NOTES

ON A. C. AITKEN'S METHOD OF INTERPOLATION*

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1. A. C. Aitken[1] has recently devised a method of practical interpolation which is particularly well adapted for computing machines; neither differences nor tables of interpolation coefficients are used, and the necessary operations are most easily performed on modern computing machines. Moreover, the degree of the interpolating polynomial decreases by two at each stage, which minimizes the amount of necessary work. Recent experience has again confirmed that the method is extremely convenient and timesaving. It would nevertheless seem that the method is not sufficiently known, and we propose therefore to give a brief outline. Our proofs seem simpler than the two proofs given by Aitken,[1] or that given by Lidstone.[3] At the same time we shall be led to a procedure which works for an odd number of data as well as for an even number (originally the method appeared to work for an even number only and special computational devices were used to reduce an odd number of data). For most practical arrangements of computations we have to refer to Aitken[1,2] and Lidstone.[3]

2. **Linear cross-means.** The full power of the method appears only with the use of quadratic cross-means, but these are in turn based on linear cross-means. Moreover, with completely unsymmetrical data only linear cross-means can be used.

Let it be required to compute the value $f(\xi)$ of a polynomial of nth degree, $f(x)$, given $f_k = f(x_k)$ for $k=0, \cdots, n$. We note that

$$f^{(1)}(x) = \begin{vmatrix} f_0 & x_0 - \xi \\ f(x) & x - \xi \end{vmatrix} \div (x - x_0) \tag{1}$$

is a polynomial of degree $n-1$, and that $f^{(1)}(\xi) = f(\xi)$. Hence we are required to find $f^{(1)}(\xi)$ knowing

$$f_k^{(1)} = \begin{vmatrix} f_0 & x_0 - \xi \\ f_k & x_k - \xi \end{vmatrix} \div (x_k - x_0) \tag{2}$$

for $k=1, \cdots, n$. Thus the problem has been reduced from n to $n-1$. In like manner the problem is further reduced to $n-2$ and so on.

All the computer has to do is to write in a column the "parts" $x_k - \xi$, and

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[1] A. C. Aitken: *On interpolation by iteration of proportional parts, without the use of differences.* Proc. Edinburgh Math. Soc. Ser. 2, **3**, 56–76 (1932).

[2] A. C. Aitken: *Studies in practical mathematics III: The application of quadratic extrapolation to the evaluation of derivatives, and to inverse interpolation* Proc Roy Soc. Edinburgh, Vol. **58**, pp. 161–175, 1938.

[3] G. J. Lidstone: *Notes on interpolation.* J. Inst. Actuar., Vol. **68**, pp. 267–296, 1938.

in an adjacent column the corresponding values f_k. Using (2), new columns are successively added to the right, the number of rows decreasing by one each time. The "parts" remain obviously the same throughout the computation. The determinant in (2) is easily computed, and the result appears in the main dials ready for division without clearing. Moreover, on most machines, the divisor $x_k-x_0=(x_k-\xi)-(x_0-\xi)$ will automatically appear on the secondary dials (provided the main keyboard has been used for the factors f_k). Actually in most cases the divisor x_k-x_0 will be a small integer. It should also be noted that as the computation proceeds the entries will tend to agree in an ever increasing number of their more important digits. These, of course, will not be copied down; this reduction of digits of f_k makes it in turn possible to drop some last digits of the "parts."

3. Quadratic cross-means. For these it is necessary that the given data be placed symmetrically with respect to some point $x=a$. Denote, then, two symmetrically placed points by x_k and x_{-k} ($k=1, \cdots, m$; $x_k-a=a-x_{-k}$). The point $x_0=a$ is included among the data only if $n=2m$. Consider

$$\phi(x) = \begin{vmatrix} f(2a-x) & 2a-x-\xi \\ f(x) & x-\xi \end{vmatrix} \div 2(x-a) \tag{3}$$

and

$$\psi(x) = \begin{vmatrix} -f(2a-x) & 2a-x-\xi \\ f(x) & x-\xi \end{vmatrix} \div 2(\xi-a). \tag{4}$$

Obviously $\phi(x)$ and $\psi(x)$ are even functions of $x-a$, and hence polynomials in $t=(x-a)^2$. Moreover, $\phi(\xi)=\psi(\xi)=f(\xi)$.

(a) If $n=2m-1$, the problem is reduced to finding the value for $t=(\xi-a)^2$ of the polynomial of $(m-1)$th degree $\phi(a+\sqrt{t})$ given its values

$$\phi_k = \begin{vmatrix} f_{-k} & x_{-k}-\xi \\ f_k & x_k-\xi \end{vmatrix} \div (x_k-x_{-k}) \tag{5}$$

for $t=(x_k-a)^2$, $k=1, \cdots, m$. Thus a simple application of (5) will reduce the number of data from $2m$ to m. From here we proceed as before using linear cross-means. It should be noticed that the "parts" now to be used are $(x_k-a)^2-(\xi-a)^2=-(x_k-\xi)(x_{-k}-\xi)$, that is to say the product of the parts already used for (5). This invariant property dispenses of the necessity to label the panels. In most practical cases the new "parts" will differ only by integers or multiples of $1/2$.

(b) If $n=2m$, we compute the values

$$\psi_k = \begin{vmatrix} -f_{-k} & x_{-k}-\xi \\ f_k & x_k-\xi \end{vmatrix} \div 2(\xi-a) \tag{6}$$

for $t=(x_k-a)^2$, $k=0, \cdots, m$, and proceed as before. Since here the denominator is the same for all k the division may be deferred to the final result. This is a slight simplication.