ON MOMENT BALANCING IN STRUCTURAL DYNAMICS*

BY

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1. The method of moment balancing. In recent years several writers in this country have developed the method of moment balancing in the analysis of continuous beams and frameworks. Mention should be made especially of the basic paper by Hardy Cross.[1,2] One could also classify as related procedures the method of balancing angle changes given in a paper by L. E. Grinter,[3] and the whole field of relaxation methods being investigated by R. V. Southwell.[4] That such interest is taken in these methods would seem to indicate that their extension to the dynamics of beams and frameworks might be desirable, and it is the purpose of this article to provide at least the beginning of this extension.

We assume that we are dealing with plane structures on which loads are acting in the plane of the structure. Members of the structure consist of uniform straight beams; and they meet in stiff joints, which are assumed to be fixed against translation. All connections to a foundation are either built-in or hinged.

The method of moment balancing depends upon three very simple ideas, namely, fixed-end moment, stiffness and carry-over factor. We give their definitions here:

> The "fixed-end moment" at the end of a member is the moment which would exist at that end if all joints to which it is connected were fixed against rotation.
>
> If one end of a member is simply-supported, its "stiffness" is the moment required to produce unit rotation of that end. The other end may be built-in, simply-supported or free.
>
> The "carry-over factor" is the numerical value of the moment induced at one end of a member by a unit moment acting at the other end.

Methods of finding these characteristics of beams and other components of a structure are numerous and well-known. Having determined them for all components

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[1] H. Cross, *Analysis of continuous frames by distributing fixed-end moments*, Trans. A.S.C.E. **96,** 1 (1932). This paper is followed by discussions, that by L. E. Grinter, pp. 11–20, being particularly informative.

[2] See also: Hardy Cross and N. D. Morgan, *Continuous frames of reinforced concrete*, John Wiley and Sons, 1932, Chapter IV, pp. 81–125; *Moment distribution applied to continuous concrete structures*, Portland Cement Association, Second Edition, 1942.

[3] L. E. Grinter, *Analysis of continuous beams by balancing angle changes*, Trans. A.S.C.E. **102,** 1020 (1937).

[4] R. V. Southwell, *Relaxation methods in engineering science*, Oxford University Press, 1940.

of a framework, we assume that all joints of the framework (in Fig. 1, for example) are fixed against rotation; and determine the resulting fixed-end moments acting at the ends of each member. Built-in, simply-supported or free ends are not considered as joints. Then at any joint, say D, a moment equal but opposite in sign to the sum of its fixed-end moments is applied, representing the effect of releasing the joint. This moment is distributed to the members AD, CD, BD, ED, meeting at D, in proportion to their stiffnesses, since all members meeting at D rotate through the same angle. The share falling to each member is called the "balancing moment" acting at the end D of this member. The joint D is now balanced, but the new balancing moment M_{DA} acting at the end D of AD will induce an additional moment

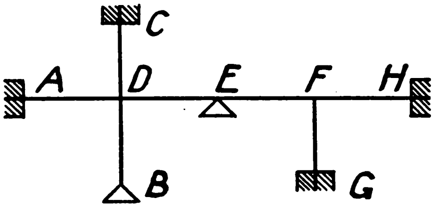

FIG. 1.

$$M_{AD} = C_{AD} M_{DA}$$

at the opposite end A. C_{AD} is the carry-over factor for the member AD, and the moment M_{DA} is said to be "carried over." Likewise, moments are carried over to C and E, but none to B since $C_{DB}=0$. The joint D is again locked—this time in its balanced position—and the process repeated for all joints of the framework until the balancing moments are negligible. The order of choosing unbalanced joints for balancing is not obligatory, but usually the joint with the largest total unbalanced moment at any given stage is balanced. Signs of the moments are chosen so that a positive moment acting on the end of the beam tends to rotate it in a clockwise direction. Likewise, a rotation in the clockwise direction is considered positive.

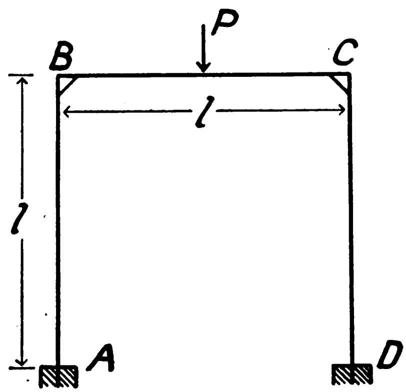

FIG. 2.

