ON THE DEFLECTION OF ANISOTROPIC THIN PLATES*

BY

VLADIMIR MORKOVIN

Brown University

1. Introduction. The elastic medium considered in this paper[1] is assumed to have at each point at least one plane of elastic symmetry parallel to the neutral plane of the plate, which is taken as the xy plane. The theory of bending of thin plates possessing this type of anisotropy has been developed mainly by Boussinesq,[2] Voigt,[3] and Lechnitzky.[4] It is based on the usual assumptions of thin plate theory, leading to the following relations which are valid throughout the (small) thickness $2h$ of the plate

$$\tau_{zz} = 0, \tag{1.1}$$

$$u = -z\frac{\partial w}{\partial x}, \qquad v = -z\frac{\partial w}{\partial y}, \tag{1.2}$$

where w denotes the (small) deflection of the neutral surface of the plate, while u and v are the displacements in the x- and y-directions respectively.

If the equilibrium conditions are used together with *Generalized Hooke's Law*, (1.1), and (1.2), it is possible to express[5] the quantities characterizing the state of stress in terms of the partial derivatives of the deflection w. Then by considerations of equilibrium w is found to satisfy the differential equation:

$$b_{11}\frac{\partial^4 w}{\partial x^4} + 3b_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(b_{12} + b_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 3b_{26}\frac{\partial^4 w}{\partial x \partial y^3} + b_{22}\frac{\partial^4 w}{\partial y^4} = \frac{3q(x, y)}{2h^3}, \tag{1.3}$$

where b_{ij} are constants depending upon the elastic material and $q(x, y)$ is the normal load per unit area on the upper face, the lower face being free.

The problem, therefore, is to find a solution $w(x, y)$ of (1.3) which satisfies

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[1] The author wishes to express his appreciation to Professor I. S. Sokolnikoff, who proposed the problem and suggested the manner of approach.

[2] M. J. Boussinesq, Journal de Mathématiques (Liouville) Ser. 3, **5**, 329 (1879).

[3] W. Voigt, *Kompendium der Theoretischen Physik*, Leipzig, 1895, pp. 436–455.

[4] S. G. Lechnitzky, Applied Mathematics and Mechanics, Leningrad, **2**, 181–210, (1938).

[5] An exposition of the theory is given in I. S. Sokolnikoff's *Mathematical Theory of Elasticity*, (mimeographed, Brown University, 1941), pp. 319–329. His notation is adopted in the present discussion.

the prescribed conditions at the boundary (edge) C_0 of the neutral surface of the plate. This general problem has not been solved even in the case of very simple boundaries. Some success was achieved by various writers[6] investigating special shapes of plates made of material with three mutually perpendicular planes of elastic symmetry (orthotropic material). In the present paper a general method of solution is indicated when the boundary C_0 is an analytic curve. Detailed solution, illustrating the procedure, is carried out in the case of a clamped elliptic plate and a polynomial loading function $q(x, y)$. The complicated form of the solutions obtained is due to the generality of the problem. For a given material the values of the parameters b_{ij} in (1.3) are specified and the computations become relatively simple.

2. Preliminary considerations. Utilizing the fact that the potential energy of any realizable state of stress is always nonnegative, Lechnitzky[7] proved that the roots μ_k of the characteristic equation

$$b_{11} + 3b_{16}\,\mu + 2(b_{12} + b_{66})\mu^2 + 3b_{26}\,\mu^3 + b_{22}\,\mu^4 = 0 \tag{2.1}$$

corresponding to (1.3), must be conjugate complex numbers, say,

$$\begin{aligned} \mu_1 &= \alpha_1 + i\beta_1; \qquad \mu_3 = \bar{\mu}_1 = \alpha_1 - i\beta_1; \qquad \beta_1 \neq 0; \\ \mu_2 &= \alpha_2 + i\beta_2; \qquad \mu_4 = \bar{\mu}_2 = \alpha_2 - i\beta_2; \qquad \beta_2 \neq 0. \end{aligned} \tag{2.2}$$

Consider the complex variables z_1 and z_2, related to x and y as follows:

$$z_k = x + \mu_k y = (x + \alpha_k y) + i(\beta_k y), \qquad k = 1, 2. \tag{2.3}$$

This formula specializes to z_0, the complex variable in the original xy plane when $\mu_0 = i$. The relationship between the complex plane of z_0 and that of z_1 or z_2 is

$$z_k = p_k z_0 + \bar{q}_k \bar{z}_0, \qquad k = 1, 2, \tag{2.4}$$

where

$$p_k = \tfrac{1}{2}(1 - i\mu_k) \quad \text{and} \quad q_k = \tfrac{1}{2}(1 - i\bar{\mu}_k). \tag{2.4a}$$

Geometrically, (2.4) corresponds to affine transformations of the z_0 plane. Since $\beta_k \neq 0$, these transformations possess inverses, given by:

$$z_0 = \frac{1}{\beta_k}(\bar{p}_k z_k - \bar{q}_k \bar{z}_k), \qquad k = 1, 2. \tag{2.5}$$

Let w_0 designate a particular integral of (1.3). Lechnitzky[8] has shown that the most general solution of (1.3) can be expressed in the form:

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[6] For instance, M. T. Huber, *Probleme der Statik technisch wichtiger orthotroper Platten*, Warszawa, 1929; W. McDaniels, thesis, University of Wisconsin, 1940; S. G. Lechnitzky, *loc. cit.*, under assumption of vanishing normal load $q(x, y)$.

