QUARTERLY OF APPLIED MATHEMATICS

Vol. I JULY, 1943 No. 2

FORCED VIBRATIONS OF SYSTEMS WITH NONLINEAR RESTORING FORCE*

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1. Introduction. The purpose of our lecture is, briefly stated, the discussion of periodic solutions of the differential equation

$$\ddot{x} + f(x) = F \cos \omega t, \tag{1}$$

in which $f(x)$ is in general a nonlinear function of x. For $f(x) = g/l \sin x$ we have, for example, the case of the simple pendulum with a periodic external force. Such a differential equation may occur in electrical problems when iron is contained in the magnetic circuit.[1] Any system with one degree of freedom[2] and consisting of a mass and an elastic restoring force will lead to a differential equation of the type (1) if the displacement x is not kept very small.

The problems to be discussed here differ essentially from those of self-excited oscillations where *negative damping* is involved, as is the case for the group of problems first treated by van der Pol. Neither the nonlinear character of the restoring force as such nor linearity or nonlinearity in possible friction forces is responsible for the differences. In the case of self-excited oscillations results are altered but little if a nonlinear restoring force is assumed, while the results to be discussed here would not be altered in general character if nonlinear *positive damping* were assumed. In other words, it appears that even a slight negative damping will dominate the entire phenomena.

Our discussion will yield, for the most part, results which are already well known. Much of the material can be found in the standard textbooks, e.g., in the books of Timoshenko [28] and den Hartog [5] cf. also [11, 20]. How-

* Lecture given at the Conference on Non-Linear Mechanics, Brown University, August 1–3, 1942. Manuscript received Oct. 15, 1942.

[1] For this subject see [32] and the list of references given there.

[2] Some cases of free oscillations with two degrees of freedom have been treated [4] and [12] (3. Mitt.). For experimental results in a case with infinitely many degrees of freedom, see [22].

ever, the problems are of sufficient intrinsic importance that they merit thorough consideration and discussion.

We are interested, to begin with, in obtaining insight into the physical phenomena connected with the case under discussion. It turns out that it is highly useful to begin by obtaining information of a *qualitative* character. Once this has been done, it is not very difficult to see how one should proceed to obtain accurate quantitative information. The validity of the methods to be discussed here could be justified rigorously in most cases, but we do not wish to take up these matters here. We prefer to stress methods which lead to the essential qualitative information in as simple a manner as possible. What might be found original in this lecture consists largely in the exploitation of this point of view.

2. Nonlinear restoring forces. The differential equation (1) has been treated for a considerable number of different restoring forces $f(x)$, cf. [5, 6, 7, 16, 24, 25, 28]. Some of these cases are indicated in fig. 1:

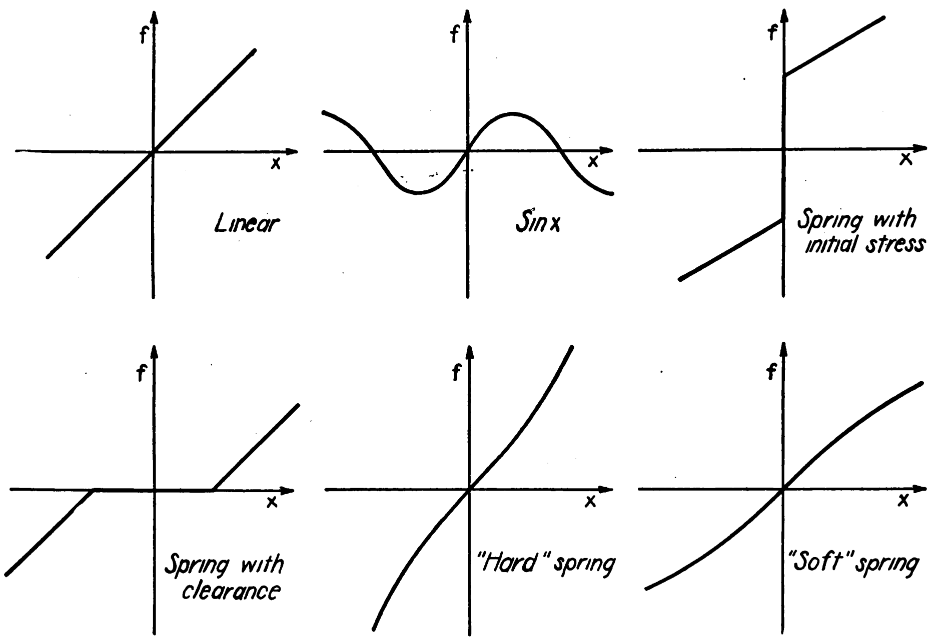

FIG. 1

The forces indicated in fig. 1 are *symmetrical*, i.e. $f(-x) = -f(x)$. This need not always be the case, of course; but we shall assume it in what follows. The last two examples, for the *hard* and *soft* springs, indicate the essential distinction between different types of springs, i.e., those for which the *stiffness* $f'(x)$ is an increasing or a decreasing function of x. The first case is typified by a mass attached to a stretched string, the second by the pendulum.

Since methods and also qualitative results appear not to depend greatly upon the special form of $f(x)$, we shall choose for this function always the expression

$$f(x) \equiv \alpha x + \beta x^3, \qquad \alpha > 0.$$

In any case there is no great loss in generality involved for moderate values of x, since our expression might be regarded as the linear and cubic terms in the power series for $f(x)$. Note also that $\beta > 0$ characterizes a hard spring (stretched string), while $\beta < 0$ characterizes a soft spring (pendulum).