Example 1. As a simple example consider the rectangular bent formed of uniform and equal bars, illustrated in Fig. 2. All of the bars are of equal stiffness and the carry-over factor in each case is 1/2. The only non-vanishing fixed-end moments are $-.125Pl$ and $.125Pl$ at the left and right ends of the horizontal bar. The calculations used in the method of moment balancing are shown in Table I. In a given column, say that headed M_{CB}/Pl, we find recorded successively the fixed-end moment and the balancing moment. These are added, and since at this stage $M_{CB}+M_{CD}=0$, the joint C is balanced. The balancing moment has been carried over to column M_{BC}/Pl, and the joint B is balanced next. The same steps are followed until after five balancings the moments to be carried over are negligible. The results obtained agree with those computed by other methods.

2. Dynamics of a simple beam. It is clear that if we can set up analogous definitions for fixed-end moment, carry-over factor and stiffness for a beam on which an oscillating force is acting, and if we can find these characteristics for the oscillating beam, it may be possible to use the method of balancing moments just as it is in the dynamic case. A procedure adapted to this purpose can be found in an article of W. Prager's,[5] the essentials of which will be given here.

TABLE I.

	B		C		
M_{AB}/Pl	M_{BA}/Pl	M_{BC}/Pl	M_{CB}/Pl	M_{CD}/Pl	M_{DC}/Pl
.000	.000	−.125	.125	.000	.000
			−.063	−.062	
		−.032			−.031
.000	.000	−.157	.062	−.062	−.031
	.078	.079			
.039			.040		
.039	.078	−.078	.102	−.062	−.031
			−.020	−.020	
		−.010			−.010
.039	.078	−.088	.082	−.082	−.041
	.005	.005			
.002			.002		
.041	.083	−.083	.084	−.082	−.041
			−.001	−.001	
.041	.083	−.083	.083	−.083	−.041

The differential equation for the deflection, $y(x, t)$, of a uniform beam with no external load is taken as

$$\mu \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0,$$

where μ is the mass per unit length of the beam and EI is its flexural rigidity (Fig. 3). Following a well-known procedure we write $y(x, t) = u(x) \cos \omega t$, $u(x)$ being the amplitude of the assumed harmonic motion and ω its circular frequency. Hence

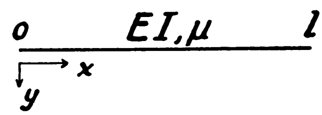

FIG. 3.

$$\frac{d^4 u}{dx^4} + n^4 u = 0,$$

where $n^4 = \omega^2 \mu / EI$; and from this equation

$$u(x) = A \cosh nx + B \sinh nx + C \cos nx + D \sin nx.$$

It is convenient to express the four constants of integration in terms of four quantities of immediate physical importance: the *amplitudes* of the moments acting on the ends of the beam, and of the displacements at the ends of the beam. This can be done by use of the relations

[5] W. Prager, *Die Beanspruchung von Tragwerken durch schwingende Lasten,* Ingenieur-Archiv **1,** 527 (1930).

$$u_0 = A + C$$

$$u_1 = A \cosh \lambda + B \sinh \lambda + C \cos \lambda + D \sin \lambda,$$

$$M_0 = -EI \left.\frac{d^2u}{dx^2}\right]_{x=0} = -A - C\} EIn^2, \{$$

$$M_1 = EI \left.\frac{d^2u}{dx^2}\right]_{x=l} = \{A \cosh \lambda + B \sinh \lambda - C \cos \lambda - D \sin \lambda\} EIn^2,$$

where $\lambda = nl$. It can be seen that the amplitudes of the deflection, angle of rotation, bending moment and shear for any value of x will involve linearly the amplitudes of the end deflections and end moments. These quantities can be expressed in much simpler form if the following functions and abbreviations are introduced:

$$\phi(\lambda) = (\coth \lambda - \cot \lambda)/2\lambda, \qquad \bar{\phi}(\lambda) = \frac{\lambda}{2} (\coth \lambda + \cot \lambda),$$

$$\psi(\lambda) = -(\operatorname{csch} \lambda - \csc \lambda)/2\lambda, \qquad \bar{\psi}(\lambda) = \frac{\lambda}{2} (\operatorname{csch} \lambda + \csc \lambda),$$

$$l' = l/EI.$$

Then we find the following expressions for the amplitudes of the angles of rotation at the end points (Fig. 4):