[7] S. G. Lechnitzky, *loc. cit.*

[8] S. G. Lechnitzky, *loc. cit.*

$$w = f_1(z_1) + \bar{f}_1(\bar{z}_1) + f_2(z_2) + \bar{f}_2(\bar{z}_2) + w_0, \tag{2.6}$$

where $f_k(z_k)$ are arbitrary functions of z_k, and $\bar{f}(\bar{z}_k)$ are their conjugates. These functions are analytic in the regions which correspond through (2.4) to the region occupied by the plate. If the roots μ_1 and μ_2 in (2.2) are not distinct, the solution w assumes the form

$$w = \bar{z}_1 f_1(z_1) + z_1 \bar{f}_1(\bar{z}_1) + g_1(z_1) + \bar{g}_1(\bar{z}_1) + w_0. \tag{2.7}$$

When the region in the xy plane is simply-connected, these analytic functions are also single-valued. The present discussion will be confined to simply-connected regions bounded by analytic curves C_0. Let the parametric representation of the boundary be:

$$C_0: \quad z_0 = x + iy = F_1(t) + iF_2(t), \qquad 0 \leqq t < 2\pi. \tag{2.8}$$

The period of the functions $F_s(t)$ is 2π; their Fourier coefficients are

$$d_{ns} = \frac{1}{\pi}\int_0^{2\pi} F_s(t)\cos nt\,dt; \qquad e_{ns} = \frac{1}{\pi}\int_0^{2\pi} F_s(t)\sin nt\,dt. \tag{2.8a}$$

The boundaries of the corresponding regions in z_k planes, by virtue of (2.3), are

$$C_k: \quad z_k = F_1(t) + \mu_k F_2(t), \qquad k = 1, 2. \tag{2.9}$$

It should be noted that the homogeneous deformations (2.4) and hence the contours C_k are determined at the outset by the anisotropy of the plate. The controlling parameters are five in number, namely b_{11}, b_{16}, $(b_{12}+b_{66})$, b_{26}, b_{22}, or b_{11}, α_1, β_1, α_2, β_2.

3. Outline of procedure. The form of the solution (2.6) suggests that some simplification could be achieved by considering the boundary conditions directly in the z_k planes. The boundary conditions usually consist of two functional relations between the deflection w and its partial derivatives of at most third order on C_0

$$G_n\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \cdots, \frac{\partial^3 w}{\partial y^3}\right) = 0, \qquad n = 1, 2. \tag{3.1}$$

If one regards z_k and $\bar{z}_k$ as the independent variables, then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z_k} + \frac{\partial}{\partial \bar{z}_k}; \qquad \frac{\partial}{\partial y} = \mu_k \frac{\partial}{\partial z_k} + \bar{\mu}_k \frac{\partial}{\partial \bar{z}_k}\cdot \tag{3.2}$$

Hence, the boundary conditions (3.1) can be formulated along either C_1 or C_2:

$$G_n^k\left(w, \frac{\partial w}{\partial z_k}, \frac{\partial w}{\partial \bar{z}_k}, \cdots, \frac{\partial^3 w}{\partial \bar{z}_k^3}\right) = 0, \qquad n, k = 1, 2. \tag{3.3}$$

It is desirable to express the boundary conditions in terms of a single variable. To effect this, suitably chosen neighborhoods of contours C_k in z_k planes are mapped conformally into some neighborhoods of circumferences γ_k of unit radius in new complex ζ_k planes in such a way that points on C_k correspond to points on γ_k.[9] Let φ_k represent the polar angle in the ζ_k planes. Identical values of φ_1 and φ_2, $0 \leqq \varphi_2 < 2\pi$, can be made to correspond to one and the same value of the parameter t on C_0, $0 \leqq t < 2\pi$, by the proper choice of the mapping functions

$$z_k = \omega_k(\zeta_k), \qquad k = 1, 2. \tag{3.4}$$

Laurent expansions for $\omega_k(\zeta_k)$ are found in terms of the Fourier coefficients of $F_s(t)$. Let the common value of ζ_1 and ζ_2 on γ_k be denoted by $\sigma \equiv e^{i\varphi}$; then by virtue of the relations

$$\frac{\partial}{\partial z_k} = \frac{1}{\omega_k'(\zeta_k)} \frac{\partial}{\partial \zeta_k}; \qquad \frac{\partial}{\partial \bar{z}_k} = \frac{1}{\bar{\omega}_k'(\bar{\zeta}_k)} \frac{\partial}{\partial \bar{\zeta}_k}, \tag{3.5}$$

the boundary conditions (3.2) can be expressed in terms of σ alone:

$$\Gamma_n\left(w, \frac{\partial w}{\partial \sigma}, \frac{\partial w}{\partial \bar{\sigma}}, \cdots, \frac{\partial^3 w}{\partial \bar{\sigma}^3}\right) = 0, \qquad n = 1, 2. \tag{3.6}$$

The deflection w may have the forms

$$w = \phi_1(\zeta_1) + \bar{\phi}_1(\bar{\zeta}_1) + \phi_2(\zeta_2) + \bar{\phi}_2(\bar{\zeta}_2) + w_0, \tag{3.7}$$

$$w = \bar{\omega}_1(\bar{\zeta}_1)\phi_1(\zeta_1) + \omega_1(\zeta_1)\bar{\phi}_1(\bar{\zeta}_1) + \psi_1(\zeta_1) + \bar{\psi}_1(\bar{\zeta}_1) + w_0, \tag{3.8}$$

where

$$\phi_k(\zeta_k) \equiv f_k\{\omega_k(\zeta_k)\} \quad \text{and} \quad \psi_k(\zeta_k) \equiv g_k\{\omega_k(\zeta_k)\}$$

are undetermined functions, analytic and single-valued in some neighborhoods of γ_k. Accordingly one has

$$\phi_k(\zeta_k) = \sum_{m=-\infty}^{+\infty} A_{mk}\zeta_k^m; \qquad \psi_k(\zeta_k) = \sum_{m=-\infty}^{+\infty} B_{mk}\zeta_k^m. \tag{3.9}$$

Expanding Γ_n in (3.6) into powers of σ and equating coefficients of like powers to zero, one may expect to obtain recursion formulae for the determination of the coefficients A_{mk} and B_{mk} in (3.9). This, in essence, solves the problem.

