < x < l) \tag{14}$$

Knowing the variation of $\theta(x)$ it is a matter of integration to obtain the required results. Now suppose that $P=0$, i.e., the beam is under no axial loads, and subject to the same boundary conditions. Thus $k=\alpha=0$ in equations (9), (10), and (11) and then

$$y''(0) = \frac{6l}{\pi^3 EI} \sum_{n=1}^{\infty} \frac{1}{n^3} \int_b^c \theta(x') \sin (n\pi x'/l)dx'.$$

Again interchanging formally the integral and summation sign,

$$y''(0) = \frac{1}{2l^2 EI} \int_b^c \theta(x')x'(x' - l)(x' - 2l)dx',$$

where

$$\sum_{n=1}^{\infty} (1/n^3) \sin (n\pi x/l) = \frac{\pi^3}{12} \left\{ 2(x/l) - 3(x/l)^2 + (x/l)^3 \right\}. \qquad (0 < x/l < 2.)$$

The equation for the elastic line becomes

$$y(x) = -\frac{1}{12EI} \left[2(x/l) - 3(x/l)^2 + (x/l)^3 \right] \int_b^c \theta(x')(x' - l)(x' - 2l)dx'$$
$$+ \frac{2l^3}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{\sin (n\pi x/l)}{n^4} \int_b^c \theta(x') \sin (n\pi x'/l)dx'. \qquad (0 < x < l)$$

To be sure, further summation in finite terms is possible, but this will lead to $y(x)$ being defined in distinct intervals in $(0, l)$, as in the solution furnished by the classical methods of differential equations; unquestionably, this is a disadvantage in engineering computations. The above results, however, remain in the desired form, with one function $y(x)$ in $(0, l)$ regardless of the discontinuities of transverse loading.

In like manner other boundary conditions may be imposed, and other beam problems, such as beams on elastic foundations, can be solved.