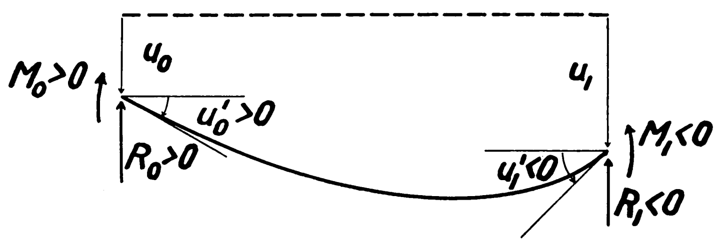

Fig. 4.

$$u_0' = -\frac{u_0\bar{\phi}(\lambda)}{l} + \frac{u_1\bar{\psi}(\lambda)}{l} + M_0 l'\phi(\lambda) - M_1 l'\psi(\lambda), \tag{1}$$

$$u_1' = -\frac{u_0\bar{\psi}(\lambda)}{l} + \frac{u_1\bar{\phi}(\lambda)}{l} - M_0 l'\psi(\lambda) + M_1 l'\phi(\lambda); \tag{2}$$

and for the amplitudes of the reactions:

$$R_0 = -EIu_0''' = u_0 \frac{\lambda^4}{l^2 l'} \phi(\lambda) + u_1 \frac{\lambda^4}{l^2 l'} \psi(\lambda) - \frac{M_0}{l} \bar{\phi}(\lambda) - \frac{M_1}{l} \bar{\psi}(\lambda), \tag{3}$$

$$R_1 = -EIu_1''' = -u_0 \frac{\lambda^4}{l^2 l'} \psi(\lambda) - u_1 \frac{\lambda^4}{l^2 l'} \phi(\lambda) - \frac{M_0}{l} \bar{\psi}(\lambda) - \frac{M_1}{l} \bar{\phi}(\lambda). \tag{4}$$

If the beam, simply supported at both ends, is loaded at its center by an oscillating load, $P \cos \omega t$ (Fig. 5),

$$M_0 = u_0 = u'\left(\frac{l}{2}\right) = 0, \qquad R\left(\frac{l}{2}\right) = P/2.$$

Then, from (2) and (4),

$$\frac{2}{l}\, u\left(\frac{l}{2}\right)\bar{\phi}\left(\frac{\lambda}{2}\right) + \frac{l'}{2}\, M\left(\frac{l}{2}\right)\phi\left(\frac{\lambda}{2}\right) = 0,$$

$$-\frac{1}{2}\, u\left(\frac{l}{2}\right)\frac{\lambda^4}{l^2 l'}\,\phi\left(\frac{\lambda}{2}\right) - \frac{2}{l}\, M\left(\frac{l}{2}\right)\bar{\phi}\left(\frac{\lambda}{2}\right) = \frac{P}{2};$$

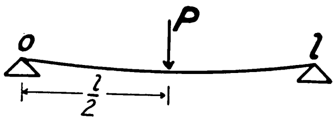

FIG. 5.

so that

$$u\left(\frac{l}{2}\right) = Pl^2 l'\,\Phi(\lambda), \tag{5}$$

$$M\left(\frac{l}{2}\right) = -\,Pl\bar{\Phi}(\lambda), \tag{6}$$

where

$$\Phi(\lambda) = -\,(\tanh \tfrac{1}{2}\lambda - \tan \tfrac{1}{2}\lambda)/4\lambda^3, \qquad \bar{\Phi}(\lambda) = (\tanh \tfrac{1}{2}\lambda + \tan \tfrac{1}{2}\lambda)/4\lambda.$$

Also, from formulas (5) and (6), and with (1)–(4), we find that

$$u_0' = Pll'\Psi(\lambda) = -\,u_1', \tag{7}$$

and

$$R_0 = P\bar{\Psi}(\lambda) = -\,R_1, \tag{8}$$

where

$$\Psi(\lambda) = -\,(\operatorname{sech} \tfrac{1}{2}\lambda - \sec \tfrac{1}{2}\lambda)4\lambda^2, \qquad \bar{\Psi}(\lambda) = (\operatorname{sech} \tfrac{1}{2}\lambda + \sec \tfrac{1}{2}\lambda)/4.$$