4. The mapping functions. The mapping functions $\omega_k(\zeta_k)$, if they exist, must be analytic and one-valued at least in narrow rings around the circumferences γ_k where they can be represented by convergent Laurent series, say

$$z_k = \sum_{m=-\infty}^{+\infty} D_{km}\zeta_k^m, \qquad k = 1, 2. \tag{4.1}$$

[9] This will be recognized as an extension of the scheme of N. I. Mushelisvili. See Mathematische Annalen, **107**, 282–312, (1932).

The correspondence between points on the curves C_k and γ_k is expressed through the equation

$$C_k:\quad z_k = \sum_{m=-\infty}^{+\infty} D_{km}\sigma_k^m = \sum_{m=-\infty}^{+\infty} D_{km}(\cos m\varphi_k + i \sin m\varphi_k). \tag{4.2}$$

The parametric representations of C_k are:

$$C_k: z_k = \tfrac{1}{2}(d_{01} + \mu_k d_{02}) + \sum_{n=1}^{\infty} \left\{ (d_{n1} + \mu_k d_{n2}) \cos nt + (e_{n1} + \mu_k e_{n2}) \sin nt \right\}. \tag{4.3}$$

Letting identical values of φ_1 and φ_2 correspond to the same value of the parameter t, $\varphi_k = t$, one can solve for the coefficients D_{km} from (4.2) and (4.3):

$$\begin{aligned} D_{kn} &= \tfrac{1}{2}\left\{ (d_{n1} + \alpha_k d_{n2} + \beta_k e_{n2}) - i(e_{n1} + \alpha_k e_{n2} - \beta_k d_{n2}) \right\}, \\ D_{k,-n} &= \tfrac{1}{2}\left\{ (d_{n1} + \alpha_k d_{n2} - \beta_k e_{n2}) + i(e_{n1} + \alpha_k e_{n2} + \beta_k d_{n2}) \right\}, \quad n \geqq 0. \end{aligned} \tag{4.4}$$

It will be shown presently that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{| D_{kn} |} < 1, \qquad \overline{\lim_{n\to\infty}} \sqrt[n]{| D_{k,-n} |} < 1, \tag{4.5}$$

so that series (4.1) with coefficients (4.4) actually converge in some rings around γ_k and the foregoing formal steps are justified. The proof rests on the following theorem:[10]

The necessary and sufficient condition that the periodic function $f(z)$ defined by the series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

be analytic in some strip parallel to and containing the real axis is

$$\overline{\lim_{n\to\infty}} \sqrt[n]{(a_n^2 + b_n^2)} < 1 \quad \text{or} \quad \overline{\lim_{n\to\infty}} \sqrt[n]{| a_n + ib_n |} < 1. \tag{4.6}$$

The functions $F_s(t) = \frac{1}{2}d_{0s} + \sum_{n=1}^{\infty}(d_{ns} \cos nt + e_{ns} \sin nt)$, $s = 1, 2$, are by hypothesis (real) analytic functions. Hence the functions defined by the series

$$\begin{aligned} &\frac{1}{2}\sum_{n=1}^{\infty} \left\{ (d_{n1} + \alpha_k d_{n2} + \beta_k e_{n2}) \cos nt - (e_{n1} + \alpha_k e_{n2} - \beta_k d_{n2}) \sin nt \right\}, \\ &\frac{1}{2}\sum_{n=1}^{\infty} \left\{ (d_{n1} + \alpha_k d_{n2} - \beta_k e_{n2}) \cos nt + (e_{n1} + \alpha_k e_{n2} + \beta_k d_{n2}) \sin nt \right\}, \end{aligned} \tag{4.7}$$

are also (real) analytic. Replacing t by the complex variable z which reduces to t on the real axis, one is led to complex functions, satisfying the conditions

[10] This is a slight modification of a theorem found on pp. 125–127 of De La Vallée Poussin's, *On the approximation of functions of a real variable and on quasi-analytic functions,* Rice Institute pamphlet, 1925.

of the preceding theorem. But since (4.6) applied to the functions (4.7) implies (4.5), the existence of regions of convergence of (4.1) containing γ_k is shown.

These regions can be narrowed to rings about[11] γ_k in which the derivatives of the mapping functions have no zeros so that the functions themselves are simple (*schlicht*). The representations of the narrowed regions on the corresponding bands about C_k are therefore one to one and conformal as desired. The foregoing discussion can be extended to any number of planes z_k. If, for instance, one sets $\mu_0 = i$, one obtains a function $\omega_0(\zeta_0)$ mapping conformally the original curve C_0 together with a surrounding band into γ_0, the circumference of unit radius in a new ζ_0 plane, and its neighborhood.

