STABILITY OF COLUMNS AND STRINGS UNDER PERIODICALLY VARYING FORCES*

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1. Introduction. It is a well known fact that a rigid body hinged at one end and standing vertically can be put into stable equilibrium by applying a vertical periodic force of proper frequency and amplitude at the lower end. The differential equation for small oscillations of the rod is a linear homogeneous equation with a periodic coefficient—it is a Mathieu equation if the applied force is a simple sine or cosine function of the time. Stability of the rod would require that all solutions of this equation be bounded; it is found that this is the case if the frequency and amplitude of the applied force are properly chosen. A more complicated problem of the same general type in a system with more than one degree of freedom has been considered by G. Hamel [4][1]; linear differential equations with periodic coefficients play the essential role in this case also.

We shall be interested here in analogous problems in elastic systems with infinitely many degrees of freedom. One of these is the problem of the column under periodic compressive forces $F(t)$ applied at the ends of the column.[2] The analogue of the problems mentioned above would be as follows: the force $F(t)$ consists of a constant part P plus a periodic part $H \cos \omega t$. Suppose that P were a compressive force larger than the lowest compressive load (the Euler load) for which the column in the original unbent position is instable. The question is, then, whether or not H and ω can be chosen in such a way that small motions in the neighborhood of the undeflected position are stable ones. We shall see that this can always be done, though, as one would expect, the quantity H must be chosen so that the total force $F(t)$ falls below the Euler value during at least part of the time. However, the time average of F (over a cycle) may be very much larger than the Euler load. On the other hand, it is quite possible that the column may be *instable* when P is a compressive force smaller than the Euler load or when P is a tension rather than a compression, if H and ω are properly chosen.[3] From the point of view of the practical applications these latter possibilities are certainly the more important ones. For the case of the column with pinned ends we give diagrams which make it possible to decide whether the column is stable or not under any of these circumstances. The stability of the stretched string under a tension which varies periodically in time is also considered.

In all of these problems the Mathieu equation[4] (more properly, a *sequence* of

* Received April 9, 1943.

[1] Numbers in square brackets refer to the bibliography at the end.

[2] A special case of this problem has been treated by I. Utida and K. Sezawa [16].

[3] Analogous problems for plates under loads in the plane of the plate have been considered by R. Einaudi [1].

[4] We consider always that the applied forces are simple harmonic functions of the time—otherwise we should have to deal with the more general Hill's equation.

Mathieu equations in the continuous systems) plays a central rôle, since the decision as to stability depends upon the character of the solutions of such equations. For this reason a brief summary of the main facts concerning the solutions of the Mathieu equation is included here. A brief treatment of the Mathieu equation with a viscous damping term added is also included because of its importance for the stability problem.

2. The column under periodic axial forces at its ends. We make the assumptions that are customary in dealing with the transverse oscillations of thin rods. Of these, the principal ones are: 1) the rod is an initially straight uniform cylinder, 2) the lateral deflection w (Fig. 1) and the cross sectional dimensions of the beam are small in comparison with the length l, 3) all stresses remain below the proportional limit,

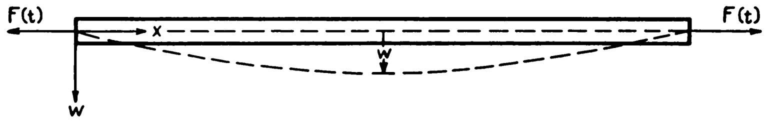

FIG. 1.

4) the effects of shear and rotary inertia are negligible.[5] In addition, we assume that the column is subjected to axial forces F depending on the time t and applied at the ends of the column; these forces are counted positive when they are tensions. With these assumptions the differential equation for the lateral deflection $w(x, t)$ is well known to be as follows:

$$EI \frac{\partial^4 w}{\partial x^4} - F(t) \frac{\partial^2 w}{\partial x^2} + m \frac{\partial^2 w}{\partial t^2} = 0. \tag{2.1}$$

In this equation E and I are Young's modulus of the column and the moment of inertia of its cross section, and m is the mass per unit length. In what follows we assume always that $F(t)$ is given by

$$F(t) = P + H \cos 2\pi f t; \tag{2.2}$$

i.e., it consists of a constant part plus a harmonic component of amplitude H and frequency f.

It should be pointed out that the derivation of (2.1) involved a tacit assumption not included among those enumerated above. This was that the forces $F(t)$ applied at the ends of the column result in forces throughout the column which are, to a sufficiently close approximation, independent of x. We proceed to show that this assumption is warranted under the circumstances normally encountered in practice. The differential equation for the longitudinal displacement $u(x, t)$ of the rod is

$$E \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2}, \tag{2.3}$$

in which ρ is the density of the rod. The total force F transmitted through any cross section of the rod of area A is given by

[5] These effects could be taken into account without difficulty, but nothing new in principle would result.

$$F = AE\frac{\partial u}{\partial x}\cdot \tag{2.4}$$

We assume as boundary conditions

$$u = 0 \quad \text{at} \quad x = 0, \tag{2.5}$$

and

$$F = AE\frac{\partial u}{\partial x} = P + H\cos 2\pi ft \quad \text{at} \quad x = l/2, \tag{2.6}$$

the origin of coordinates being taken at the midpoint of the rod in order to take advantage of symmetry. We seek the forced oscillation and neglect the free oscillation. The result for the quantity F is readily found to be

$$F(x, t) = P + H\frac{\cos \lambda x}{\cos (\lambda l/2)}\cos 2\pi ft, \tag{2.7}$$

with

$$\lambda = 2\pi f(\rho/E)^{1/2}. \tag{2.8}$$

It is convenient to introduce the fundamental frequency f_0 of the free longitudinal vibration of the rod which has a single node at the center. This is given by

$$f_0 = (1/2l)(E/\rho)^{1/2}. \tag{2.9}$$

Upon introducing this into (2.7) we obtain

$$F(x, t) = P + H\frac{\cos (\pi fx/f_0 l)}{\cos (\pi f/2f_0)}\cos 2\pi ft. \tag{2.10}$$

If f is small compared with f_0 it is clear that F will be nearly independent of x. For steel or aluminum $(E/\rho)^{1/2} = 17000$ ft./sec., while for brass, concrete, stone, or wood this quantity is about 12000 ft./sec. For any column of usual length f_0 will therefore be of the order of 500 cycles/sec. or more. Hence if the applied axial force $F(t)$ is one of frequency below say 50 cycles/sec. it is reasonable to assume that the variation of the axial force with x may be neglected.

We introduce new independent variables replacing t and x in (2.1) by the equations

$$\vartheta = 2\pi ft \quad \text{and} \quad \xi = \pi x/l. \tag{2.11}$$

In addition, it is convenient to introduce new parameters as follows:

$$P_E = \pi^2 EI/l^2, \qquad \epsilon_0 = P_E/EA, \tag{2.12}$$

$$p = P/P_E, \qquad h = H/P_E. \tag{2.13}$$

The quantity P_E is the negative of the Euler load for the column and ϵ_0 is the tensile strain due to that load. The quantities p and h are the ratios of the constant part and of the amplitude of the oscillating part of the applied load to the negative Euler load. With these new quantities the differential equation (2.1) becomes

$$\frac{\partial^4 w}{\partial \xi^4} - (p + h\cos\vartheta)\frac{\partial^2 w}{\partial \xi^2} + (f^2/f_0^2\epsilon_0)\frac{\partial^2 w}{\partial \vartheta^2} = 0. \tag{2.14}$$

The quantity f_0 is the fundamental frequency of longitudinal vibration of the column given by (2.9).