Now, if the beam in question is on unyielding supports but has moments acting on its ends in addition to the load acting at its center, we find by addition of (1) and (7) that

$$u_0' = M_0 l'\phi(\lambda) - M_1 l'\psi(\lambda) + Pll'\Psi(\lambda); \tag{9}$$

and, similarly

$$u_1' = -\,M_0 l'\psi(\lambda) + M_1 l'\phi(\lambda) - Pll'\Psi(\lambda). \tag{10}$$

Obviously, with the equations derived, problems in dynamics of frameworks are reduced to problems in statics of frameworks. To facilitate this work, tables of the functions $\phi(\lambda)$, $\bar{\phi}(\lambda)$, $\psi(\lambda)$, $\bar{\psi}(\lambda)$, $\Phi(\lambda)$, $\bar{\Phi}(\lambda)$, $\Psi(\lambda)$ and $\bar{\Psi}(\lambda)$ are available.[6]

3. Dynamic moment balancing. By substituting "moment-amplitude" and "rotation-amplitude" for "moment" and "rotation," respectively, wherever they occur in the definitions of fixed-end moment, stiffness and carry-over factor, we arrive at suitable definitions for the corresponding dynamic quantities. These three quantities will give us a basis for the application of the moment balancing method to problems in dynamics of frameworks.

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[6] K. Hohenemser and W. Prager, *Dynamik der Stabwerke*, Julius Springer, Berlin, 1933.

First let us consider the amplitudes of the moments acting at the ends of a centrally loaded built-in beam (Fig. 6). Equations (9) and (10) can be applied, and we find that

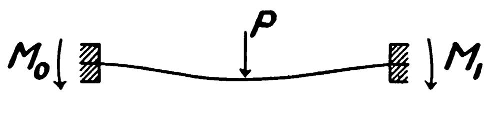

FIG. 6.

$$M_0\phi(\lambda) - M_1\psi(\lambda) = -Pl\Psi(\lambda),$$

and

$$M_0\psi(\lambda) - M_1\phi(\lambda) = -Pl\Psi(\lambda).$$

Then, since $\phi(\lambda)+\psi(\lambda)=2\bar{\Phi}(\lambda)$, the amplitudes of the moment are given by the relations

$$M_0 = -M_1 = -\frac{Pl\Psi(\lambda)}{2\bar{\Phi}(\lambda)}. \tag{11}$$

These quantities give the amplitudes of the fixed-end moments for a beam loaded at its center with a load of amplitude P.

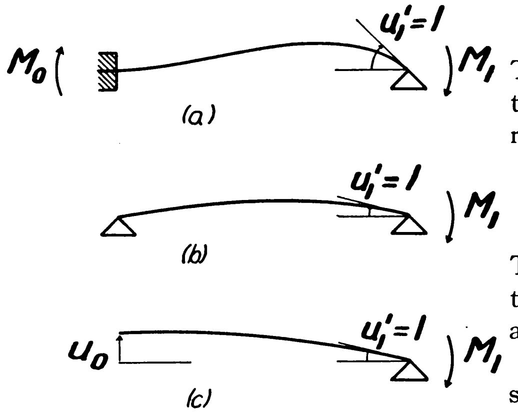

FIG. 7.

The problem of finding the dynamic stiffness is illustrated in Fig. 7. If the far end of the beam is built-in (Fig. 7*a*), we find from (2) that

$$-M_0l'\psi(\lambda) + M_1l'\phi(\lambda) = 1,$$

and from (9)

$$M_0\phi(\lambda) - M_1\psi(\lambda) = 0.$$

Since M_1 is by definition the stiffness, K,

$$K = \frac{\phi(\lambda)}{l'\{[\phi(\lambda)]^2 - [\psi(\lambda)]^2\}} = -\frac{\lambda B(\lambda)}{l'D(\lambda)}, \tag{12}$$

where

$$B(\lambda) = \cosh\lambda\sin\lambda - \cos\lambda\sinh\lambda,$$
$$D(\lambda) = \cosh\lambda\cos\lambda - 1.$$

Tables exist for these functions and for the quotient $B(\lambda)/D(\lambda)$.[6]

For a beam on two simple supports (Fig. 7*b*), equation (2) gives

$$1 = M_1l'\phi(\lambda), \quad \text{so} \quad K = 1/l'\phi(\lambda).$$

To find the stiffness of the cantilever beam (Fig. 7*c*), we find from (2) and (3) that

$$-\frac{u_0}{l}\bar{\psi}(\lambda) + M_1l'\phi(\lambda) = 1,$$

$$u_0\frac{\lambda^4}{l^2l'}\phi(\lambda) - \frac{M_1}{l}\bar{\psi}(\lambda) = 0.$$

Hence,

$$K = \frac{\lambda^4\phi(\lambda)}{l'\{\lambda^4[\phi(\lambda)]^2 - [\bar{\psi}(\lambda)]^2\}}. \tag{13}$$

The carry-over factor needs to be found only for the case illustrated in Fig. 7*a*, since it is zero in the other two cases.