5. Mapping of elliptic regions. The procedure described in the preceding section constitutes a practical scheme for the determination of the mapping functions $\omega_k(\zeta_k)$ once the shape of the plate is known. In the case of the elliptic boundary

$$C_0: \quad z_0 = a \cos t + ib \sin t, \tag{5.1}$$

formulae (4.1) and (4.4) yield the mapping function

$$z_k = \omega_k(\zeta_k) = s_k\left(\zeta_k + \frac{R_k}{\zeta_k}\right), \tag{5.2}$$

where

$$R_k = \frac{r_k}{s_k}; \qquad r_k = \tfrac{1}{2}(a + i\mu_k b); \qquad s_k = \tfrac{1}{2}(a - i\mu_k b).$$

It follows from elementary considerations that the deformed contours in z_k planes

$$C_k: \quad z_k = a \cos \varphi + \mu_k b \sin \varphi \tag{5.3}$$

are ellipses whose major axes make angles δ_k with the real x_k axes, where

$$\tan (2\delta_k) = \frac{2\alpha_k\beta_k b^2}{a^2 + b^2(\alpha_k^2 - \beta_k^2)} = \operatorname{am} \left\{4r_k s_k\right\}. \tag{5.4}$$

It also follows from (5.2) that the families of circles concentric with γ_k in the ζ_k planes correspond to the families of confocal ellipses with foci at $2\sqrt{r_k s_k}$ in the z_k planes. In particular, to the circumferences γ_k correspond the elliptical contours (5.3) and to the smaller[12] circumferences with radii $\sqrt{|R_k|}$, the degenerate ellipses of the families, i.e., the double segments joining the foci. These degenerate ellipses can be represented in the form

[11] γ_k may well be the outer boundary of these annuli when the derivatives $F_s'(t)$, $s=1, 2$, vanish simultaneously. Thus the class of curves to which this discussion is applicable is somewhat larger than the class of analytic functions as usually defined.

[12] If μ_1 and μ_2 are defined to be the roots whose imaginary parts are positive, $\beta_k > 0$, then $\sqrt{|R_k|} < 1$.

$$z_k = 2\sqrt{(r_k s_k)} \cos (\varphi - \delta_k') \tag{5.5}$$

where δ_k' are the amplitudes of $\sqrt{R_k}$, the ζ_k plane images of the foci $2\sqrt{r_k s_k}$. Since the derivatives $\omega_k'(\zeta_k)$ vanish only at $\sqrt{R_k}$, one may choose for the rings in which the mapping functions are to be *schlicht* the annuli $1 \leqq \zeta_k < \sqrt{|R_k|}$. The corresponding regions in z_k planes comprise the interiors of the ellipses C_k made doubly-connected by the introduction of slits (5.5) between the two foci.

From (5.5) it is clear that every point z_k on the slit corresponds to two distinct points, $\sqrt{|R_k|}e^{i\varphi}$ and $\sqrt{|R_k|}e^{i(2\delta_k'-\varphi)}$, on the inner boundary of the above ring. Since the analytic functions $f_k(z_k)$ are single valued inside C_k, the transformed functions $\phi_k(\zeta_k)$ must assume identical values at the two points. Hence,

$$\sum_{m=-\infty}^{+\infty} A_{mk}\sqrt{|R_k|^m}\, e^{im\varphi} = \sum_{m=-\infty}^{+\infty} A_{mk}\sqrt{|R_k|^m}\, e^{im(2\delta_k'-\varphi)}, \tag{5.6}$$

so that

$$A_{-n,k} = R_k^n A_{nk}, \qquad n > 0,$$

and finally

$$\phi_k(\zeta_k) = \sum_{n=0}^{\infty} A_{nk}'\left(\zeta_k^n + \frac{R_k^n}{\zeta_k^n}\right); \qquad \psi_k(\zeta_k) = \sum_{n=0}^{\infty} B_{nk}'\left(\zeta_k^n + \frac{R_k^n}{\zeta_k^n}\right). \tag{5.7}$$

The question arises whether the restriction (5.6) is sufficient to insure that the functions $F_k(z_k)$ obtained by the inverse transformation

$$\zeta_k = \frac{1}{2s_k}\left\{z_k + \sqrt{(z_k^2 - 4r_k s_k)}\right\} \tag{5.8}$$

from the functions $\phi_k(\zeta_k)$ in (5.7) are single-valued in the full, simply-connected interiors of the ellipses C_k. An affirmative answer is reached after application of *Schwarz's Reflection Principle* to the potentially different branches of $F_k(z_k)$ on the opposite sides of the slits. Therefore, if the unknown coefficients in (5.7) are determined from similarly transformed boundary conditions, the resulting functions will correspond to the solutions of the original problem.

6. Uniqueness of solutions. If in (2.7) the functions $f_1(z_1)$ and $g_1(z_1)$ were replaced by $f_1(z_1) + f_1^*(z_1)$ and $g_1(z_1) + g_1^*(z_1)$ where

$$Re\left\{\bar{z}_1 f_1^*(z_1) + g_1^*(z_1)\right\} = 0, \tag{6.1}$$

the value of w would remain unaltered. The extent of arbitrariness in the choice of $f_1(z_1)$ and $g_1(z_1)$ is determined next by finding the most general analytic functions $f_1^*(z_1) = u_1 + iv_1$ and $g_1^*(z_1) = u_2 + iv_2$ satisfying (6.1), or the equivalent condition

$$x_1u_1 + y_1v_1 + u_2 = 0. \tag{6.2}$$

Utilizing the fact that u_1, v_1, u_2, and v_2 are harmonic functions, conjugate in pairs, one readily finds from (6.2) that

$$\frac{\partial u_1}{\partial x_1} = 0, \qquad \frac{\partial v_1}{\partial y_1} = 0,$$

and therefore

$$f_1^*(z_1) = \lambda_1 i z_1 + (\lambda_2 + i\lambda_3), \tag{6.3a}$$

where λ_j are arbitrary real constants. Substituting this expression into (6.1) one arrives at

$$g_1^*(z_1) = -(\lambda_2 - i\lambda_3)z_1 + i\lambda_4. \tag{6.3b}$$

The foregoing functions are transformed by (5.2) into

$$\begin{aligned} \phi_1^*(\zeta_1) &= \lambda_1 i s_1\left(\zeta_1 + \frac{R_1}{\zeta_1}\right) + \lambda_2 + i\lambda_3; \\ \psi_1^*(\zeta_1) &= -(\lambda_2 - i\lambda_3)s_1\left(\zeta_1 + \frac{R_1}{\zeta_1}\right) + i\lambda_4. \end{aligned} \tag{6.4}$$