The differential equation (first treated by Duffing [8]) which we want to consider in the following is, thus,

$$\ddot{x} + (\alpha x + \beta x^3) = F \cos \omega t. \tag{2}$$

3. Periodic solution. We seek periodic solutions of (2) with frequency ω. (The term frequency will be used here in place of the more correct term circular frequency, since no confusion is likely to result). That solutions of (2) other than periodic ones exist is certainly true (even unbounded ones exist for $\beta < 0$). However, the literature of the subject is almost entirely devoted to the periodic solutions, which are the interesting ones in the engineering applications. Apparently the experimenters always find periodic motions, at least after some transient motions have died out. Viscous damping, which is always present in any actual case, seems to act in such a way that the motions in a wide variety of cases tend, as $t \to \infty$, to periodic ones. It would be of considerable interest to prove that the solutions of (2) with a damping term added (and under appropriate conditions) are of this character.

We turn, then, to the problem of finding periodic solutions of (2). No explicit solutions of this differential equation are known and we are forced to turn to approximate methods. Perhaps the simplest of these is the iteration method. Let us write (2) in the form

$$\ddot{x} = -(\alpha x + \beta x^3) + F \cos \omega t \tag{3}$$

and insert

$$x_0 = A \cos \omega t \tag{4}$$

as a first approximation in the right-hand side. This means, in effect, that we assume β to be small so that a motion not greatly different from a simple harmonic motion can be expected. In fact, we assume in all of our methods but one that β is small. Upon using the identity

$$\cos^3 \omega t \equiv \tfrac{3}{4} \cos \omega t + \tfrac{1}{4} \cos 3\omega t \tag{5}$$

we find

$$\ddot{x}_1 = -(\alpha A + \tfrac{3}{4}\beta A^3 - F) \cos \omega t - \tfrac{1}{4}\beta A^3 \cos 3\omega t \tag{6}$$

as equation for the next approximation x_1. Integrating this one obtains

$$x_1 = \frac{1}{\omega^2}\left(\alpha A + \tfrac{3}{4}\beta A^3 - F\right)\cos \omega t + \frac{1}{36}\frac{\beta A^3}{\omega^2}\cos 3\omega t, \tag{7}$$

the integration constants being taken as zero to insure that x_1 and the next following approximation x_2 be periodic.

So far this is quite straightforward. What to do from now on is not so clear. One might seek to continue the process by inserting x_1 in the right side of (3) in order to find an x_2, etc., which would be a rather natural procedure. The significance of this procedure can be illustrated by the following figure which indicates the well known *response* curve for the linear forced vibration, i.e., a curve showing the amplitude $|A|$ of the steady forced vibration as a function of the frequency ω, the force amplitude F being a parameter (fig. 2):

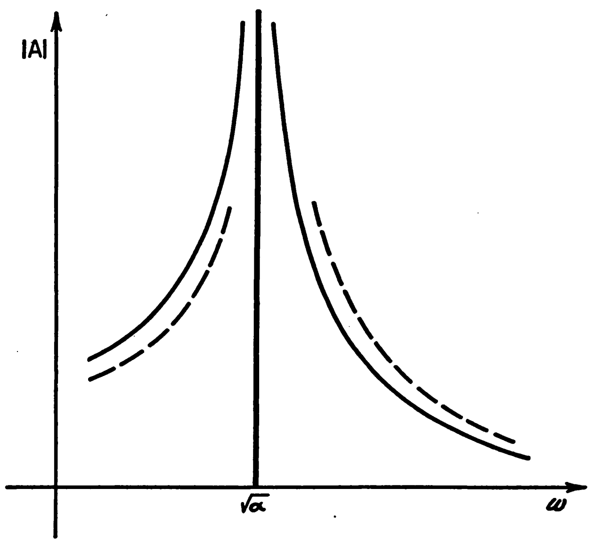

FIG. 2

The procedure just outlined would mean that ω is held fixed while $|A|$ is left open, for $\beta \neq 0$. This would yield certain curve (shown dotted) in the neighborhood of the linear response curve. It is clear that there would be difficulties near $\omega = \sqrt{\alpha}$. But worst of all, the really essential features of the response curves for $\beta \neq 0$ would not be obtained at all.

The method used by Duffing [8], who seems to have been the first to obtain the significant results for the differential equation (2) was as follows: the coefficient A_1 of $\cos \omega t$ in (7) is taken equal to A in (4), on the ground that A_1 should differ but little from A if (4) is truly a reasonable first approximation. Also, Duffing argues, such a procedure would furnish the exact result in the linear case ($\beta = 0$) and might hence be expected to yield good results for β small, which we assume here anyway so that convergence can be expected. This reasoning of Duffing leads thus to what proves to be the decisive relation (for $A \neq 0$)

$$\omega^2 = \alpha + \tfrac{3}{4}\beta A^2 - \frac{F}{A} \tag{8}$$

between the *amplitude* A and frequency ω of the periodic solution. The relation (8) has been written purposely so that ω^2 is given as a function of A: as we shall see it is decisive to consider A as a prescribed quantity in terms of which ω is to be determined.

Before discussing the significance and interpretation of (8), let us first obtain the same relation by a procedure which is perhaps a more systematic one. Equation (3) is modified by adding a term $\omega^2 x$ to both sides:

$$\ddot{x} + \omega^2 x = -\left[(\alpha - \omega^2)x + \beta x^3\right] + F \cos \omega t. \tag{9}$$

As a first approximation x_0 to a periodic solution we begin with the solution of (9) for $\alpha = \omega^2$, $\beta = 0$, $F = 0$, i.e., with a free undamped linear oscillation. This leads at once to

$$x_0 = A \cos \omega t, \tag{10}$$

with A arbitrary. Upon insertion in the right-hand side of (9) and use of (5) once more we find

$$\ddot{x}_1 + \omega^2 x_1 = \left\{(\omega^2 - \alpha)A - \tfrac{3}{4}\beta A^3 + F\right\} \cos \omega t - \tfrac{1}{4}\beta A^3 \cos 3\omega t. \tag{11}$$