The general problem which we wish to investigate can now be stated: for given boundary conditions there are certain values of p, h, and f for which all solutions $w(\xi, \vartheta)$ of (2.14) remain bounded when arbitrary initial conditions are prescribed and other values of these quantities for which unbounded solutions exist. In the former case we say that the column is stable and refer to p, h, and f in this case as stable values. Our problem is to separate the stable from the instable values of p, h, and f.

We do not solve the problem in this generality; we choose rather a special case with regard to the boundary conditions to be imposed.

3. Formulation of the stability problem for the column with pinned ends. The boundary conditions we choose are those corresponding to the case of a column with pinned ends; that is, we assume that the deflection w and bending moment $M = EI(\partial^2 w/\partial x^2)$ are both zero at $x=0$ and $x=l$. We have, therefore, as boundary conditions for (2.14):

$$w = \frac{\partial^2 w}{\partial \xi^2} = 0 \quad \text{for} \quad \xi = 0 \quad \text{and} \quad \xi = \pi. \tag{3.1}$$

These boundary conditions can be satisfied by taking for w a solution in the form of a Fourier sine series:

$$w = \sum_{n=1}^{\infty} F_n(\vartheta) \sin n\xi. \tag{3.2}$$

The series (assuming that it converges properly) is a solution of (2.14) provided that the function $F_n(\vartheta)$ satisfies the differential equation

$$\frac{d^2 F_n}{d\vartheta^2} + (\alpha_n + \beta_n \cos \vartheta) F_n = 0, \qquad n = 1, 2, 3, \cdots, \tag{3.3}$$

in which

$$\alpha_n = n^2(f_0^2/\epsilon_0/f^2)(n^2 + p) \tag{3.4}$$

and

$$\beta_n = n^2(f_0^2\,\epsilon_0/f^2)(h). \tag{3.5}$$

The quantities f, f_0, ϵ_0, p, and h have been defined by equations (2.2), (2.9), (2.12), and (2.13) respectively. The differential equation (3.3) is, of course, a Mathieu equation.

We can now see why the choice of the boundary conditions (3.1) brings with it essential simplifications. To begin with, it is not possible to separate the variables in (2.14) in the usual way: if we insert for w in (2.14) an expression of the form $w = f(\xi)F(\vartheta)$ we do not obtain a pair of ordinary differential equations for f and F alone. By assuming for w the *special* form given in (3.2) we are able to satisfy (2.14) by virtue of the fact that only even ordered derivatives of w with respect to ξ occur in it. This form of solution is, however, not useful for boundary conditions other than those given by (3.1).[6] The reason for this is as follows: since w satisfies (2.14) we

[6] The problem can be solved for other boundary conditions, but with much more difficulty. It is not possible, for example, to make use of the theory of the Mathieu equation in other cases. For a possible approach, see R. Einaudi [1], and S. Lubkin [8].

must require that $\partial^4 w/\partial\xi^4$ be continuous, since w and $\partial^2 w/\partial\xi^2$ (the bending moment within a constant factor) should be assumed continuous on physical grounds. But the sine series (3.2) can be differentiated four times with respect to ξ if, and only if, w and $\partial^2 w/\partial\xi^2$ vanish at $\xi=0$ and $\xi=\pi$, the end points of the column.[7]

Our definition of stability requires that $w(\xi, \vartheta)$ be bounded for $0 \leqq \vartheta < \infty$ when arbitrary initial conditions are prescribed. Hence we must require for stability that all solutions $F_n(\vartheta)$ of (3.3) for $n=1, 2, 3, \cdots$ and $0 \leqq \vartheta < \infty$ remain bounded when arbitrary initial conditions are prescribed. This is, of course, only a necessary condition for stability. However, we show in an appendix that the Fourier series (3.2) will, roughly speaking, converge for all ϑ if it converges for $\vartheta=0$ and if each $F_n(\vartheta)$ is a stable solution of the Mathieu equation. Such a question does not arise in the more usual type of initial value problem, since the functions analogous to $F_n(\vartheta)$ are generally of the form $e^{-r_n\vartheta}(A_n \cos n\vartheta + B_n \sin n\vartheta)$, $r_n \geqq 0$.

4. The Mathieu equation. The problem of the stability of the column with pinned ends has been reduced to that of determining whether all solutions of the Mathieu equation

$$\frac{d^2F}{d\vartheta^2} + (\alpha + \beta \cos \vartheta)F = 0, \tag{4.1}$$

i.e., of Eq. (3.3) without subscripts, are bounded for given values of α and β or not.

We summarize briefly the known theory of this equation in so far as it is needed for our purposes; more extended discussions and proofs can be found in the pamphlets of M. J. O. Strutt [15] and P. Humbert [5], and in the books of E. L. Ince [7] and Whittaker and Watson [17]. We have also made use of papers of S. Goldstein [2], E. L. Ince [6], and M. J. O. Strutt [14]. The notation we have chosen for the Mathieu equation has been taken to fit our problem; we compare it with the notation used by others:

Strutt	Goldstein	Ince and Whittaker and Watson	Here
u	y	y	F
$2x$	$2x$	$2x$	ϑ
$\lambda/4$	α	$a/4$	α
$-h^2/2$	$-4q$	$4q$	β

It can be shown (theorem of Floquet) that there exist in general two linearly independent solutions F_1 and F_2 of (4.1) which satisfy the relations

$$\begin{aligned} F_1(\vartheta + 2\pi) &= K_1 F_1(\vartheta), \\ F_2(\vartheta + 2\pi) &= K_2 F_2(\vartheta). \end{aligned} \tag{4.2}$$

The quantities K_1 and K_2 are either conjugate complex or real constants which satisfy the relation

[7] The analogous problem of the rectangular plate with simply supported edges can be treated in the same way as the column with pinned ends. The only essential difference would be that the relations corresponding to (3.4) and (3.5) would contain more parameters.

$$K_1 \cdot K_2 = 1. \tag{4.3}$$

Hence all solutions of (4.1) will be bounded only if

$$|K_1| = |K_2| = 1. \tag{4.4}$$

In case (4.4) is not satisfied, it follows from (4.3) that K_1 and K_2 are both real—a fact of which we make use later on. For certain values of α and β there exist solutions for which the values of K are $+1$ or -1; such solutions are therefore periodic of period 2π or 4π respectively.[8] The pairs of values (α, β) for which such periodic solutions of (4.1) exist can be shown to fill out curves in an α, β-plane which divide that plane into "stable" regions in which (4.4) holds and "instable" regions in which it does not hold. The boundary curves themselves belong to the instable region, the general solution of (4.1) corresponding to (α, β) on such a curve consisting of the sum of a periodic function plus ϑ times a periodic function. Fig. 2 indicates these regions, the stable ones being shaded.

It is of some interest to note that in the stable regions the relation

$$\alpha + |\beta| > 0 \tag{4.5}$$

must hold since otherwise $d^2F/d\vartheta^2$ would always have the sign of F and a solution not identically zero could not remain bounded for $\vartheta \to +\infty$ as well as for $\vartheta \to -\infty$; this would mean instability since $F(-\vartheta)$ is evidently a solution of (4.1) if $F(\vartheta)$ is.