It is desirable to eliminate the arbitrariness inherent in the form of solutions by choosing the parameters λ_j so as to simplify the expansions (5.7) of $\phi_1(\zeta_1)$ and $\psi_1(\zeta_1)$. If

$$\lambda_1 = \frac{-2 \text{ Im } \{A'_{11}\}}{a + b\beta_1}; \qquad \lambda_2 + i\lambda_3 = -2A'_{01}; \qquad \lambda_4 = -2 \text{ Im } \{B'_{01}\};$$

the functions $\phi_1 + \phi_1^*$ and $\psi_1 + \psi_1^*$ reduce to

$$\sum_{n=1}^{\infty} A_{n1}\left(\zeta_1^n + \frac{R_1^n}{\zeta_1^n}\right) \quad \text{and} \quad \sum_{n=0}^{\infty} B_{n1}\left(\zeta_1^n + \frac{R_1^n}{\zeta_1^n}\right), \tag{6.5}$$

with $A_{11} = \overline{A}_{11}$ and $B_{01} = \overline{B}_{01}$. That the solutions of the form (6.5) are uniquely determined by the boundary conditions is shown in Section 8.

A similar argument shows that one needs to consider only the functions

$$\phi_1(\zeta_1) = \sum_{n=0}^{\infty} A_{n1}\left(\zeta_1^n + \frac{R_1^n}{\zeta_1^n}\right); \qquad \phi_2(\zeta_2) = \sum_{n=2}^{\infty} A_{n2}\left(\zeta_2^n + \frac{R_2^n}{\zeta_2^n}\right), \tag{6.6}$$

with $A_{01} = \overline{A}_{01}$ and $A_{22} = \overline{A}_{22}$, when the roots μ_1 and μ_2 are distinct.

7. Boundary conditions. Let θ designate the angle between the outward normal to the boundary C_0 and x axis. Then,

$$\begin{aligned} \sin\theta &= -\frac{1}{2ds}(dz_0 + d\bar{z}_0) = -\frac{1}{2ds\beta_k}(i\bar{\mu}_k dz_k - i\mu_k d\bar{z}_k), \\ \cos\theta &= \frac{i}{2ds}(dz_0 - d\bar{z}_0) = \frac{i}{2ds\beta_k}(dz_k - d\bar{z}_k), \end{aligned} \tag{7.1}$$

where dz_0 and ds are differentials along C_0 and dz_k the corresponding (dependent) differentials in the z_k planes. The equations (7.1) and (3.2), together with the mapping functions (3.4) make it possible to formulate the boundary conditions in terms of σ on the circumferences γ_k. In the case of a clamped plate one has the conditions:

$$w(x, y) = 0, \qquad \frac{\partial w}{\partial n} = \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right) w(x, y) = 0, \tag{7.2}$$

on C_0, which are equivalent to the vanishing, on γ_k, of the expressions[13]

$$w(\sigma, \bar{\sigma}) = 0, \tag{7.3a}$$

$$\begin{aligned} -\frac{i}{\beta_k\sigma}\frac{d\sigma}{ds}\Bigg[&\left\{\sigma(p_k\bar{p}_k + q_k\bar{q}_k) + \frac{1}{\sigma}\frac{\bar{\omega}_k'(\bar{\sigma})}{\omega_k'(\sigma)} 2p_k\bar{q}_k\right\}\frac{\partial}{\partial\sigma} \\ &+ \left\{\sigma\frac{\omega_k'(\sigma)}{\bar{\omega}_k'(\bar{\sigma})} 2\bar{p}_k q_k + \frac{1}{\sigma}(p_k\bar{p}_k + q_k\bar{q}_k)\right\}\frac{\partial}{\partial\bar{\sigma}}\Bigg] w(\sigma, \bar{\sigma}) = 0, \end{aligned} \tag{7.3b}$$

where $d\sigma$ is dependent on dz_0. Obviously, to the part of $w(\sigma, \bar{\sigma})$ obtained by transformation of $f_i(z_i)$ and $g_i(z_i)$ one applies the operator in (7.3b) with $k = i$. It is expedient to make use of the mapping function $z_0 = \omega_0(\zeta_0)$ for the transformation of the particular integral w_0. When $k = 0$ the operator in (7.3b) reduces to

$$-\frac{i}{\sigma}\frac{d\sigma}{ds}\left(\sigma\frac{\partial}{\partial\sigma} + \frac{1}{\sigma}\frac{\partial}{\partial\bar{\sigma}}\right) w_0(\sigma, \bar{\sigma}), \tag{7.4}$$

and (5.2) has the form

$$z_0 = \omega_0(\zeta_0) = \tfrac{1}{2}(a - b)\left(k^2\zeta_0 + \frac{1}{\zeta_0}\right), \qquad k^2 = \frac{a + b}{a - b}\cdot \tag{7.5}$$

If the normal load $q(x, y)$ in (1.3) is expressed as a polynomial[14] in x and y of degree $(n_1 - 4)$, w_0 is a polynomial of degree n_1. For instance, when the load is linear

$$q(x, y) = \frac{2h^3}{3}(c_0 + c_1 x + c_2 y), \tag{7.6}$$

where c_i are real parameters, one can take for w_0,

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[13] It is noted that, as a consequence of (3.2) and (3.4), σ and $\bar{\sigma}$ are treated as independent variables in the partial differentiation process, whereas the result can be simplified by $\bar{\sigma} = 1/\sigma$, an identity on γ_k.