From equation (1)

$$M_0\phi(\lambda) - M_1\psi(\lambda) = 0,$$

so that the carry-over factor is defined by

$$C = \frac{M_0}{M_1} = \frac{\psi(\lambda)}{\phi(\lambda)} \cdot \tag{14}$$

We are now equipped to apply the moment balancing method to problems in dynamics of frameworks.

Example 2. Consider again the bent illustrated in Fig. 2, but now suppose that the frequency ω has a value such that $\lambda = 3.30$ for each bar. Then we have fixed-end moment-amplitudes of $-.169Pl$ and $.169Pl$ at the left and right ends of the horizontal bar, equal carry-over factors of 1.22, and equal stiffnesses for each bar. Table II gives the calculations involved in solving this problem. The values obtained from the 12 balancings are correct to two significant figures.

4. Dynamic balancing of angle changes. The application of the results of Section 2 to L. E. Grinter's method of balancing angle changes[3] is not difficult. In balancing a given joint, the members of the framework meeting at the joint are assumed to be simply-supported and disconnected there. Then rotations are forced by means of applied moments until the angular discontinuities between the members are negligible. To work with rotations rather than moments we require two more definitions.

By "angle-change" will be meant the change in slope produced at the end of a member either by loads or by an applied end moment.

The "angle carry-over factor" is the numerical value of the angle change induced at one end of a member by a unit angle-change imposed upon the other end.

The amplitudes of the angle changes, at the ends of a simply supported beam, due to a central load of amplitude P are seen from (7) to be

$$u_0' = - u_1' = Pll'\Psi(\lambda).$$

The angle carry-over factor can be found by consideration of a simply supported beam, one of whose ends is rotated by means of an applied moment-amplitude (Fig. 8). From equations (1) and (2), $u_1' = -u_0' \ \psi(\lambda)/\phi(\lambda)$ so that the angle carry-over factor is $\psi(\lambda)/\phi(\lambda)$.

Similarly, by consideration of equations (1), (2) and (4) we can arrive at an angle carry-over factor for a cantilever beam:

FIG. 8.

$$C = \frac{\bar{\phi}(\lambda)\bar{\psi}(\lambda) + \lambda^4\phi(\lambda)\psi(\lambda)}{[\bar{\psi}(\lambda)]^2 - \lambda^4[\phi(\lambda)]^2} \cdot$$

Since $\bar{\phi}(0) = \bar{\psi}(0) = 1$, it is seen that this reduces to unity in the static case.

Continuity is established between a member and a joint by giving the joint a rotation-amplitude $K_i\theta_i/\Sigma K$, where θ_i is the amplitude of the angle change in the ith

TABLE II.

	B		*C*		
M_{AB}/Pl	M_{BA}/Pl	M_{BC}/Pl	M_{CB}/Pl	M_{CD}/Pl	M_{DC}/Pl
.000	.000	−.169	.169	.000	.000
			−.084	−.085	
		−.103			−.103
———	———	———	———	———	———
.000	.000	−.272	.085	−.085	−.103
	.136	.136			
.166			.166		
———	———	———	———	———	———
.166	.136	−.136	.251	−.085	−.103
			−.083	−.083	
		−.101			−.101
———	———	———	———	———	———
.166	.136	−.237	.168	−.168	−.204
	.051	.050			
.062			.062		
———	———	———	———	———	———
.228	.187	−.187	.230	−.168	−.204
			−.031	−.031	
		−.038			−.038
———	———	———	———	———	———
.228	.187	−.225	.199	−.199	−.242
	.019	.019			
.023			.023		
———	———	———	———	———	———
.251	.206	−.206	.222	−.199	−.242
			−.011	−.012	
		−.014			−.014
———	———	———	———	———	———
.251	.206	−.220	.211	−.211	−.256
	.007	.007			
.009			.009		
———	———	———	———	———	———
.260	.213	−.213	.220	−.211	−.256
			−.005	−.004	
		−.005			−.005
———	———	———	———	———	———
.260	.213	−.218	.215	−.215	−.261
	.002	.003			
.003			.003		
———	———	———	———	———	———
.263	.215	−.215	.218	−.215	−.261
			−.001	−.002	
		−.002			−.002
———	———	———	———	———	———
.263	.215	−.217	.217	−.217	−.263
	.001	.001			
———	———	———	———	———	———
.263	.216	−.216	.217	−.217	−.263