The stable regions are connected at the points $\alpha = k^2/4$, $\beta = 0$, $k = 1, 2, 3, \cdots$, for which the solutions of (4.1) are evidently bounded. As indicated earlier, the boundary curves separating stable and instable regions are characterized by the fact that a periodic solution of period 2π or 4π exists for any pair of values (α, β) on such a curve. This can be made the basis of a method (due to Ince [6]) for determining these curves, as follows: a Fourier series with undetermined coefficients is assumed as a solution of (4.1). Upon substitution in (4.1) an infinite set of linear equations in the coefficients is obtained, each of which involves only three successive coefficients. Each equation may then be solved for the ratio of two successive coefficients in terms of the next higher or of the next lower coefficients. By successive substitution in these relations one is in this way led to two expressions for any such ratio, one of which is a finite and the other an infinite continued fraction. By equating the two, a relation between α and β is obtained which holds at the boundary points separating the stable and instable regions. For a given value of β and with α ranging from $-\infty$ to $+\infty$ one comes first upon the boundary curve C_0 which begins at $\alpha = 0$, $\beta = 0$ (cf. Fig. 2)[9]; the periodic solutions corresponding to points on this curve are of period 2π. Following this, the next two curves, C_1 and S_1, starting at $\alpha = 1/4$, $\beta = 0$ correspond to solutions of period 4π, followed by two, S_2 and C_2, starting at $\alpha = 1$, $\beta = 0$ corresponding to solutions of period 2π, etc. The letters C and S refer to developments in cosine series (for the even solutions) and in sine series (for the odd solutions). The points between two successive curves for which the periods of the corresponding solutions are different are stable points. For small β the boundary curves are given by the following expressions, solutions of type C_{2k} and S_{2k} having the period 2π, while those of type C_{2k+1}, S_{2k+1} have period 4π:

[8] For a given value of β, say, the problem of determining values of α for which such solutions exist is obviously a linear eigenvalue problem.

[9] Essentially the same figure appears in the book of Strutt [15].

$$
\left.\begin{aligned}
C_0:&\quad \alpha = -\beta^2/2 + O(\beta^4),\\
C_1:&\quad \alpha = 1/4 - \beta/2 - \beta^2/8 + \beta^3/32 + O(\beta^4),\\
S_1:&\quad \alpha = 1/4 + \beta/2 - \beta^2/8 - \beta^3/32 + O(\beta^4),\\
S_2:&\quad \alpha = 1 - \beta^2/12 + O(\beta^4),\\
C_2:&\quad \alpha = 1 + 5\beta^2/12 + O(\beta^4),\\
C_3:&\quad \alpha = 9/4 + \beta^2/16 - \beta^3/32 + O(\beta^4),\\
S_3:&\quad \alpha = 9/4 + \beta^2/16 + \beta^3/32 + O(\beta^4),\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
S_k, C_k:&\quad \alpha = k^2/4 + \beta^2/2(k^2-1) + O(\beta^4), \qquad k = 4, 5, 6, \cdots
\end{aligned}\right\} \tag{4.6}
$$

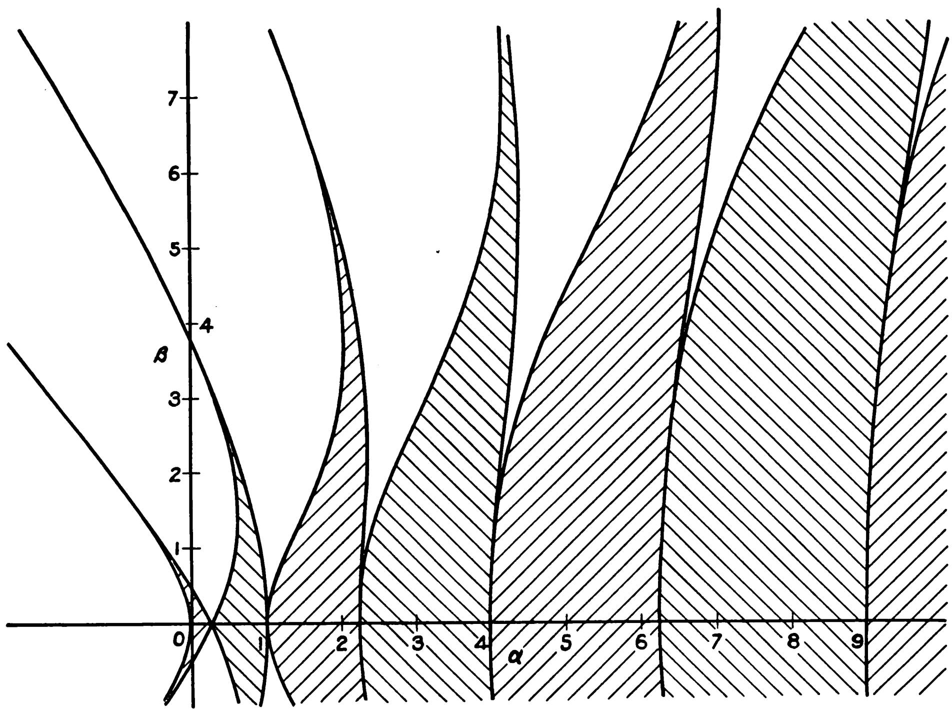

FIG. 2.

Curves of type C_{2k} and S_{2k} have contact of order $2k$ at the points $\alpha = k^2$, $\beta = 0$, while curves of type C_{2k+1} and S_{2k+1} have contact of order $2k+1$ at the points $\alpha = (2k+1)^2/4$, $\beta = 0$. This behavior is clearly indicated in Fig. 2. A table of values of α and β for points on these curves is given at the end of the paper. These values were calculated by means of the procedure outlined above and were checked against values given by S. Goldstein [2] and E. L. Ince [6] where possible.

For large positive values of α the points for which $|\beta| < \alpha$ are stable except for very narrow strips which lie near the lines $\alpha = k^2/4$. For large values of β it has been shown that all boundary curves tend to have the slope -1 (for $\beta > 0$). The stable regions are in general very narrow for $\alpha<0$ and grow narrower as $|\beta|$ increases. These observations are all borne out by Fig. 2.

5. The stability of the column with pinned ends. We may now conclude that the column with pinned ends will be stable only if the applied force $F=P+H\cos\omega t$ is such that all points (α_n, β_n) given by (3.4) and (3.5) fall within the shaded region of Fig. 2. In other words, a set of values (p, h, f) is stable only if every point of the sequence (α_n, β_n) determined by (p, h, f) is stable.

Suppose, for example, that $P=P_E$ (i.e., the steady part of the load is a tension equal in value to that of the Euler load) and that the harmonic part of the load has a frequency $f=f_0(\epsilon_0/2)^{1/2}$. We find that $\alpha_1=1$ and that the column (it is, rather, a tensile member in this case) is instable even for small amplitudes H of the oscillatory part of the load (i.e., for $|\beta_1|$ small), since the points $(1, \beta_1)$ are clearly seen with reference to Fig. 2 to be instable if $|\beta_1|\neq 0$ is small. We could expect the column to be set into motion with heavy lateral oscillations.

On the other hand, let us assume the steady load P to be a compression of twice the Euler value, while the harmonic part of the load has a frequency $f=2f_0\epsilon_0^{1/2}$ and an amplitude such that $h=H/P_E=3.1$. We find in this case:

$$\begin{aligned}
\alpha_1 &= -0.25, & \beta_1 &= 0.775,\\
\alpha_2 &= 2.00, & \beta_2 &= 3.10,\\
\alpha_3 &= 15.75, & \beta_3 &= 6.975,\\
&\vdots & &\vdots\\
\alpha_n &= n^2(n^2-2)/4, & \beta_n &= 0.775n^2.
\end{aligned}$$