[14] This may be considered as an approximation to more complicated loading functions.

$$w_0(x, y) = \frac{c_0}{4!\,b_{11}} x^4 + \frac{c_1}{5!\,b_{11}} x^5 + \frac{c_2}{5!\,b_{22}} y^5. \tag{7.7}$$

The following abbreviations referring to values on the original curve C_0 but expressed on γ_k are introduced:

$$\begin{aligned} w_0 &= -K(\sigma) = -\sum_0^{n_1}\left(b_m\sigma^m + \bar{b}_m \frac{1}{\sigma^m}\right), \\ i\sigma \frac{ds}{d\sigma}\frac{\partial w_0}{\partial n} &= -L(\sigma) = -\sum_0^{n_1}\left(a_m\sigma^m + \bar{a}_m \frac{1}{\sigma^m}\right). \end{aligned} \tag{7.8}$$

One readily finds the coefficients a_m and b_m for the special case of the linear load applied to an elliptic plate:

$$\begin{gathered} a_4 = -\frac{4ba^3c_0}{2^4\,4!\,b_{11}}; \qquad a_5 = -\frac{5ab}{2^5\,5!}\left(\frac{c_1a^3}{b_{11}} - i\frac{c_2b^3}{b_{22}}\right), \\ a_0 = 3a_4, \qquad a_2 = 4a_4; \qquad a_3 = 5\bar{a}_5, \qquad a_1 = 10a_5, \\ b_{2m} = \frac{a}{4b}a_{2m}; \qquad (b_{2m+1} \pm \bar{b}_{2m+1}) = \frac{a}{5b}(a_{2m+1} \pm \bar{a}_{2m+1}). \end{gathered} \tag{7.9}$$

The preceding transformation of the particular integral w_0 is valid whether the deflection w has the form (2.6) or (2.7). However, the parts of w involving the unknown functions $f_k(z_k)$ and $g_k(z_k)$ have to be treated somewhat differently in the cases of equal and distinct roots. To avoid repetition only the first case is discussed here.

8. The case of equal roots. Substituting[15] (3.8) into the boundary conditions (7.3a) and (7.3b) and simplifying gives:

$$\bar{\omega}\left(\frac{1}{\sigma}\right)\phi(\sigma) + \omega(\sigma)\bar{\phi}\left(\frac{1}{\sigma}\right) + \psi(\sigma) + \bar{\psi}\left(\frac{1}{\sigma}\right) = K(\sigma), \tag{8.1a}$$

$$\begin{aligned} &\frac{1}{\beta}\left[\sigma(p\bar{p} + q\bar{q}) + \frac{1}{\sigma}\,\frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\omega'(\sigma)}\,2p\bar{q}\right] \\ &\qquad\cdot\left[\bar{\omega}\left(\frac{1}{\sigma}\right)\phi'(\sigma) + \omega'(\sigma)\bar{\phi}\left(\frac{1}{\sigma}\right) + \psi'(\sigma)\right] \\ &+\frac{1}{\beta}\left[\sigma\,\frac{\omega'(\sigma)}{\bar{\omega}'\left(\frac{1}{\sigma}\right)}\,2\bar{p}q + \frac{1}{\sigma}(p\bar{p} + q\bar{q})\right] \\ &\qquad\cdot\left[\omega(\sigma)\bar{\phi}'\left(\frac{1}{\sigma}\right) + \bar{\omega}'\left(\frac{1}{\sigma}\right)\phi(\sigma) + \bar{\psi}'\left(\frac{1}{\sigma}\right)\right] = L(\sigma). \end{aligned} \tag{8.1b}$$

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[15] In this section the subscripts are omitted since the distinction between z_1 and z_2 planes is not involved in the discussion.

In order to avoid a system of an infinite number of equations in infinitely many unknowns the following device is used. The condition (8.1b) is replaced by a combination of (8.1b) and (8.1a). The first boundary condition (8.1a) is an identity in σ and will therefore yield an equality upon differentiation with respect to σ. This equation is multiplied by

$$\frac{\sigma}{\beta}\left\{\frac{\sigma^2\omega'(\sigma)}{\bar{\omega}'\left(\frac{1}{\sigma}\right)}2\bar{p} + (p\bar{p}+q\bar{q})\right\}$$

and added to (8.1b). In the resulting identity, one solves for the expression involving the unknown functions and obtains[16]

$$\begin{aligned}\bar{\omega}\left(\frac{1}{\sigma}\right)\sigma\phi'(\sigma) + \sigma\omega'(\sigma)\bar{\phi}\left(\frac{1}{\sigma}\right) + \sigma\psi'(\sigma)\\ = \frac{1}{2}\left\{\frac{p}{\frac{p}{\sigma}\bar{\omega}'\left(\frac{1}{\sigma}\right) + q\sigma\omega'(\sigma)} - \frac{\bar{q}}{\bar{p}\sigma\omega'(\sigma) + \frac{\bar{q}}{\sigma}\bar{\omega}'\left(\frac{1}{\sigma}\right)}\right\}\\ \cdot\left\{\frac{L(\sigma)}{\sigma}\bar{\omega}'\left(\frac{1}{\sigma}\right) + \frac{\sigma K'(\sigma)}{\beta}\left(\sigma\omega'(\sigma)2\bar{p}q + \frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\sigma}(p\bar{p}+q\bar{q})\right)\right\}.\end{aligned} \tag{8.2}$$