We require always that the solution be periodic; hence it is necessary that the coefficient P_1 of $\cos \omega t$ in the right hand side of (11) should vanish in order to avoid the resonance case and hence the occurrence of nonperiodic terms in the solution of (11). The vanishing of this coefficient yields, obviously, the same relation as (8), which is to be regarded as an equation to determine ω after A has been prescribed. Once this relation has been satisfied, the solution of (11) will be

$$x_1 = A_1 \cos \omega t + B_1 \sin \omega t + \frac{\beta A^3}{32\omega^2} \cos 3\omega t, \tag{12}$$

in which A_1 and B_1 are arbitrary. The method of fixing A_1 and B_1 is, however, now clear. We simply set $A_1 = A$ and $B_1 = 0$, the value of ω being left open. If we stop with the degree of approximation implied in x_1 we should have, then, as approximate solution

$$x = A \cos \omega t + \frac{1}{32} \frac{\beta A^3}{\alpha + \tfrac{3}{4}\beta A^2 - \dfrac{F}{A}} \cos 3\omega t. \tag{13}$$

It is perhaps worth while to consider how the iterations should proceed: x_1 from (13) is inserted again in the right-hand side of (9). A term $P_2 \cos \omega t$ will again occur and we must require, as before, that $P_2 = 0$. The equation $P_2 = 0$ yields an improved relation between ω and A that takes the place of (8). After integration the coefficient of $\cos \omega t$ is again prescribed to have the value A while the coefficient of $\sin \omega t$ is taken as zero.

One could interpret this method as meaning that the first coefficients in the Fourier series for $x(t)$ are prescribed once and for all to be A and zero, while the frequency is left open. In what follows it is convenient to speak of the quantity A as the *amplitude* of the oscillation, though it should be referred to rather as the first Fourier coefficient.

Our procedures, in any case, require that A be fixed while ω, the frequency, is to be determined by relation (8). Why this at first sight seemingly unnatural procedure leads to the desired results can be best seen through a discussion of relation (8), which yields a set of curves, the *response* curves (in first approximation), with the force amplitude F as parameter. Usually the sign of A is not considered essential (it means a phase shift) so that only $|A|$ is plotted. The curves for $|A|$ against ω are then readily seen to be of the form shown in fig. 3, where curves for $\beta > 0$, $\beta = 0$ and $\beta < 0$ are given:

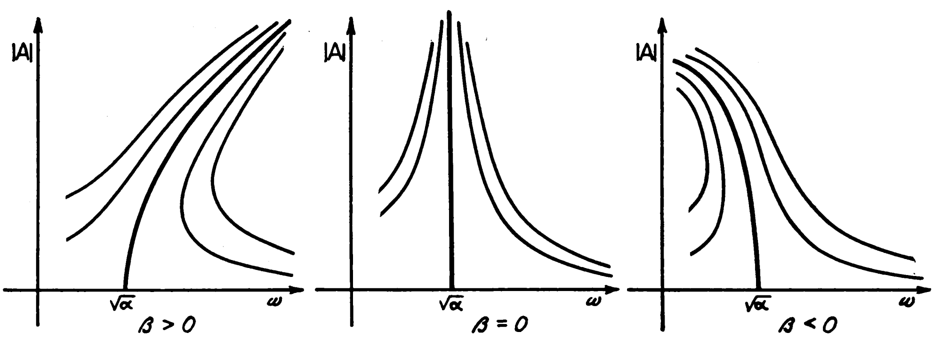

FIG. 3

The interpretation of the effect of the nonlinearity on the response curves is quite clear: The entire family of response curves (with F as parameter) for the linear forced oscillations is bent left or right depending upon whether the spring is a soft or a hard one. That these curves mirror the essential facts, at least for β and A not too large, cannot be doubted.

We can now see the reasons why A, instead of ω, should be prescribed. To begin with, we started our approximation with $\omega = \sqrt{\alpha}$, $\beta = 0$, $F = 0$, that is, at point 1 in fig. 4, which superimposes curves for $\beta = 0$ upon curves for $\beta > 0$. One sees that, for $\beta = 0$, A is completely arbitrary while ω is fixed and it is therefore necessary to prescribe A but leave ω open for $\beta \neq 0$.

The further steps in the iterations correspond to the passage from this state to one indicated by point 2 in fig. 4. One sees also that to hold ω fixed while varying A could yield at most one of the branches of the response curves.

This discussion indicates that the guiding principle to be followed independent of the particular scheme of approximation used is to observe what is arbitrary and what is fixed in the linear case, i.e., for $\beta=0$. For $\beta\neq 0$, any quantity that is arbitrary for $\beta=0$ should be prescribed, and vice versa.

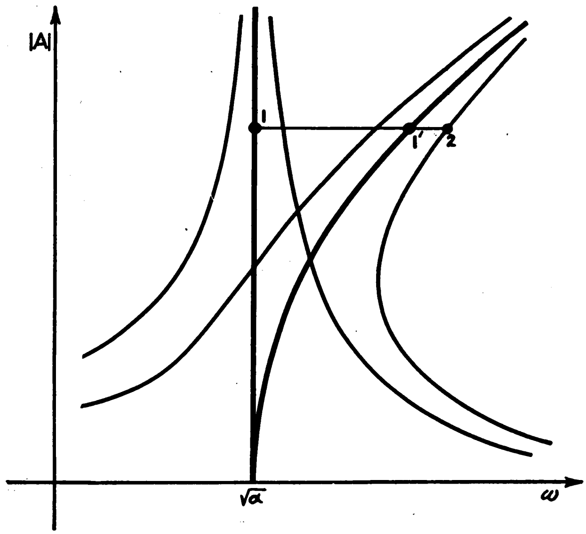

FIG. 4

4. Other methods. Before proceeding to draw the interesting physical conclusions from the response curves, we prefer to discuss some other methods of approximating the periodic solutions of (2).