member, K_i its stiffness, and the summation extends over all bars meeting at the joint; and at the same time the end of the member itself is given a rotation-amplitude $-[\theta_i - K_i\theta_i/\Sigma K]$. This is done for each member meeting at the joint, thereby balancing that joint; then the assigned rotation-amplitudes are carried over, and the balancing process continues.

After the rotation-amplitudes for all joints have been found with the desired accuracy, the moment-amplitudes can be found from a combination of (9) and (10):

$$M_0 = \frac{u_0'\phi(\lambda) + u_1'\psi(\lambda)}{l'\{[\phi(\lambda)]^2 - [\psi(\lambda)]^2\}} - \frac{Pl\Psi(\lambda)}{2\bar{\Phi}(\lambda)}, \tag{15}$$

$$M_1 = \frac{u_0'\psi(\lambda) + u_1'\phi(\lambda)}{l'\{[\phi(\lambda)]^2 - [\psi(\lambda)]^2\}} + \frac{Pl\Psi(\lambda)}{2\bar{\Phi}(\lambda)}. \tag{16}$$

TABLE III.

B		*C*	
θ_{BA}/Pll'	θ_{BC}/Pll'	θ_{CB}/Pll'	θ_{CD}/Pll'
.0000	.0625	−.0625	.0000
		.0357	−.0268
	−.0178		
.0000	.0447	−.0268	−.0268
.0192	−.0225		
		.0128	
.0192	.0192	−.0140	−.0268
		−.0073	.0055
	.0036		
.0192	.0228	−.0213	−.0213
.0016	−.0020		
		.0010	
.0208	.0208	−.0203	−.0213
		−.0006	.0004
	.0003		
.0208	.0211	−.0209	−.0209
.0001	−.0002		
		.0001	
.0209	.0209	−.0208	−.0209
		−.0001	
.0209	.0209	−.0209	−.0209

$M_{AB} = .0418\ Pl$ $M_{BA} = .0836\ Pl$ $M_{CB} = .0832\ Pl$

$M_{DC} = -.0419\ Pl$ $M_{BC} = -.0832\ Pl$ $M_{CD} = -.0836\ Pl$

It is interesting to observe that when $\lambda=0$ equations (15) and (16) reduce to the slope-deflection equations for a centrally loaded beam.[7]

Example 3. As an illustration let us solve Example 1 by the method of balancing angle changes. For this method, the stiffness of the horizontal bar will be the moment required to produce unit rotation of one end while the other end is simply-supported; while the stiffness of a vertical bar requires the other end to be built-in. Hence the ratio of the stiffness of the horizontal bar to the stiffness of a vertical bar is 3/4. If all joints are assumed to be pin-connected, we have angle changes $\theta_{BC}=.0625Pll'$ and $\theta_{CB}=-.0625Pll'$ due to the load P. For simplicity, u_i' is replaced by θ_i.

In Table III we find the computation used in solving this example. Joint C is balanced first by rotating the member CB through the angle $-(1-\frac{3}{7})(-.0625)Pll'$ $=.0357Pll'$ and the other members of the joint (that is, CD) through the angle $\frac{3}{7}(-.0625)Pll'=-.0268Pll'$. Continuity at that joint is then established, but the rotation of CD induces a rotation at the other end of the beam, $\theta_{BC}=-.178Pll'$. This leaves a total unbalance of $.0447Pll'$ at joint B, which is balanced next. These balancings continue until the angle changes to be carried over are negligible. The resulting moments, computed from (15) and (16) are also listed, and compare favorably with the results obtained for Example 1 by moment balancing.