We can readily convince ourselves that all points (α_n, β_n) lie in the stable region of Fig. 2. The points (α_1, β_1) and (α_2, β_2) are stable, as one sees from Fig. 2 and the table of values of α and β for points on the boundary curves given at the end of the paper. (Note particularly the values of α and β on C_0 and C_1 for $\alpha\simeq-0.25$ and the values on C_2 for $\alpha\simeq 2.0$). The numbers α_n can be written in the form $\alpha_n=(n^2-1)^2/4-1/4 \doteq k^2/4-1/4$, with $k=n^2-1$; in other words the abscissae α_n lie always a distance 1/4 to the left of the points $(k^2/4, 0)$ where the boundary curves delimiting the stable regions cross the α-axis. The points $(\alpha_n, 0)$ for $n>1$ are therefore stable points. Also, for β not too large the boundary curves lie to the right of the straight lines $\alpha=k^2/4$, as one sees from (4.6). Hence all points (α_n, β_n) will be stable if each β_n is not too large in comparison with α_n, and this condition is certainly fulfilled in our case for $n\geqq 2$. Note, for example, that β must be taken larger than 8 for a point of instability when $\alpha=8.75$ (that is, a value 1/4 less than 9). For $\alpha_3=15.75$ we have β_3 only 6.975 in value so that (α_3, β_3) is certainly stable. Since the α_n increase like n^4 while the β_n increase only like n^2, it becomes obvious that all (α_n, β_n) are stable. *The column is therefore stable even though the steady value of the load is twice that of the Euler load.*[10] However, the total compressive load always, as in this case, drops below P_E in value during at least part of the cycle if the column is stable: we have seen (cf. 4.5)) that the inequality $\alpha_n+|\beta_n|>0$ holds for stable solutions; in particular, for $n=1$ this leads to

[10] A. Stephenson [13] appears to have been the first to point out the possibility of such phenomena in general. This paper appeared in 1908.

$$p + |h| > -1, \tag{5.1}$$

as one sees from (3.4) and (3.5), and our statement follows from (2.13).

Thus there exist both stable and instable sets of values (p, h, f). However, our definition of stability leaves out of account a possibility which is always inherent in any physical problem, i.e., that slight changes in the parameters of the problem (p, h, and f in our case) may be sufficient to cause a stable motion to become an instable one. *A set of values (p, h, f) should be considered stable in any proper physical sense only if a complete neighborhood of these values exists which is made up entirely of what we have defined as stable sets of values.*

We proceed to show that the problem of the column never has a stable solution in this more restricted sense; i.e., we show that arbitrarily small changes δf in f and δp in p, for example, can always be found such that $(p+\delta p, h, f+\delta f)$ is instable no matter what values are chosen for p, h, and f. This is done by showing that a certain pair of values (α_n, β_n) becomes instable when properly chosen but arbitrarily small changes are made in f and p. Our statement follows from (3.4) and (3.5) and the character of the instable regions of the Mathieu equation for high values of α. We write equation (3.4) in the form

$$\alpha_n^{1/4}/n = (\epsilon_0 f_0^2/f^2)^{1/4}(1 + p/n^2)^{1/4}, \tag{5.2}$$

and show first that this equation can always be satisfied by taking for α_n the square of an integer, provided only that f is changed by a small amount δf and n is a sufficiently large integer: the real number $(\epsilon_0 f_0^2/f^2)^{1/4}$ can be approximated as accurately as desired by a rational number N/n. It is clear that n can always be chosen so large that an arbitrarily small change δf in f will suffice to make the right hand side of (5.2) exactly equal to N/n. Hence $\alpha_n = N^4$ and our statement is proved. It is also evident that an α_n of the form $n^2/4$ could have been determined in the same manner. We have thus determined a point (α, β) for which $\alpha = n^2/4$, n and $f+\delta f$ being now considered as fixed. We recall the fact that the instable regions of the Mathieu equation cross the α-axis at right angles at the points where $\alpha = n^2/4$ and that these regions for high values of n are narrow strips which remain (for not too large values of β) very near to the vertical straight lines $\alpha = n^2/4$. Since the values of β_n increase like n^2, while those of α_n increase like n^4 it becomes evident that a small change δp in the value of p in (3.4) will be sufficient to cause the point (α', β) corresponding to the values $p+\delta p$, h, $f+\delta f$ to fall inside an instable region of the Mathieu equation. We repeat: no values of p, h, and f ($h, f \neq 0$) can be found such that the column is stable when small variations in these quantities are permitted.

In the actual physical problem, however, there is an important element present, i.e., viscous damping, which has been neglected so far. In a later section we shall show that the presence of even the slightest amount of viscous damping will suffice to make all values (α, β) stable for which $\alpha \geqq \alpha_0 > 0$, and $|\beta| < \alpha$, when α_0 is a certain constant which may be large. In other words, damping acts in such a way as to cut out the narrow instable strips which occur for large α in the regions for which $|\beta| < \alpha$. Under these circumstances it becomes sufficient to test only a certain *finite* number of the points (α_n, β_n) for stability. Thus the column may be stable if viscous damping is present even when small variations in the quantities p, h, and f take place, though, as we have seen, this is not the case without damping.

Figures 3, 4, and 5 show the stable values of f and h (frequency and relative amplitude $h=H/P_E$ of the vibratory part of the load) for the values $p=P/P_E=-1.5$, -1.0, and 1.0 respectively. The stable regions are shaded.[11] These diagrams have been constructed on the assumption that the amount of viscous damping is large enough that values of α larger than 10 can be ignored. In other words, Figs. 3, 4 and 5 were constructed by combining the stability regions of Fig. 2, which includes values of α up to 10 only, for a suitable number of values of n.

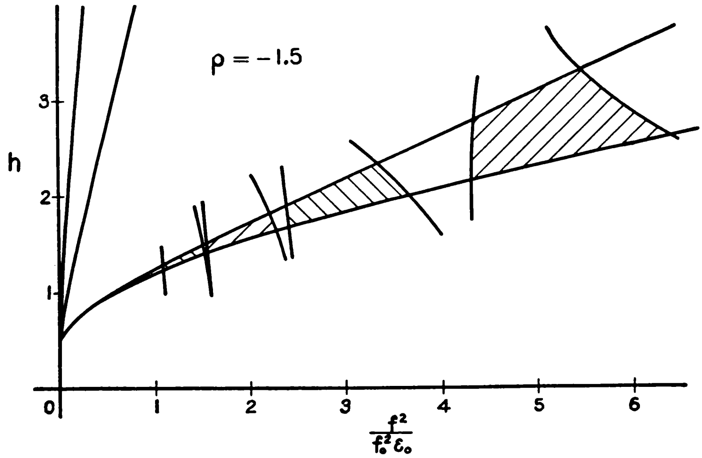

FIG. 3.

The general character of Figs. 3 and 4 is typical for the cases in which $p<-1$, i.e., in which the steady part of the load is a compression larger than the Euler load. We note that the shaded stable regions for $p=-1.5$ are much smaller than

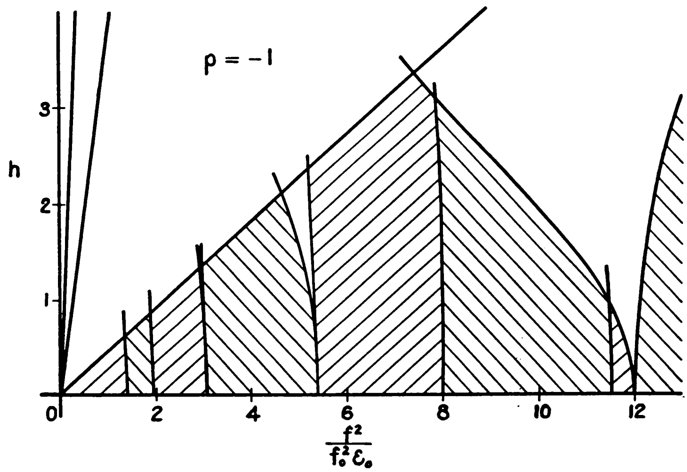

FIG. 4.

those for $p=-1.0$, as was to be expected; for the higher values of the steady compressive load beyond the Euler load it is necessary to make more accurate

[11] Without damping, as we have seen, there could be no stable *regions* though there are stable points. It would have a certain mathematical interest to investigate the set of stable points in detail in this case.

adjustments in the frequency and amplitude of the oscillatory part of the load in order to obtain stability. The full lines which cut through the shaded regions in the figures are not really curves; they represent, rather, narrow instable regions. However, the two curves in Figs. 3 and 4 which appear to be straight lines running near the h-axis indicate narrow stable regions. Fig. 5 is typical for all cases in which $p > -1.0$, i.e., for cases in which the steady part of the load is either a tension or a compression less in value than the Euler load. In these cases the column is stable for all frequencies when $h=0$; it is in fact stable almost everywhere in the neighborhood of the axis $h=0$.