One of the best known and most often used methods for nonlinear problems is the perturbation method, cf. [28], which consists for our problems in developing the solution $x(t)$ in powers of a parameter, say ϵ, with coefficients prescribed to be periodic functions of t:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots . \tag{14}$$

However, as we have seen, the frequency should not be prescribed but rather left open, and this requires that ω should also be developed with respect to ϵ:

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots , \tag{15}$$

where the ω_i are constants. It is awkward to work with functions which have a variable period, but one can avoid this by introducing a new variable $\theta = \omega t$, so that (2) becomes

$$\omega^2 x'' + (\alpha x + \beta x^3) = F \cos \theta, \tag{16}$$

the prime meaning differentiation with respect to θ. The functions x_i in (14) can now be prescribed to have the fixed period 2π. The perturbation series (14) and (15) are now to be inserted in (16), and the coefficients of the powers of ϵ in the resulting series are equated to zero. The result is a set of linear differential equations for the x_i. The parameter ϵ is to a certain degree arbitrary. A very reasonable choice for it would be to take $\epsilon = \beta$, so that (14) could be considered a development in the neighborhood of the solutions of the linearized vibration problem. The first few steps lead to essentially the same result as those obtained above by iteration. One finds, for example, that relation (8) results from the periodicity requirement. It should be said that the convergence of the perturbation series could be established with no great difficulty.

Since the solutions which interest us are periodic ones, they will possess Fourier series developments. This suggests writing the solutions $x(t)$ at the outset as such a series with undetermined coefficients:

$$\begin{aligned} x = A_0 + A_1 \cos \omega t + A_2 \cos 2\omega t + \cdots \\ + B_1 \sin \omega t + B_2 \sin 2\omega t + \cdots . \end{aligned} \tag{17}$$

Upon insertion into (2) one obtains an infinite set of nonlinear algebraic equations for the A_i and B_i which in their turn are to be solved by iterations or perturbations. Our previous experience indicates, however, that we should not prescribe ω but rather the first coefficients in (17) and this is decisive in fixing the correct set-up for iterations, say. Probably the Fourier series method is the simplest for actual computation once one has the clue to the correct procedure.

The approximation methods considered so far have all taken the solution for the linear free oscillation as a starting point. Although it is true that the significant qualitative features of the response curves can thus be obtained, it is clear that other procedures could be expected to yield more accurate results. Instead of beginning with the free linear oscillation, for example, one might begin with the free nonlinear oscillation, that is, at point 1′ rather than point 1 in fig. 4. For a given value of the force amplitude F it is quite obvious that the solution corresponding to point 1′ is much nearer to the desired solution (point 2) than is point 1. It is, of course, quite feasible to begin with a solution corresponding to point 1′, i.e., with a solution of $\ddot{x} + f(x) = 0$, since such a differential equation can be solved by explicit integration. With this solution as a basis one might then proceed by the perturbation method using F as small parameter, as for example has been done by G. Hamel [9] and R. Iglisch [12, 13]. The method of variation of parameters has also been used, for example, by E. W. Brown [4], and A. N. Krylov and N. Bogoliùboff [18, 19] have used such methods in a considerable number of papers.

The method of M. Rauscher [24, 25] is an iteration procedure which also takes the solution corresponding to point 1′ as its starting point. It is convenient here also to introduce the variable $\theta=\omega t$ in order to work with functions which have a fixed period (as in the perturbation method). With this variable our differential equation is

$$\omega^2 x'' + f(x) = F\cos\theta. \tag{18}$$

The idea of Rauscher is to take as first approximation the periodic solution $x_0(\theta)$ of

$$\omega_0^2 x'' + f(x) = 0 \tag{19}$$

which has a prescribed amplitude A (i.e., $(x_0)_{\max}=A$). The quantity ω_0 is to be determined in such a way that x_0 will have the period 2π. This is a problem that can be solved by explicit integration, which may be carried out graphically if necessary. The quantity ω_0 is, of course, taken as a first approximation to the unknown frequency ω. The essentially new idea of Rauscher consists then in replacing θ in the right hand side of (18) by the function $\theta_0(x)$ obtained by inverting the solution $x_0(\theta)$ of (19), over a half period in which x_0 is a monotone function of θ. This yields as equation for the next approximation:

$$\omega_1^2 x'' + f(x) - F\cos[\theta_0(x)] = 0, \tag{20}$$

which is an equation of the same type as (19): it differs only in that the restoring force is different. Its solution, for the same fixed A and same period 2π, yields a second approximation $x_1(\theta)$ and a new value ω_1 for ω, which will in general be different from ω_0. The general scheme of this iteration method thus consists in finding the solution $x_n(\theta)$ of period 2π of

$$\omega_n^2 x'' + f(x) - F\cos[\theta_{n-1}(x)] = 0 \tag{21}$$

for $(x_n)_{\max}=A$, where $\theta_{n-1}(x)$ has been obtained by inverting the solution of the equation for $(n-1)$. At the same time the nth approximation ω_n for ω is found. Actually, there is no need to invert the solution each time: The Rauscher method makes direct use of the fact that the solution of an equation of the type (19) is obtained most conveniently at the outset in the form $\theta=\theta(x)$. In the case of the pendulum, for example, the time is obtained as an elliptic integral which must be inverted to obtain the displacement as an elliptic function of the time.

The amount of numerical computation is not small in this method, but the convergence is so rapid that it may well be the best method of obtaining accurate quantitative results. Rauscher finds that the second step usually furnishes a very close approximation. It might be objected that the method of Rauscher could not be used if viscous damping were present since the differential equation

$$x'' + cx' + f(x) = 0$$

cannot be integrated explicitly. However, the term cx' can be treated in much the same manner as the term $F \cos \theta$ once a phase shift has been introduced.