Example 4. If, now, ω has a value such that $\lambda=3.00$, we find angle changes $\theta_{BC}=.381Pll'=-\theta_{CB}$ due to the load P $\cos\omega t$. Furthermore the angle carry-over factor for the horizontal bar is $-.872$, and as to stiffnesses, $K_{BC}=.549\,l'$, $K_{AB}=K_{CD}$ $=3.102\,l'$. Table IV gives the computation involved in 12 balancings of angle changes in this case. The values of the moment-amplitudes obtained are compared with those obtained by moment balancing.

5. Convergence of the moment balancing process. Convergence of the process of moment balancing can be assured if the frequency of the forced vibration is smaller than the first natural frequency of the structure. The first step of the method of moment balancing leads to the determination of the amplitudes of the unbalanced moments. For the following steps these unbalanced moments are considered as exterior couples acting on the joints of the structure. In the type of structure considered here (joints fixed against translation) the amplitudes of displacement and bending moment of any member are completely determined by the frequency ω and the rotation-amplitudes at the two ends of the member. If a set of values of the rotation-amplitudes at the n joints of the structure is assumed, it is therefore possible to compute the amplitudes of the periodic couples which must be applied to the joints in order to produce the assumed rotation-amplitudes. Let $\theta_i=u_i'$, $(i=1, 2, \cdots, n)$, be the rotation-amplitudes and A_i the corresponding amplitudes of the couples. Furthermore, let B_i be the amplitudes of the exterior couples obtained by the first step of the method of moment balancing. Then, if the assumed θ_i represent the actual configuration enforced by the loads B_i, $A_i-B_i=0$; but in general

$$A_i - B_i = C_i, \tag{17}$$

where C_i is the residual moment-amplitude.

Amongst all possible systems θ_i the actual one minimizes the energy function

[7] See, for example, J. I. Parcel and G. A. Maney, *Statically indeterminate stresses*, John Wiley and Sons, 1936, p. 149.

TABLE IV.

θ_{BA}/Pll'	θ_{BC}/Pll'	θ_{CB}/Pll'	θ_{CD}/Pll'
.000	.381	−.381	.000
		.307	−.074
	−.268		
———	———	———	———
.000	.113	−.074	−.074
.022	−.091		
		.079	
———	———	———	———
.022	.022	.005	−.074
		−.064	.015
	.055		
———	———	———	———
.022	.077	−.059	−.059
.011	−.044		
		.038	
———	———	———	———
.033	.033	−.021	−.059
		−.030	.008
	.027		
———	———	———	———
.033	.060	−.051	−.051
.005	−.022		
		.019	
———	———	———	———
.038	.038	−.032	−.051
		−.015	.004
	.013		
———	———	———	———
.038	.051	−.047	−.047
.003	−.010		
		.008	
———	———	———	———
.041	.041	−.039	−.047
		−.006	.002
	.006		
———	———	———	———
.041	.047	−.045	−.045
.001	−.005		
		.005	
———	———	———	———
.042	.042	−.040	−.045
		−.004	.001
	.003		
———	———	———	———
.042	.045	−.044	−.044
.001	−.002		
———	———	———	———
.043	.043	−.044	−.044

$\theta_{AB}=\theta_{DC}=0.$

	Balancing angle changes	Balancing moments
M_{AB}/Pl	.116	.118
M_{BA}/Pl	.134	.135
M_{BC}/Pl	−.138	−.135
M_{CB}/Pl	.132	.135
M_{CD}/Pl	−.136	−.135
M_{DC}/Pl	−.119	−.118

$$H = \tfrac{1}{2}\sum_{i,k=1}^{n} \alpha_{ik}\theta_i\theta_k - \sum_{k=1}^{n} B_k\theta_k. \tag{18}$$

The first term on the right side represents the internal energy,

$$\tfrac{1}{2}\sum_{i,k=1}^{n} \alpha_{ik}\theta_i\theta_k = \sum\left\{EI\int_0^l (u'')^2dx - \mu\omega^2\int_0^l u^2dx\right\}, \tag{19}$$

where the right hand sum is to be taken over all members of the structure. The relation (19) arises from the fact that, for any member, u'' and u can be expressed linearly in terms of the rotation-amplitudes at the ends of this member. Note that $\alpha_{ik}=\alpha_{ki}$ and $A_i=\sum_{k=1}^{n}\alpha_{ik}\theta_k$.