When the plate is elliptic, (8.1a) and (8.2) read:

$$\begin{aligned}\bar{s}\left(\frac{1}{\sigma}+\bar{R}\sigma\right)\sum_1^\infty A_m\left(\sigma^m+\frac{R^m}{\sigma^m}\right) + s\left(\sigma+\frac{R}{\sigma}\right)\sum_1^\infty \bar{A}_m\left(\frac{1}{\sigma^m}+\bar{R}^m\sigma^m\right)\\ + \sum_0^\infty B_m\left(\sigma^m+\frac{R^m}{\sigma^m}\right) + \sum_0^\infty \bar{B}_m\left(\frac{1}{\sigma^m}+\bar{R}^m\sigma^m\right)\\ = \sum_0^{n_1}\left(b_m\sigma^m + \bar{b}_m\frac{1}{\sigma^m}\right),\end{aligned} \tag{8.3}$$

$$\begin{aligned}\bar{s}\left(\frac{1}{\sigma}+\bar{R}\sigma\right)\sum_1^\infty mA_m\left(\sigma^m-\frac{R^m}{\sigma^m}\right) + s\left(\sigma-\frac{R}{\sigma}\right)\sum_1^\infty \bar{A}_m\left(\frac{1}{\sigma^m}+\bar{R}^m\sigma^m\right)\\ + \sum_0^\infty mB_m\left(\sigma^m-\frac{R^m}{\sigma^m}\right) = \frac{1}{\sigma\beta(a+b)}\left[\frac{p\sigma^2}{1-\sigma^2/k^2} - \frac{\bar{q}}{1-1/k^2\sigma^2}\right]\\ \cdot\left[\bar{s}\left(\frac{1}{\sigma}-\bar{R}\sigma\right)\sum_0^{n_1}\left(a_m\sigma^m+\frac{\bar{a}_m}{\sigma^m}\right) + \tfrac{1}{2}\{(b-i\bar{\mu}a)\sigma\right.\\ \left. + \frac{1}{\sigma}(b+i\bar{\mu}a)\}\sum_0^{n_1} m\left(b_m\sigma^m - \bar{b}_m\frac{1}{\sigma^m}\right)\right].\end{aligned} \tag{8.4a}$$

[16] The expressions in the denominators, $1/\beta(\bar{p}\sigma\omega'(\sigma)+\bar{q}/\sigma\ \bar{\omega}'(1/\sigma))$ and $1/\beta(p/\sigma\ \bar{\omega}'(1/\sigma)+q\sigma\omega'(\sigma))$, never vanish, since $p\bar{p}-q\bar{q}=\beta\neq 0$.

For a circular plate $a=b$, $1/k^2$ vanishes, and the last equation is already expanded in series of powers of σ. Otherwise, the expansion of the right hand member is infinite, since

$$\frac{1}{\sigma}\left\{\frac{p\sigma^2}{1-\sigma^2/k^2}-\frac{\bar{q}}{1-1/k^2\sigma^2}\right\}=p\sigma\sum_{l=0}^{\infty}\left(\frac{\sigma}{k}\right)^{2l}-\frac{\bar{q}}{\sigma}\sum_{l=0}^{\infty}\left(\frac{1}{k\sigma}\right)^{2l}. \tag{8.5}$$

Combining the terms $\bar{s}(1/\sigma-\overline{R}\sigma)$ and $\frac{1}{2}((b-i\bar{\mu}a)\sigma+1/\sigma(b+i\bar{\mu}a))$ with the expansion (8.5), one simplifies the right-hand member of (8.4a):

$$\frac{1}{2\beta(a+b)}\left[(a+\mu\bar{\mu}b)\sum_{0}^{n_1}\left(a_m\sigma^m+\bar{a}_m\frac{1}{\sigma^m}\right)+i(\bar{\mu}a-\mu b)\sum_{0}^{n_1}m\left(b_m\sigma^m-\bar{b}_m\frac{1}{\sigma^m}\right)\right]$$

$$-\frac{2ab}{\beta(a^2-b^2)}\left[pq\left(\sum_{0}^{n_1}\left\{(a_m-mb_m)\sigma^m+(\bar{a}_m+m\bar{b}_m)\frac{1}{\sigma^m}\right\}\right)\sum_{1}^{\infty}\left(\frac{\sigma}{k}\right)^{2l}\right.$$

$$\left.+\bar{p}\bar{q}\left(\sum_{0}^{n_1}\left\{(a_m+mb_m)\sigma^m+(\bar{a}_m-m\bar{b}_m)\frac{1}{\sigma^m}\right\}\right)\sum_{1}^{\infty}\left(\frac{1}{\sigma k}\right)^{2l}\right]. \tag{8.4b}$$

Let Q_n denote the coefficient of σ^n in (8.4b). This coefficient is evaluated by means of relations similar to (7.9). It is found that Q_n vanishes for $n>n_1$.