Finally, brief mention should be made of two other methods. The Ritz method has been used [8, 9, 27, 31]. It consists in minimizing the expression

$$\int_0^T \left\{ \dot{x}^2 - \alpha x^2 - \frac{\beta}{2} x^4 + 2xF \cos \omega t \right\} dt$$

with respect to periodic functions $x(t)$ of period $T = 2\pi/\omega$. The expression attains a minimum, however, only for soft springs ($\beta < 0$).

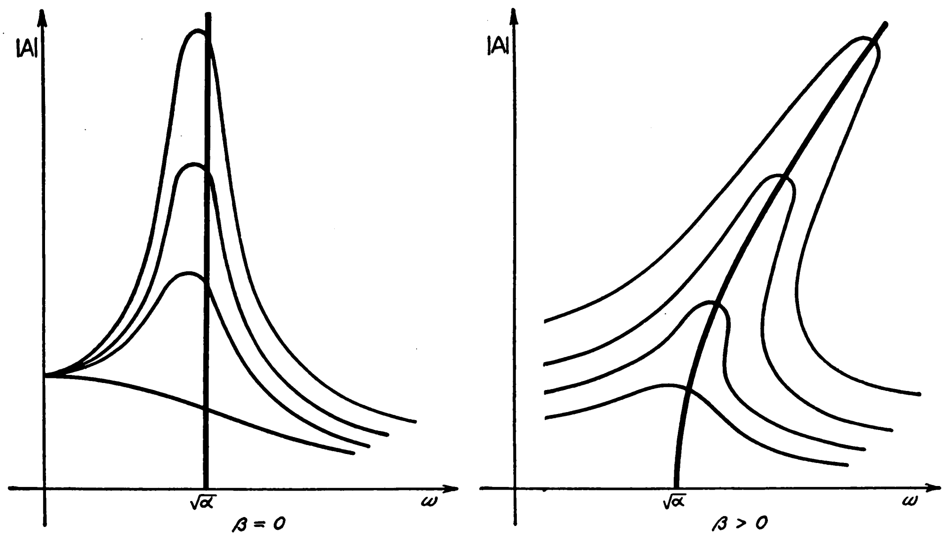

FIG. 5

In case one is interested in obtaining a solution satisfying given initial conditions (such a solution need not be periodic), the graphical method of E. Meissner [23] is to be recommended. The method is particularly well designed for just the type of differential equation considered here.

5. Jump phenomena. We turn now to the physical conclusions which can be drawn from the form of the response curves. Instead of discussing these phenomena on the basis of the curves of fig. 3, we prefer to consider the analogous curves which result under the assumption of slight viscous damping in the system. By analogy with the response curves in the linear case with damping, one would expect the curves to be rounded off as indicated in fig. 5: This is a point to which we shall return a little later.

In the following fig. 6, we show a typical response curve (i.e., for a given value of the force amplitude F) for the case of a hard and a soft spring:

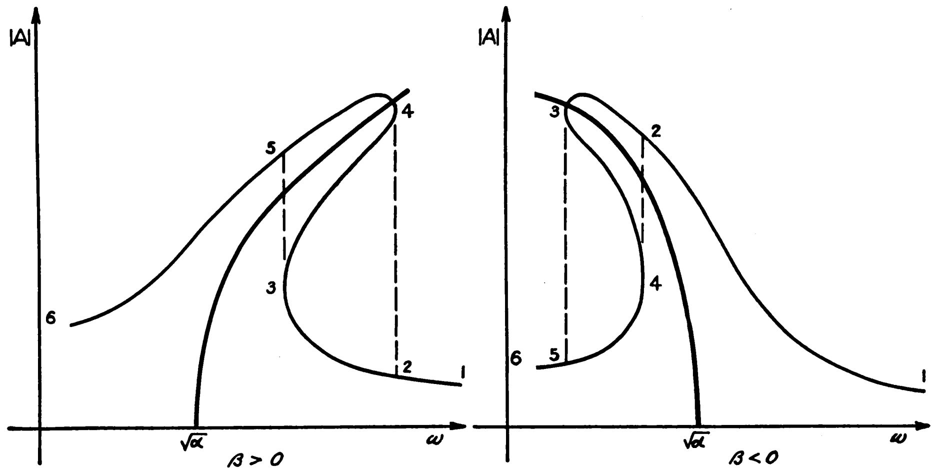

FIG. 6

Let us imagine an experiment performed in which the amplitude F of the external force is held constant, but its frequency ω is slowly varied. (If ω were held fixed and F slowly varied, the results would be of the same general character.) Consider first the case of a hard spring ($\beta > 0$) and suppose that ω is rather large at the beginning of our experiment, i.e., we start at point 1 on the curve. As ω is decreased A slowly increases through point 2 until point 3 is reached. Since F is held constant, a further decrease in ω would require a jump from point 3 to point 5 with an accompanying increase in the amplitude A, after which A decreases with ω. Upon performing the experiment in the other direction, i.e., starting at point 6 and increasing ω, the amplitude follows the $6 \rightarrow 5 \rightarrow 4$ portion of the curve, then jumps to point 2 and afterwards slowly decreases. The circumstances are quite similar with a soft spring, but the jumps in amplitude take place in the reverse direction. It would not have been necessary, we see now, to consider the influence of damping in order to conclude that a jump from point 3 to point 5 should take place (for $\beta > 0$) on decreasing ω, but the jump from point 4 to point 2 on increasing ω would be inexplicable on the basis of the curves of fig. 3.