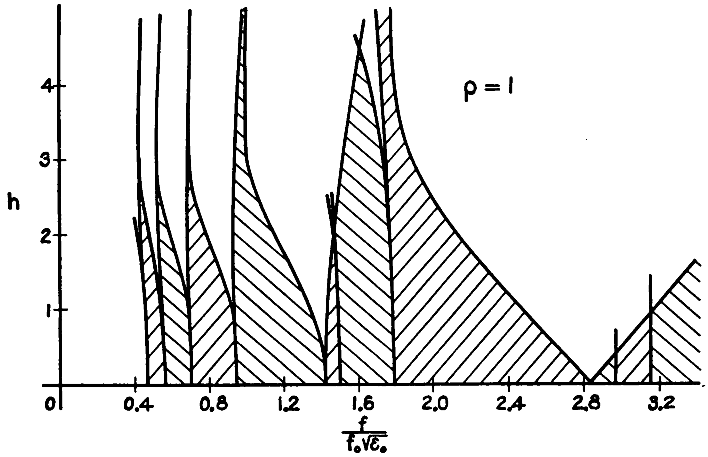

FIG. 5.

It is of some interest to consider the special case in which the amplitude H of the oscillatory part of the applied load is very small so that the values of β_n are small (for n not too large). We note that the natural frequencies f_n of the free lateral oscillations of the rod under steady load (that is, in this case, for $H=0$) are given by $f_n = f\alpha_n^{1/2}$ as one can readily verify. From Fig. 2 we observe that the rod is instable for small values of β when $\alpha_n = k^2/4$, k being any integer. Hence instability occurs for small amplitudes of the oscillatory part of the load whenever

$$f = 2f_n/k, \qquad k = 1, 2, 3, \cdots, \tag{5.3}$$

that is, whenever the load frequency is twice any integral submultiple of a natural frequency of oscillation. At such frequencies one could expect that heavy oscillations would be built up.[12] However, the most favorable case for the production of oscillations is, in general, that for which $n=k=1$. Consider, for example, the case $p=1$. For $n=k=1$ we find readily that $f/f_0\epsilon_0^{1/2} = 8^{1/2} = 2.83$, and one readily sees from Fig. 5 that this furnishes the most favorable frequency for instability at small amplitudes of the oscillatory force.

6. The flexible string under harmonically varying tension. With only slight modifications our preceding results can be used to discuss the problem of the vibrating

[12] This problem has been considered both experimentally and theoretically by I. Utida and K. Sezawa [16].

string subjected to a harmonically varying tension.[13] We have only to set $I=0$ in (2.1) to obtain the fundamental differential equation. The tension $F(t)$ in the string is assumed given by (2.2) and the same independent variables as before are introduced. However, the parameters p and h in (2.3) can obviously not be used here. Instead, we introduce the quantities

$$p_s = P/EA, \qquad h_s = H/EA. \tag{6.1}$$

We may assume for w the expansion (3.2) for a string with fixed ends and will obtain (3.3) as differential equation for the quantities $F_n(\vartheta)$ if we now define α_n and β_n by the equations

$$\alpha_n = n^2 f_0^2 p_s/f^2, \qquad \beta_n = n^2 f_0^2 h_s/f^2. \tag{6.2}$$

The investigation of stability involves the same considerations as for the column, and much the same general remarks might be made as were made in the case of the column. For example, if $P>0$ and $P>|H|$, i.e., if the force applied to the string is never a compression, viscous damping acts in such a way as to cut out the instable regions of Fig. 2 for sufficiently large values of α. Hence it is possible to construct a diagram for the determination of the stable values of p, h, and f in the same manner as for the column. Figure 6 shows the stable regions (shaded); the quantity $f/f_0 p_2^{1/2}$ is taken as abscissa and $H/P = \beta_n/\alpha_n = h_s/p_s$ as ordinate.

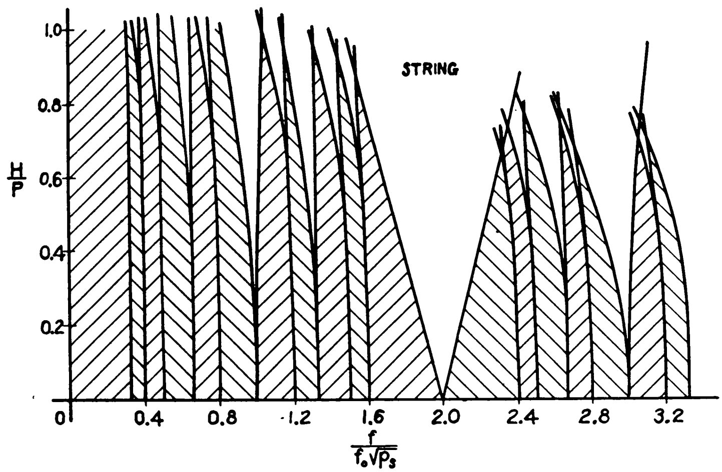

FIG. 6.

It is readily seen that the natural frequencies f_n for the free lateral oscillation of the string (under constant tension) are given by $f_n = f\alpha_n^{1/2}$, just as in the case of the column. The string is instable for low amplitudes of the oscillatory part of the tension when $\alpha_n = k^2/4$, $k=1, 2, 3, \cdots$. In this case we know in addition that $f_n = nf_1$, in which f_1 is the fundamental frequency of the string. Hence "resonance," that is, heavy oscillations for low amplitudes of the applied oscillatory force, will occur when

[13] This problem was first discussed by Lord Rayleigh [11]. The problem was discussed later by A. Stephenson [12], and [13], and by C. V. Raman [10].

$$f = 2nf_1/k, \qquad n, k = 1, 2, 3, \cdots, \tag{6.3}$$

that is, at twice any rational multiple of the fundamental frequency of free lateral oscillation of the string. However, the most favorable case for the production of oscillations is readily seen to be that for which $n=k=1$ (i.e., that corresponding to $f/f_0p_s^{1/2}=2.0$). In Melde's experiment lateral oscillations of a string are produced in accordance with (6.3) by attaching one end of the string to the prong of a tuning fork.

There is one marked (though not unexpected) difference between the behavior of the column and that of the string: it could be shown that the string is never stable even with viscous damping if the load on it becomes a compression during any part of the cycle. For stability of the string we must always require $P \geqq |H|$.

7. The effect of damping. If it is assumed that there is a lateral damping force acting on the column that is proportional to the velocity $\partial w/\partial t$, the differential equation (2.1) is readily seen to be modified by the addition of a term $\delta(\partial w/\partial t)$, $\delta > 0$, to its left hand side. With the same notation as before we find as differential equation for the functions $F_n(\vartheta)$:

$$\frac{d^2F}{d\vartheta^2} + 2\nu\frac{dF}{d\vartheta} + (\alpha + \beta\cos\vartheta)F = 0, \tag{7.1}$$

where

$$\nu = \delta/4\pi mf, \tag{7.2}$$

and subscripts have been dropped.

The general theory of equation (7.1) could be developed in the same way as that for the Mathieu equation without damping (for a treatment which includes a damping term, see the papers of G. Gorelik [3]). In particular, the α, β-plane could be divided into stable and instable regions. We confine ourselves here to one special problem, i.e., to a discussion of the behavior of the solutions of (7.1) for a given value of ν and large positive values of α. We assume also that $|\beta| < \alpha$.