Recalling that $A_0=0$, $A_1=\overline{A}_1$, and $B_0=\overline{B}_0$, one concludes from the relations between the coefficients of σ^0 in (8.4a) and (8.3):

$$A_1=\frac{Q_0}{(1-P)(s+\bar{s})},\quad B_0=\frac{1}{2}b_0-\frac{1}{4}\frac{1+P}{1-P}Q_0,\ \text{where } P=R\overline{R}. \tag{8.6}$$

The system of equations between the coefficients of σ^n, $n>1$,

$$\begin{aligned}\bar{s}A_{n+1}+sR\overline{R}^{n+1}\overline{A}_{n+1}+B_n+\overline{R}^n\overline{B}_n&=b_n-\bar{s}\overline{R}A_{n-1}-s\overline{R}^{n-1}\overline{A}_{n-1},\\ \bar{s}(n+1)A_{n+1}-sR\overline{R}^{n+1}\overline{A}_{n+1}+nB_n&=Q_n-\bar{s}\overline{R}(n-1)A_{n-1}-s\overline{R}^{n-1}\overline{A}_{n-1},\end{aligned} \tag{8.7}$$

and of the conjugate equations has a non-vanishing determinant:

$$s\bar{s}D_{n+1}(P)=s\bar{s}\left\{1-(n+1)^2P^n+2n(n+2)P^{n+1}-(n+1)^2P^{n+2}+P^{2n+2}\right\},\quad n>1,$$

because $P<1$. The system, therefore, possesses a unique solution given by the expressions:

$$\begin{aligned}A_{n+1}=\frac{1}{\bar{s}D_{n+1}}&\left[Q_n(1-(n+1)P^n+nP^{n+1})-\overline{Q}_n\overline{R}^n(n-(n+1)P+P^{n+1})\right.\\ &-nb_n(1-P^{n+1})+n(n+1)\bar{b}_n\overline{R}^n(1-P)+\bar{s}\overline{R}D_n(P)A_{n-1}\\ &\left.+s\overline{R}^{n-1}E_n(P)\overline{A}_{n-1}\right],\end{aligned} \tag{8.8}$$

$$B_n = \frac{-1}{D_{n+1}} [Q_n\{1 - (n+2)P^{n+1} + (n+1)P^{n+2}\}$$
$$- \overline{Q}_n\overline{R}^n\{(n+1) - (n+2)P + P^{n+2}\} - (n+1)b_n(1 - P^{n+2})$$
$$+ \bar{b}_n\overline{R}^n\{(n+1)^2 - n(n+2)P - P^{n+2}\}$$
$$+ \bar{s}\overline{R}G_{n+1}(P)A_{n-1} + s\overline{R}^{n-1}E_{n+1}(P)\overline{A}_{n-1}], \tag{8.9}$$

where the following abreviations are used:

$$E_n(P) = \{(n-1) - 2nP + (n+1)P^2 + (n+1)P^n - 2nP^{n+1} + (n-1)P^{n+2}\},$$
$$G_{n+1}(P) = \{2 - n(n+1)P^{n-1} + (n-1)(n+2)P^n + (n-1)(n+2)P^{n+1}$$
$$- n(n+1)P^{n+2} + 2P^{2n+1}\}.$$

One can calculate any number of coefficients in the expansions of $\phi(\zeta)$ and $\psi(\zeta)$ from those already known, A_0 and A_1, and the recurrence formulae (8.8) and (8.9). If (8.8) is rewritten for $n > n_1$ in the form

$$\frac{A_{n+1}}{A_{n-1}} = \frac{1}{D_{n+1}(P)}\left\{\overline{R}D_n(P) + \frac{s}{\bar{s}}\overline{R}^{n+1}E_n(P)\frac{\overline{A}_{n-1}}{A_{n-1}}\right\}, \tag{8.10}$$

One sees that $\lim_{n\to\infty}|A_{n+1}/A_{n-1}| = |\overline{R}|$. Consequently, $\phi(\zeta)$ and $\psi(\zeta)$ converge and represent analytic functions in the ring $|R|^{3/2} < |\zeta| < |R|^{-1/2}$, which extends beyond the ring corresponding to the interior of the ellipse C_0. The expansions are finite if A_{n-1} and A_n vanish for any $n > n_1$. In the case of the linear load one finds $A_5 = A_6 = 0$, $B_5 \neq 0$, so that the solution is a polynomial of fifth degree. Specific examples indicate that the solution is a polynomial when the loading function $q(x, y)$ is a polynomial.

9. Conclusion. The discussion in the case of distinct roots is similar to that of Section 8. For details and results the reader is referred to the author's doctoral thesis.[17]

When $\mu_1 = \mu_2 = i$, the solution reduces to the results for an isotropic clamped elliptic plate bent by uniformly varying load.[18] Another important special case is that of an orthotropic elliptic plate bent by a linear load, for which the solution is new. There are two possibilities obtained by setting either $\mu_1 = i\beta_1$, $\mu_2 = i\beta_2$, or $\mu_1 = \alpha_1 + i\beta_1$, $\mu_2 = -\alpha_1 + i\beta_1$. The calculations in these and other specific cases are comparatively simple. It is clear that by specifying the values of some or all of the parameters characterizing the material

[17] *On the deflection of anisotropic thin plates*, University of Wisconsin, 1942.

[18] See A. E. H. Love, *Treatise on the Mathematical Theory of Elasticity*, Cambridge, 1927, p. 486.

of the plate, its shape, and the type of loading, a considerable simplification is achieved.

The foregoing method can be extended to the cases of (a) different boundary conditions, (b) different shapes of plates. Many of the devices and formulae, such as those of Sections 4, 6, and 7, remain valid. The extension to the interesting case of infinite doubly-connected regions involves little more than the additional feature of determining the character of the multiple-valued functions $f_k(z_k)$ from the fact that the deflection w remains single-valued and continuous.