This curious behavior of nonlinear systems has often been observed by experiment. Probably the first to discuss it experimentally was O. Martienssen [21]. Martienssen carried out his experiments with an electrical apparatus involving a condenser and an inductance with an iron core; he ascribed the results to the nonlinearity.

6. Effect of viscous damping. We wish to discuss briefly a method of treating the effect of viscous damping, i.e., the modifications which result upon adding a term $c\dot{x}$, $c>0$, to the left-hand side of (3):

$$\ddot{x} + c\dot{x} + (\alpha x + \beta x^3) = F \cos \omega t - G \sin \omega t. \tag{22}$$

The amplitude $H=\sqrt{F^2+G^2}$ of the external force is prescribed, but the ratio F/G is not prescribed in order to permit a phase difference between the applied force and the resulting periodic displacement.

The iteration method can be applied in much the same manner as above by beginning with $x_0=A \cos \omega t$ as a first approximation. In place of the relation (8), we now have the following two relations:

$$(\alpha - \omega^2)A + \tfrac{3}{4}\beta A^3 = F, \tag{23}$$

$$c\omega A = G. \tag{24}$$

Once can draw an interesting, though not unexpected, conclusion immediately. It is that no periodic motion except the state of rest can result if there is damping but no external force, i.e., if $F=G=0$ with $c\neq 0$. In fact, (24) is satisfied in this case only for $A=0$ which implies $x_0\equiv 0$.

Equations (23) and (24) are squared and added to obtain

$$[(\alpha - \omega^2)A + \tfrac{3}{4}\beta A^3]^2 + c^2\omega^2A^2 = F^2 + G^2 = H^2, \tag{25}$$

which gives the relation between the amplitude A and frequency ω of the vibration which results from the force amplitude H. Equation (25) replaces (8) for this case; it of course reduces to (8) when $c=0$.

The discussion of the response curves furnished by (25) turns out to be quite easy and to yield the expected results. One finds that the curves have only one branch instead of two as in the undamped case, and that each curve shows a single maximum for A. The locus of the maxima (for different H) is a curve which runs close to the response curve for the free undamped oscillation ($H=c=0$), always remaining to the left of it. With no damping there is only one vertical tangent on each response curve. With damping a second vertical tangent appears; the new point of tangency lies near to the curve for free undamped oscillations. These results have already been indicated in figs. 5 and 6.

It is of some interest to note that the curves for values of H less than a certain limit value will have no vertical tangents, which means that for such amplitudes no jump phenomena would occur.

The effect of viscous damping on the response curves appears to have been discussed first by E. V. Appleton [1]. Appleton uses the method of variation of parameters. He investigates also in detail the stability of the solutions. It is perhaps of interest to note that Appleton was led to the problem through observing the peculiar behavior of a certain galvanometer at the Cavendish laboratory at Cambridge. This behavior corresponds to what was called jump phenomena above.

7. Stability questions. We shall discuss briefly the stability question for the cases without damping. Probably the most difficult question in any consideration of dynamic stability is that of the definition of stability itself in a reasonable way. Into this question we do not wish to enter here. We take, simply, the following often used definition of stability: let $x(t)$ and $x(t)+\delta x(t)$ be two solutions of our differential equation. Consider the *variational equation* which results when $x+\delta x$ is inserted in the original differential equation and powers of δx above the first are neglected. If *all* solutions δx of this equation are bounded, then $x(t)$ is said to be stable, otherwise unstable.

In particular, the variational equation for Duffing's equation (2) is

$$\delta\ddot{x} + (\alpha + 3\beta x^2)\delta x = 0. \tag{26}$$

In the cases under discussion here $x(t)$ is a periodic function and thus (26) is a Hill's equation. The question is then whether for a given $x(t)$ all solutions of (26) are bounded or not. Or we might put the problem a little differently by referring to the A-ω-plane, the points of which characterize the periodic solutions $x(t)$. The problem thus is to divide this plane into regions which correspond to stable solutions and others which correspond to unstable ones. Questions of this kind have been discussed, cf. for example [1, 18, 29, 31], but the problem has apparently not been solved completely.

Some conclusions can, however, be reached without much difficulty. Consider, for example, fig. 7, which indicates the response curve for $F=0$ (i.e., for the free oscillation) together with the locus of vertical tangents on the response curves for $F>0$. (In connection with the latter curve see [12, 14, 15]). Our discussion of the jump phenomena would lead us to expect that the shaded area between these two curves corresponds to solutions which are unstable in some sense—quite possibly in the sense of the above definition. It can at least be shown that the two curves in question really do represent boundary curves between regions of stable and unstable solutions in the sense of the definition.

If one were to be contented with taking $A \cos \omega t$ as a sufficiently close approximation to $x(t)$, then equation (26) would be a Mathieu equation. For the special case of the Mathieu equation the problem of separating stable and unstable solutions has been completely solved. If one makes use of this theory it turns out that our two curves have contact of the second order at $\omega=\sqrt{\alpha}$, $|A|=0$ with curves delimiting a region of unstable solutions of the Mathieu equation, the unstable region being that within the cusp.