Upon making the substitutions

$$F = e^{-\nu\vartheta}G, \qquad \alpha' = \alpha - \nu^2 \tag{7.3}$$

Eq. (7.1) becomes

$$\frac{d^2G}{d\vartheta^2} + (\alpha' + \beta\cos\vartheta)G = 0. \tag{7.4}$$

Obviously, if G is bounded, F is not only bounded but approaches zero as ϑ increases. Also, even at boundary points (α', β) separating stable and instable regions of (7.4), the corresponding solutions F tend to zero since no solution G of (7.4) increases faster than ϑ in this case. If the amount of damping is slight (that is, if ν is small), the boundary curves for (7.1) would lie near those for $\nu=0$, but they would not intersect the α-axis except at the origin since all solutions F of (7.1) are clearly bounded for $\beta=0$, $\alpha>0$. This reasoning makes it seem rather evident that the narrow instable regions which occur for large positive values of α when $|\beta| < \alpha$ are cut out when a damping term is added.

We proceed to give a proof of the following statement: *if* $\nu > 0$ *and* $|\beta| < \alpha$, *all* *solutions of* (*7.1*) *are stable for all values of* α *which exceed a certain value* $\alpha_0 > 0$. It was pointed out earlier that there exist two linearly independent solutions G_1 and G_2 of (7.4) such that $G_1(\vartheta+2\pi)=KG_1(\vartheta)$ and $KG_2(\vartheta+2\pi)=G_2(\vartheta)$ with $|K|>1$ in case (α', β) is in an instable region for (7.4). We know also that K is a real number in this case. The solutions of (7.1) will, however, remain bounded even in such an instable case for (7.4) provided that

$$e^{2\pi\nu} > |K|, \tag{7.5}$$

as one sees from (7.3). Consequently our statement will be proved if we can show that $|K| \to 1$ as $\alpha' \to \infty$. This we prove through the use of the following asymptotic formula for K, valid under our assumptions, which has been given by Strutt [14]:

$$K + 1/K = 2 \cosh \lambda \cos \zeta + O(1/\sqrt{\alpha'}) \tag{7.6}$$

in which

$$\zeta + \lambda\sqrt{-1} = \int_0^{2\pi} (\alpha' + \beta \cos \vartheta) d\vartheta, \tag{7.7}$$

and $O(1/\sqrt{\alpha'})$ means that all terms neglected are of order $1/\sqrt{\alpha'}$ or higher. Since we assume that $|\beta| < \alpha'$ the integral in (7.7) is real and $\lambda = 0$. We have, therefore:

$$|K + 1/K| < |2 \cos \zeta| + O(1/\sqrt{\alpha'}) < 2 + O(1/\sqrt{\alpha'}). \tag{7.8}$$

Since K is real it is readily seen that

$$2 \leqq |K + 1/K|, \tag{7.9}$$

equality holding only for $|K| = 1$. From this and inequality (7.8) it follows at once that

$$|K| \to 1 \quad \text{when} \quad \alpha' \to \infty. \tag{7.10}$$

In the case of the column we note from Eqs. (3.4) and (3.5) that $|\beta_n| < \alpha_n$ for sufficiently large n and that $\alpha_n \to \infty$ with n. The assumptions under which (7.10) was derived are thus fulfilled in this case. When damping is present we are therefore justified in neglecting all values of α larger than a certain positive value α_0 in discussing the stable values for the column. Our diagrams were drawn under the assumption that $\alpha_0 = 10$. In the case of the string, α_n and β_n increase at the same rate with increase of n; consequently our conclusions regarding the effect of damping in this case are valid only when $P > |H|$ (which ensures that $|\beta_n| < \alpha_n$,) i.e., when P is a tension and H is such that the total force in the string is always a tension.

Appendix

Sufficient conditions for stability. For stability we required always that the solution

$$w = \sum_{n=1}^{\infty} F_n(\vartheta) \sin n\xi \tag{A1}$$

of our problems be bounded for arbitrary initial conditions; it is thus necessary to assume for stability that each $F_n(\vartheta)$ be bounded for $0 \leqq \vartheta < \infty$ (ϑ is essentially the time variable). In this appendix we prove a statement made at the end of section (3) to the effect that the series will converge for all ϑ if it converges for $\vartheta=0$ and if the $F_n(\vartheta)$ are all stable solutions of the Mathieu equation.

In order to state our theorem precisely we introduce the series

$$\sum_n D_n \sin n\xi, \qquad \sum_n V_n \sin n\xi \tag{A2}$$

in which D_n and V_n are defined by

$$D_n = F_n(0), \qquad V_n = \left.\frac{dF_n}{d\vartheta}\right|_{\vartheta=0}. \tag{A3}$$

We assume that the series (A2) are such that

$$\sum_n \left\{ \rho_n \left| D_n \right| + \rho_n \alpha_n^{-1/2} \left| V_n \right| \right\} < \infty, \tag{A4}$$

in which ρ_n is a certain positive quantity and α_n is one of the two parameters in the Mathieu equation for the functions $F_n(\vartheta)$:

$$\frac{d^2F_n}{d\vartheta^2} + (\alpha_n + \beta_n \cos \vartheta)F_n = 0. \tag{A5}$$

We assume in addition that the $F_n(\vartheta)$ are stable solutions of (A5) for which

$$\left| \beta_n \right| < k\alpha_n, \qquad 0 \leqq k < 1, \tag{A6}$$

at least for all $n>N$, say.[14] Under these assumptions we show that: *the series*

$$\sum_n F_n(\vartheta) \sin n\xi \quad \textit{and} \quad \sum_n \frac{dF_n(\vartheta)}{d\vartheta} \sin n\xi$$

converge for $0 \leqq \vartheta < \infty$ *in the same sense as the series* (*A*2), *i.e., the convergence relation*

$$\sum_n \left\{ \rho_n \left| F_n(\vartheta) \right| + \rho_n \alpha_n^{-1/2} \left| \frac{dF_n(\vartheta)}{d\vartheta} \right| \right\} < \infty \tag{A7}$$

holds for $0 \leqq \vartheta < \infty$.

If it were assumed that $\rho_n = 1$ in (A4) then $\sum_n F_n(\vartheta) \sin n\xi$ would converge, but its derivative with respect to ϑ would not necessarily converge. If ρ_n were assumed to be $\alpha_n^{1/2}$, the differentiated series would converge. In our cases $\alpha_n^{1/2}$ is of order n for the string and of order n^2 for the column. To assume $\rho_n = \alpha_n^{1/2}$ in (A4) would therefore not seem unduly restrictive when it is considered that the series (A1) should be assumed to converge when it is differentiated twice with respect to ξ in the case of the string and four times with respect to ξ in the case of the column.

We prove our theorem by showing that every stable solution of the Mathieu equation

$$\frac{d^2F}{d\vartheta^2} + (\alpha + \beta \cos \vartheta)F = 0 \tag{A8}$$

[14] These latter conditions are fulfilled in the stable cases for both column and string. This follows from (3.4) and (3.5) for the column, and from (6.2) and the fact that $|h_s| < p_s$ in the case of the string.

for which

$$|\beta| < k\alpha, \qquad 0 \leqq k < 1, \tag{A9}$$

and

$$F(0) = D, \qquad \left.\frac{dF(\vartheta)}{d\vartheta}\right|_{\vartheta=0} = V \tag{A10}$$

satisfies the inequality

$$|F(\vartheta)| + \alpha^{-1/2}\left|\frac{dF(\vartheta)}{d\vartheta}\right| \leqq C\{|D| + \alpha^{-1/2}|V|\}, \tag{A11}$$

for $0 \leqq \vartheta < \infty$, C being a constant which depends only upon k. Upon reintroduction of the subscript n in (A11) followed by multiplication with $\rho_n > 0$ and a summation with respect to n, it is clear that (A7) would result from (A4), since C is independent of n.