Let us denote the first natural frequency of the structure by ω_1. The values $\alpha_{11}, \alpha_{22}, \cdots, \alpha_{nn}$ then can be shown to be positive as long as $\omega<\omega_1$. Indeed, by Rayleigh's principle

$$\omega_1^2 \leqq \left[\sum EI\int_0^l (u'')^2\,dx\right]\Big/\left[\sum \mu\int_0^l u^2dx\right], \tag{20}$$

where the sums are to be extended over all members of the structure. As the function u in (20) let us take the displacements corresponding to $\theta_1=1$, $\theta_2=\theta_3=\cdots=\theta_n=0$. From (19) and (20) together with the condition $\omega<\omega_1$ it is then clear that $\alpha_{11}>0$. Similarly $\alpha_{22}>0, \alpha_{33}>0, \cdots, \alpha_{nn}>0$.

Let a first set of values $\theta_i=\theta_i^{(1)}$ be given and compute the corresponding residual moment-amplitudes $C_i^{(1)}$. Suppose the subscripts $1, 2, \cdots, n$ to be arranged in such a manner that $|C_1^{(1)}|\geqq|C_i^{(1)}|$, $(i=2, 3, \cdots, n)$. We now define a second set of values $\theta_i^{(2)}$ which differs from the first one only in so far as the value of θ_1 is concerned:

$$\theta_1^{(2)} = \theta_1^{(1)} + \phi, \qquad \theta_i^{(2)} = \theta_i^{(1)}, \qquad (i = 2, 3, \cdots, n).$$

We propose to determine ϕ in such a manner that the value of H is decreased.[8] We have

$$H(\theta^{(2)}) - H(\theta^{(1)}) = \left[\sum_{k=1}^{n} \alpha_{1k}\theta_k^{(1)} - B_1\right]\phi + \frac{\alpha_{11}}{2}\phi^2 = C_1^{(1)}\phi + \frac{\alpha_{11}}{2}\phi^2. \tag{21}$$

Taking

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[8] See G. Temple, *The General theory of relaxation methods applied to linear systems*, Proc. R. Soc. of London, Ser. A, **169**, 476, (1938–1939).

$$\phi = -\frac{C_1^{(1)}}{\alpha_{11}} \tag{22}$$

we obtain

$$H(\theta^{(2)}) - H(\theta^{(1)}) = -\tfrac{1}{2}[C_1^{(1)}]^2/\alpha_{11} \tag{23}$$

which is certainly negative as long as $\omega < \omega_1$. The residual moment-amplitude $C_1^{(2)}$ corresponding to the new values $\theta_i^{(2)}$ equals

$$C_1^{(2)} = \sum_{k=1}^{n} \alpha_{1k}\theta_k^{(2)} - B_1 = C_1^{(1)} + \alpha_{11}\phi = 0.$$

This shows that the choice of ϕ according to (22) corresponds precisely to the process of moment balancing where in each step the greatest absolute residual moment is "liquidated." For the next step the subscripts i have to be rearranged, so that $C_1^{(2)}$ is the greatest absolute residual moment. Continuing in this way we obtain a decreasing sequence of values of H. If we simplify our notation by writing $H^{(p)}$ instead of $H(\theta^{(p)})$, this sequence becomes

$$H^{(1)} > H^{(2)} > \cdots > H^{(p)} > \cdots > H_{\min},$$

with

$$H^{(p+1)} - H^{(p)} = -\tfrac{1}{2}[C_1^{(p)}]^2/\alpha_{11}^{(p)} < 0.$$

Here $\alpha_{11}^{(p)}$ has been written instead of the α_{11} of (23), since as a consequence of the rearrangement of the subscripts the value of this quantity changes from step to step. Now $\alpha_{11}^{(p)}$ is positive and can assume only a finite number of different values (n at the most). Furthermore, the sequence $H^{(p)}$ is decreasing monotonically and is bounded from below by $H_{\min}$. Therefore

$$\lim_{p\to\infty} [C_1^{(p)}]^2 = 0.$$

Since $C_1^{(p)}$ is the greatest absolute residual moment in the pth step, this means that ultimately all residual moments will disappear. The structure is then completely balanced.

This convergence may be rather slow, especially if ω is near ω_1. For example, compare the 12 balancings used in Example 2, when $\lambda = 3.30$, to the 5 needed in Example 1, for the same structure when $\lambda = 0$. For this structure $\lambda_1 \doteq 3.55$.

The method of balancing angle changes may not always converge when $\omega < \omega_1$, as will be seen if Example 4 is attempted when $\lambda = 3.30$. Usually the method of balancing moments converges more rapidly than the method of balancing angle changes.