8. Subharmonic response. Up to now we have considered periodic solutions of (3) for which the frequency is always exactly the same as that of the external force $F \cos \omega t$. Permanent oscillations with a frequency $\frac{1}{2}, \frac{1}{3}, \cdots 1/n, \cdots$ of that of the applied force can, however, occur in nonlinear systems, in particular in our case of the Duffing equation. To this phenomenon the term subharmonic resonance is usually applied.[3]

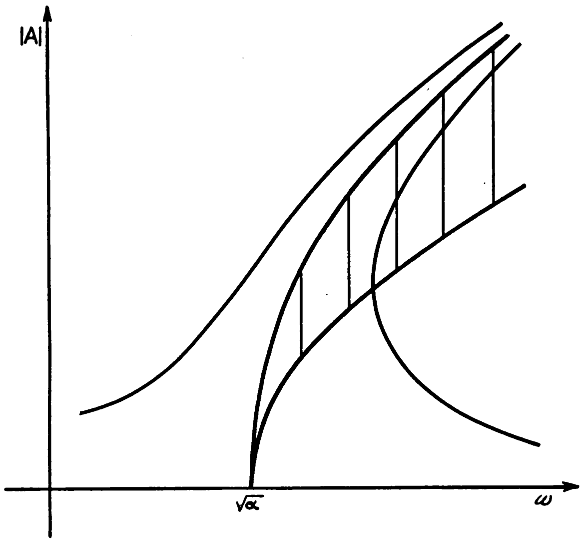

Fig. 7

The fact that subharmonic oscillations occur can hardly be denied since they have been so often observed. But it is not an entirely simple matter to give a plausible physical explanation for their occurrence. Let us recall the behavior of linear systems. If the frequency of the free oscillation of a linear system is ω/n (n an integer, say) then a periodic external force of frequency ω can excite the free oscillation in addition to the forced oscillation of frequency ω. But since some damping is always present the free oscillation is damped out so that the eventual *steady state* consists solely of the oscillation of frequency ω. Why should the situation be different in a nonlinear system? The explanation usually offered is as follows: Any oscillation of a nonlinear system contains the higher harmonics in profusion. Hence it is possible that an external force with a frequency the same as one of these might be able to excite and sustain the harmonic of lowest frequency. Of course this requires that the damping (more precisely the ratio c/H) be not too great and that proper precautions of various kinds be taken.

We shall not attempt to present a solution of the problem of subharmonic

[3] For literature, see [2, 3, 17, 18].

response for the Duffing equation in all generality. Rather, we shall treat only one special case. Also, damping will be neglected even though it should be shown that subharmonic response occurs in spite of damping. This could be done, but it would complicate our calculations unnecessarily without causing any significant qualitative changes in the results, particularly if the damping is small. In any case we are interested here, as in our previous discussion, in qualitative rather than accurate quantitative results.

There is some advantage in introducing once more the variable $\theta = \omega t$ as the new independent variable. The differential equation is

$$\omega^2 x'' + (\alpha x + \beta x^3) = F \cos \theta, \tag{27}$$

in which it is to be remembered that ω represents the frequency of the applied force $F \cos \omega t$. Our object is to find a periodic solution with frequency ω/n. We restrict ourselves to the case $n = 3$. We apply the Fourier series method and set

$$x = A_{1/3} \cos \frac{\theta}{3} + A_1 \cos \theta + A_{5/3} \cos \frac{5\theta}{3} + \cdots$$

by analogy with the usual or *harmonic* case. (Terms in even multiples of $\theta/3$ and sine terms drop out). Upon substitution of the series in (2) one finds the relations

$$\left(\alpha - \frac{\omega^2}{9}\right) A_{1/3} + \tfrac{3}{4}\beta\left[A_{1/3}^3 + A_{1/3}^2 A_1 + 2A_{1/3}A_1^2 + \cdots\right] = 0, \tag{28}$$

$$(\alpha - \omega^2) A_1 + \tfrac{1}{4}\beta\left[A_{1/3}^3 + 6A_{1/3}^2 A_1 + 3A_1^3 + \cdots\right] = F. \tag{29}$$

. . . .

Equations (28) and (29) take the place of equation (8) which was fundamental for the *harmonic* case.

Again we are faced with a problem of interpretation. Our previous experience offers a guide: we set $\beta = 0$ (i.e., we consider the linear case) and observe that $A_{1/3}$ must be taken zero unless $\omega = 3\sqrt{\alpha}$. If, however, $\omega = 3\sqrt{\alpha}$ then $A_{1/3}$ can be taken arbitrarily, while $A_1 = -(1/8\alpha)F$ is determined for a given value of F. The term $A_{1/3} \cos \theta/3$ is evidently the free oscillation of arbitrary amplitude which may be superimposed on the forced vibration $-(F/8\omega) \cos \theta$ in the linear case. Hence we should prescribe $A_{1/3}$ arbitrarily for $\beta \neq 0$ but then hold it fixed. The quantity F should also be prescribed, but the quantities ω and A_1 which were fixed for $\beta = 0$ should now be considered as functions of $A_{1/3}$ and F.

With this in mind we rewrite equations (28) and (29) in slightly different form in order to perform iterations conveniently. This yields, after a division of (28) by $A_{1/3} \neq 0$:

$$\omega^2 = 9\alpha + \tfrac{27}{4}\beta[A_{1/3}^2 + A_{1/3}A_1 + 2A_1^2 + \cdots], \tag{30}$$
$$-8\alpha A_1 = F + (\omega^2 - 9\alpha)A_1 - \tfrac{1}{4}\beta[A_{1/3}^3 + 6A_{1/3}^2A_1 + 3A_1^3 + \cdots],$$

or

$$-8\alpha A_1 = F - \tfrac{1}{4}\beta[A_{1/3}^3 - 21A_{1/3}^2A_1 - 27A_{1/3}A_1^2 - 51A_1^3 + \cdots], \tag{31}$$

the last equation resulting through elimination of ω^2.