We proceed to establish the validity of the inequality (A11). For this purpose it is convenient to introduce a new independent variable φ as well as a new dependent variable f in (A8) as follows:[15]

$$\phi(\vartheta) = \int_0^\vartheta \chi^{1/2} d\vartheta, \qquad \chi = \alpha + \beta\cos\vartheta, \tag{A12}$$

$$f = \chi^{1/4} F. \tag{A13}$$

In these variables the differential equation (A8) becomes

$$\frac{d^2 f}{d\varphi^2} + \left(1 + \frac{\beta\cos\vartheta}{4\chi^2} + \frac{5\beta^2\sin^2\vartheta}{16\chi^3}\right) f = 0, \tag{A14}$$

or, as we prefer to write it

$$\frac{d^2 f}{d\varphi^2} + f = \alpha^{-1}\gamma f, \tag{A15}$$

with

$$\gamma = -\alpha\left(\frac{\beta\cos\vartheta}{4\chi^2} + \frac{5\beta^2\sin^2\vartheta}{16\chi^3}\right). \tag{A16}$$

From now on we consider $f(\varphi)$ to be the solution of (A15) which satisfies the initial conditions

$$f(0) = 1, \qquad \frac{df}{d\varphi} = i, \qquad i = \sqrt{-1}. \tag{A17}$$

It is then readily verified that $f(\varphi)$ and its derivative satisfy the integral equations

$$f(\varphi) = e^{i\varphi} - \frac{i}{2\alpha}\left(e^{i\varphi}\int_0^\varphi \gamma(\tau)f(\tau)e^{-i\tau}d\tau - e^{-i\varphi}\int_0^\varphi \gamma(\tau)f(\tau)e^{i\tau}d\tau\right), \tag{A18}$$

$$\frac{df(\varphi)}{d\varphi} = ie^{i\varphi} + \frac{1}{2\alpha}\left(e^{i\varphi}\int_0^\varphi \gamma(\tau)f(\tau)e^{-i\tau}d\tau - e^{-i\varphi}\int_0^\varphi \gamma(\tau)f(\tau)e^{i\tau}d\tau\right). \tag{A19}$$

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[15] This transformation is frequently used in the treatment of various questions relating to the asymptotic behavior of the solutions of certain types of second order ordinary differential equations.

From the general theory of the Mathieu equation it is known that every stable solution $F(\vartheta)$ of (A8) can be expressed in the form $H(\vartheta)e^{ia\vartheta}$, in which $H(\vartheta)$ is a periodic function of period 2π and a is a real constant. It follows from (A13) that $f(\varphi(\vartheta))$ can be expressed in the form $h(\vartheta)e^{ia\vartheta}$ with $h=H\chi^{1/4}$; $h(\vartheta)$ is thus also periodic of period 2π in ϑ. Consequently we may write

$$G = \max \left| f(\vartheta) \right| = \max \left| h(\vartheta) \right| = \max_{|\vartheta|\leqq\pi} \left| h(\vartheta) \right| = \max_{|\vartheta|\leqq\pi} \left| f(\vartheta) \right|. \qquad (A20)$$

The validity of (A20) is the essential point in our proof; because of it, bounds for our quantities in the interval $-\pi \leqq \vartheta \leqq \pi$ hold also for $0 \leqq \vartheta < \infty$.

We find from (A16), the definition of χ in (A12), and (A9) that

$$\left| \gamma \right| \leqq k/4(1 - k^2) + 5k^2/16(1 - k^2) = \Gamma. \qquad (A21)$$

We note also that

$$\varphi(\pi) \leqq \pi\sqrt{\alpha + \beta} \leqq \pi\sqrt{\alpha}\sqrt{1 + k}, \qquad (A22)$$

as one sees from (A12). Finally we obtain from (A18) the following inequality for $G = \max \left| f(\vartheta) \right|$:

$$G \leqq 1 + \frac{\Gamma\pi\sqrt{1 + k}}{\sqrt{\alpha}} G. \qquad (A23)$$

In view of our purpose it is permissible to assume from now on that

$$\alpha \geqq \alpha_0 > \Gamma^2\pi^2(1 + k) = \alpha_1; \qquad (A24)$$

once this is done (A23) may be written in the form

$$G \leqq 1/(1 - \sqrt{\alpha_1/\alpha_0}) = G_0. \qquad (A25)$$

In a similar fashion we can show that

$$\max \left| \frac{df(\varphi)}{d\varphi} \right| \leqq G_0, \qquad (A26)$$

since $df/d\varphi$ satisfies (A19) and, like $f(\varphi)$ itself, can be written in the form $h(\vartheta)e^{ia\vartheta}$ with h of period 2π in ϑ.

Since the function $f(\varphi(\vartheta))$ given by (A18) and its complex conjugate are linearly independent solutions of (A15) it follows that we may write the general real solution F of (A8) in the form

$$F(\vartheta) = \operatorname{Re} C\chi^{-1/4}f(\vartheta), \qquad C = A - iB, \qquad (A27)$$

in which Re means that the real part of what follows is to be taken, and A and B are real but otherwise arbitrary constants. The quantity $dF/d\vartheta$ is then given by the expression

$$\frac{dF}{d\vartheta} = \operatorname{Re} \left\{ (\alpha + \beta \cos \vartheta)(A - iB) \frac{df}{d\varphi} + 1/4 \frac{(A - iB)f(\varphi(\vartheta))\beta \sin \vartheta}{(\alpha + \beta \cos \vartheta)^{5/4}} \right\}. \qquad (A28)$$

We find at once, since

$$\varphi(0) = 0 \quad \text{and} \quad \left.\frac{df}{d\varphi}\right|_{\varphi=0} = i,$$

$$D = F(0) = (\alpha + \beta)^{-1/4}A, \qquad V = \left.\frac{dF}{d\vartheta}\right|_{0=0} = (\alpha + \beta)^{1/4}B, \tag{A29}$$

from which we obtain

$$A = (\alpha + \beta)^{1/4}D, \qquad B = (\alpha + \beta)^{-1/4}V. \tag{A30}$$

For $|F(\vartheta)|$ we then have the inequality

$$\begin{aligned} |F(\vartheta)| &\leqq \left\{\left(\frac{1 + k}{1 - k}\right)^{1/4} |D| + \frac{1}{\alpha^{1/2}(1 - k^2)^{1/4}} |V| \right\}G_0 \\ &< p_0|D| + q_0\alpha^{-1/2}|V|, \end{aligned} \tag{A31}$$

in which p_0 and q_0 depend only upon the constant k introduced in (A9), and G_0 is the bound for max $|f(\vartheta)|$ given in (A25). From (A28) we find

$$\begin{aligned} \left|\frac{dF}{d\vartheta}\right| &< 1/4 \frac{k}{1 - k} (\alpha - \beta)^{-1/4}|C|G_0 + (\alpha + \beta)^{1/4}|C|G_0 \\ &\leqq \alpha^{1/2}p_1|D| + q_1|V|, \end{aligned} \tag{A32}$$

where

$$p_1 = \left\{\frac{1}{4\alpha_0^{1/2}} \frac{k}{1 - k}\left(\frac{1 + k}{1 - k}\right)^{1/4} + (1 + k)^{1/2}\right\}G_0 \tag{A33}$$

and q_1 is of similar nature. The quantities p_1 and q_1, like p_0 and q_0 in (A31), depend only upon k. Division of both sides of (A32) by $\sqrt{\alpha}$, followed by addition to (A31) yields

$$|F| + \alpha^{-1/2}\left|\frac{dF}{d\vartheta}\right| \leqq p|D| + \alpha^{-1/2}q|V|, \tag{A34}$$

which establishes the validity of (A11) and thus completes the proof of our theorem.

Coordinates of Points on the Boundary Curves of Fig. 2.

β	$\alpha(C_0)$	$\alpha(C_1)$	$\alpha(S_1)$	$\alpha(S_2)$
0.0	0.00000	0.25000	0.25000	1.00000
0.2	−0.01966	0.14525	0.34475	0.99667
0.4	−0.07510	0.03191	0.42796	0.98670
0.6	−0.15836	−0.08872	0.49816	0.97018
0.8	−0.26148	−0.21555	0.55906	0.94724
1.0	−0.37849	−0.34767	0.59480	0.91806
1.2	−0.50535	−0.48430	0.62006	0.88284
1.4	−0.63942	−0.62480	0.63015	0.84183
1.6	−0.77898	−0.76867	0.62592	0.79529
1.8	−0.92281	−0.91545	0.60857	0.74349
2.0	−1.07013	−1.06480	0.57950	0.68672

β	$\alpha(C_0)$	$\alpha(C_1)$	$\alpha(S_1)$	$\alpha(S_2)$
2.2	−1.22031	−1.21640	0.54012	0.62526
2.4	−1.37291	−1.37002	0.49174	0.55938
2.6	−1.52760	−1.52544	0.43554	0.48935
2.8	−1.68410	−1.68248	0.37253	0.41542
3.0	−1.84221	−1.84098	0.30357	0.33785
3.2	−2.00175	−2.00081	0.22938	0.25684
3.4	−2.16258	−2.16185	0.15057	0.17263
3.6	−2.32457	−2.32402	0.06763	0.08541
3.8	−2.48764	−2.48720	−0.01901	−0.00468
4.0	−2.65168	−2.65134	−0.10899	−0.09734
4.4	−2.98242	−2.98220	−0.29781	−0.29009
4.8	−3.31627	−3.31614	−0.49688	−0.49171
5.2	−3.65286	−3.65277	−0.70474	−0.70124
5.6	−3.99186	−3.99180	−0.92026	−0.91787
6.0	−4.33302	−4.33298	−1.14253	−1.14088
6.4	−4.67611	−4.67609	−1.37085	−1.36970
6.8	−5.02097	−5.02096	−1.60460	−1.60383
7.2	−5.36744	−5.36743	−1.84328	−1.84271
7.6	−5.71537	−5.71537	−2.08644	−2.08607
8.0	−6.06467	−6.06466	−2.33382	−2.33353
8.4	−6.41522	−6.41522	−2.58498	−2.58478
8.8	−6.76694	−6.76694	−2.83970	−2.83955
9.2	−7.11974	−7.11974	−3.09772	−3.09761
9.6	−7.47357	−7.47357	−3.35883	−3.35875
10.0	−7.82835	−7.82835	−3.62283	−3.62277
11.0	−8.71911	−8.71911	−4.29436	−4.29434
12.0	−9.61474	−9.61474	−4.98065	−4.98064
13.0	−10.51465	−10.51465	−5.67983	−5.67982
14.0	−11.41834	−11.41834	−6.39044	−6.39043
15.0	−12.32542	−12.32542	−7.11126	−7.11126
16.0	−13.23556	−13.23556	−7.84129	−7.84129
18.0	−15.06389	−15.06389	−9.32566	−9.32566
20.0	−16.90154	−16.90154	−10.83807	−10.83807

β	$\alpha(C_2)$	$\alpha(C_3)$	$\alpha(S_3)$	$\alpha(S_4)$
0.0	1.00000	2.25000	2.25000	4.00000
0.2	1.01633	2.25225	2.25275	4.00133
0.4	1.06171	2.25808	2.26203	4.00530
0.6	1.12806	2.26622	2.27933	4.01181
0.8	1.20733	2.27554	2.30589	4.02075
1.0	1.29317	2.28515	2.34258	4.03192

β	$\alpha(C_2)$	$\alpha(C_3)$	$\alpha(S_3)$	$\alpha(S_4)$
1.2	1.38126	2.29429	2.38967	4.04512
1.4	1.46860	2.30233	2.44680	4.06010
1.6	1.55305	2.30878	2.51308	4.07660
1.8	1.63302	2.31323	2.58723	4.09433
2.0	1.70727	2.31536	2.66777	4.11301
2.2	1.77487	2.31495	2.75314	4.13236
2.4	1.83509	2.31175	2.84194	4.15212
2.6	1.88745	2.30568	2.93284	4.17199
2.8	1.93163	2.29660	3.02467	4.19175
3.0	1.96752	2.28448	3.11640	4.21115
3.2	1.99517	2.26925	3.20712	4.22997
3.4	2.01478	2.25092	3.29604	4.24800
3.6	2.02665	2.22950	3.38247	4.26507
3.8	2.03118	2.20500	3.46578	4.28099
4.0	2.02881	2.17748	3.54547	4.29563
4.4	2.00521	2.11356	3.69216	4.32053
4.8	1.95947	2.03826	3.81969	4.33886
5.2	1.89487	1.95216	3.92636	4.34996
5.6	1.81419	1.85589	4.01149	4.35338
6.0	1.71968	1.75014	4.07538	4.34881
8.0	1.09281	1.09947	4.12172	4.20467
10.0	0.28857	0.29018	3.84895	3.87349
12.0	−0.63494	−0.63452	3.38071	3.38817
14.0	−1.64702	−1.64690	2.77777	2.78016
16.0	−2.72859	−2.72855	2.07287	2.07367
18.0	−3.86669	−3.86668	1.28641	1.28668
20.0	−5.05198	−5.05198	0.43241	0.43251

β	$\alpha(C_4)$	$\alpha(C_5)$	$\alpha(S_5)$	$\alpha(S_6)$
0.0	4.00000	6.25000	6.25000	9.00000
0.2	4.00134	6.25083	6.25083	9.00057
0.4	4.00538	6.25333	6.25333	9.00229
0.6	4.01226	6.25750	6.25751	9.00515
0.8	4.02215	6.26334	6.26337	9.00915
1.0	4.03530	6.27084	6.27094	9.01430
1.2	4.05204	6.27999	6.28025	9.02060
1.4	4.07273	6.29077	6.29134	9.02806
1.6	4.09776	6.30317	6.30427	9.03667
1.8	4.12755	6.31714	6.31911	9.04643
2.0	4.16245	6.33264	6.33594	9.05735
2.2	4.20283	6.34961	6.35487	9.06943
2.4	4.24889	6.36800	6.37604	9.08267

β	$\alpha(C_4)$	$\alpha(C_5)$	$\alpha(S_5)$	$\alpha(S_6)$
2.6	4.30085	6.38773	6.39956	9.09705
2.8	4.35867	6.40871	6.42560	9.11259
3.0	4.42220	6.43085	6.45432	9.12927
3.2	4.49121	6.45406	6.48591	9.14707
3.4	4.56533	6.47821	6.52052	9.16600
3.6	4.64406	6.50321	6.55837	9.18603
3.8	4.72688	6.52893	6.59962	9.20714
4.0	4.81318	6.55525	6.64444	9.22930
4.4	4.99383	6.60921	6.74533	9.27671
4.8	5.18127	6.66411	6.86185	9.32798
5.2	5.37113	6.71898	6.99394	9.38281
5.6	5.55951	6.77289	7.14093	9.44078
6.0	5.74803	6.82500	7.30201	9.50150
8.0	6.50217	7.03409	8.23272	9.82875
10.0	6.89864	7.11706	9.16125	10.14742
12.0	6.97136	7.05384	9.87814	10.40143
14.0	6.82083	6.85144	10.30874	10.55621
16.0	6.51561	6.52721	10.48838	10.59848
18.0	6.09463	6.09902	10.48167	10.52959
20.0	5.58132	5.58302	10.33749	10.35813

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