We begin the iterations with the values for $\beta=0$, i.e., with $A_{1/3}$ prescribed, $\omega=3\sqrt{\alpha}$, and $A_1=-F/8\alpha=A$. The next step yields

$$\omega^2 = 9\alpha + \tfrac{27}{4}\beta[A_{1/3}^2 + A_{1/3}A + 2A^2], \tag{32}$$
$$A_1 = A + \tfrac{1}{32}\beta[A_{1/3}^3 - 21A_{1/3}^2A - 27A_{1/3}A^2 - 51A^3]. \tag{33}$$

Equation (32) is readily discussed: it represents an ellipse or a hyperbola in an ω-$A_{1/3}$-plane, depending on the sign of β. Also ω has an extremum for $A_{1/3}=-\tfrac{1}{2}A$ and ω^2 has as value there

$$\omega^2 = 9(\alpha + \tfrac{21}{16}\beta A^2). \tag{34}$$

Thus the subharmonic vibration can exist only if

$$\omega \lessgtr 3\sqrt{\alpha + \tfrac{21}{16}\beta A^2} \quad \text{for} \quad \begin{matrix}\beta < 0 \\ \beta > 0.\end{matrix} \tag{35}$$

If $\beta\neq 0$, we may conclude that no vibration with $\omega=3\sqrt{\alpha}$ can exist; i.e., no subharmonic response with exactly the frequency of the linearized problem can exist. Some authors describe subharmonic response as an oscillation with exactly the frequency of the linearized problem excited by a force with three times this frequency. At least in the case of the Duffing equation, such an oscillation cannot occur.

Interesting conclusions can also be drawn from relation (33), which determines the second Fourier coefficient of the subharmonic vibration as a function of $A_{1/3}$. For $A_{1/3}=0$ relation (28) is satisfied identically while relation (29) reduces to relation (8) for the "harmonic" case. Hence the subharmonic vibration results *through bifurcation from the harmonic one.* This takes place for

$$A_1 = A - \tfrac{51}{32}\beta A^3. \tag{36}$$

In addition we find for $A_{1/3}=0$

$$dA_1/d(\omega^2) = -\frac{A}{8} > 0 \tag{37}$$

and this determines the direction in which the new branch leaves the original one.

Fig. 8 indicates the nature of the curves for $A_{1/3}$ and $|A_1|$ for both a hard and a soft spring.

Our calculations are accurate only for values of $A_{1/3}$ that are small in comparison with those for A_1, i.e., in the neighborhood of the bifurcation point. More interesting from the point of view of the applications are the solutions for which $A_{1/3}$ is large in comparison with A_1, i.e., those for which the subharmonic component dominates. It is very plausible that these solutions will, like their *harmonic* counterparts, lie in the neighborhood of those for the free oscillation if the force amplitude F is small. Of course, the response curve for the free oscillation in question would be the one beginning at $\omega=3\sqrt{\alpha}$ and not that beginning at $\omega=\sqrt{\alpha}$.

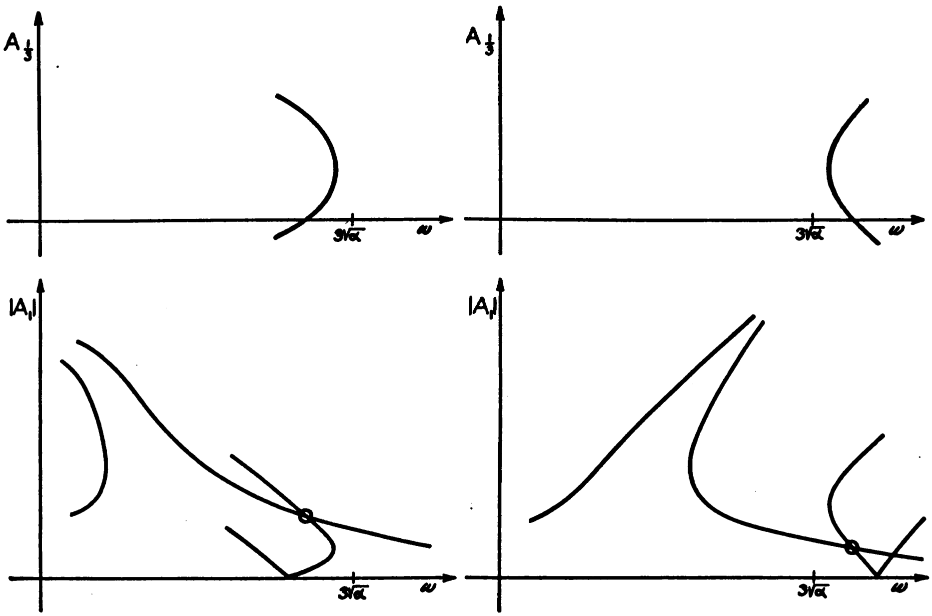

FIG. 8

If the subharmonic component dominates sufficiently so that the resultant solution $x(\theta)$ is monotone in a half period from the maximum to the minimum, then it is possible to apply the method of Rauscher to obtain it.

There are cases which are extreme in another sense, i.e., those in which the dominating harmonic component has a frequency that is a very small fraction (instead of 1/3) of the impressed frequency. In cases involving nonlinear damping such phenomena occur—subharmonic oscillations up to the 200th have been observed, apparently.

9. Other periodic solutions. The harmonic and subharmonic oscillations so far considered do not exhaust the possibilities of periodic solutions. There exists a great variety of other periodic solutions which arise near $\omega=\sqrt{\alpha}/n$, $n=2, 3, \cdots$, in contrast with the subharmonic oscillations which arise near

$\omega = n\sqrt{\alpha}$. Among them are periodic solutions in which the harmonic component of frequency $n\omega$ is dominant. Since the fundamental frequency is ω in such oscillations, they are not exact counterparts of the subharmonic ones; nevertheless they might be called superharmonic oscillations. They can be obtained by a perturbation method beginning with the solution of the linearized problem ($\beta = 0$) $x = A_1 \cos \omega t + A_n \cos n\omega t$ for $\omega = \sqrt{\alpha}/n$, $A_1 = n^2/(n^2-1)F$, while A_n is prescribed arbitrarily. Such solutions have been investigated theoretically, cf. [10, 12, 13], by a different method. They do not appear to have been observed experimentally.

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