CONCERNING THE ACCELERATION POTENTIAL*

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The following lines aim at indicating the possibility of a more rigorous approach to Prandtl's method of the acceleration potential for two dimensional flow.[1]

We consider a steady incompressible potential flow past an airfoil of infinite span. We assume the profile, P, to be given by

$$z = x + iy = Z(s), \qquad 0 \leq s \leq s_T, \qquad (P)$$

where the sense of increase of the arc length s corresponds to the counterclockwise direction and the sharp trailing edge, T, is given by $T = Z(0) = Z(s_T)$. The position of the stagnation point, S, near the nose of the airfoil shall be given by $S = Z(s_S)$. We also set

$$-\frac{dZ}{ds} = e^{i\beta(s)},$$

$\beta(s)$ being a continuous function of s and such that on the upper bank of the wing near T, $-\pi/2 \leq \beta \leq \pi/2$.

We denote by u and v the velocity components in the x and y directions respectively and assume that

$$u = U > 0, \qquad v = 0, \text{ at infinity.}$$

Then $u - iv$ is an analytic function of $z = x + iy$ and so is

$$\Phi + i\Psi = \log (u - iv).$$

At S the function $\Phi + i\Psi$ possesses a singularity. (There also is a singularity at T, unless the angle there is 0.) $\Phi + i\Psi$ may be determined as a solution of the following boundary value problem:

A. *To determine a one-valued analytic function* $\Phi + i\Psi$ *defined on the region exterior to* P *and satisfying on* P *the boundary condition*

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[1] Cf. L. Prandtl, *Beitrag zur Theorie der tragenden Fläche*, Zeitschrift f. ang. Math. u. Mech. **16**, 360–361 (1936); *Über neuere Arbeiten zur Theorie der tragenden Fläche*, Proc. Intern. Congress for Appl. Mech., Cambridge, U.S.A., (1938), pp. 478–482, (See also: N.A.C.A. Technical Memorandum No. 962); M. A. Biot, *Some Simplified Methods in Airfoil Theory*, Journ. of the Aeronaut. Sci., **9**, 185–190 (1942); W. C. Randels, *A New Derivation of Munk's Formulae*, this Quarterly **1**, 88 (1943).

$$\Psi = \begin{cases} - \beta(s) & \text{for} \quad 0 \leqq s \leqq s_S \\ \pi - \beta(s) & \text{for} \quad s_S \leqq s \leqq s_T \end{cases} \tag{1}$$

as well as the condition

$$\Phi = \log U, \qquad \Psi = 0, \quad \text{at} \quad z = \infty. \tag{2}$$

Equation (1) expresses that P is a streamline of the flow and takes care of the Kutta-Joukowsky condition (no flow around T). The *unknown* position of S is uniquely determined by (2). For instance, let $z = f(\zeta)$ map $|\zeta| > 1$ conformally into the exterior of P, taking $\zeta = \infty$ into $z = \infty$ and $\zeta = 1$ into $z = T$. Set $f(e^{i\theta}) = Z[\sigma(\theta)]$, $S = f(e^{i\tau})$. Then the condition (2) may be written in the form

$$\int_0^{2\pi} \Psi\{Z[\sigma(\theta)]\}\, d\theta = 0.$$

In view of (1) we obtain

$$\tau = 2\pi - \frac{1}{\pi}\int_0^{2} \beta[\sigma(\theta)]\, d\theta. \tag{3}$$

From Φ we can calculate the pressure p. In fact, we have by Bernoulli's equation

$$p + \tfrac{1}{2}\rho(u^2 + v^2) = p_\infty + \tfrac{1}{2}\rho U^2 = p_0, \tag{4}$$

where ρ is the density, p_∞ the pressure at infinity and p_0 the stagnation pressure. Since $\Phi = \log|u - iv|$,

$$p = p_0 - \frac{\rho}{2}\, e^{2\Phi}. \tag{5}$$

If the wing is infinitely thin, say given by

$$x = x, \qquad y = Y(x), \qquad -1 \leqq x \leqq 1, \tag{P_1}$$

the boundary value problem A takes the form:

B. *To determine a one-valued analytic function $\Phi + i\Psi$ defined on the region exterior to P_1 and satisfying on P_1 the boundary condition*

$$\Psi = \begin{cases} - \text{arc tan } Y'(x) \text{ on the upper bank of } P_1, \\ - \text{arc tan } Y'(x) - \pi \text{ on the lower bank of } P_1, \quad -1 \leqq x \leqq x_S \\ - \text{arc tan } Y'(x) \text{ on the lower bank of } P_1, \quad x_S \leqq x \leqq 1 \end{cases}$$

$(x_S + iY(x_S)$ being the stagnation point) *as well as the condition* (2).

If the wing is very slightly curved and very slightly inclined, the above rigorous but inconvenient treatment can be simplified as follows. The dis-

tance between the leading edge $L = -1 + iY(-1)$ and the stagnation point S is small of second order as compared with the angle of attack. In fact, the general character of the flow around P_1 will be similar to that of a flow around a straight line, say P_2,

$$y = - x \tan \alpha, \qquad - \cos \alpha \leqq x \leqq \cos \alpha. \tag{P_2}$$

By

$$z = \zeta + \frac{1}{4} e^{-2i\alpha} \frac{1}{\zeta} \tag{6}$$

the exterior of P_2 is mapped into $|\zeta| > \frac{1}{2}$ and T is taken into $\frac{1}{2}e^{-i\alpha}$. S is taken into $-\frac{1}{2}e^{-i\alpha}$ (this follows for instance from (3)). Therefore

$$S = - \tfrac{1}{2}(e^{i\alpha} + e^{-3i\alpha})$$

and, since in this case $L = -e^{-i\alpha}$, we have for small values of α

$$\left| L - S \right| \sim 2\alpha^2.$$

Now, $\Phi + i\Psi$ possesses singularities at L and at S. For small angles of attack we may assume that we will make a very slight error if we replace these two singularities by a single singularity situated at L. In order to determine the character of this singularity, we again consider the flow around P_2. The complex potential, say for $U = 1$, is given by

$$w = \zeta + \frac{1}{4\zeta} + (i \sin \alpha) \log \zeta \tag{7}$$

so that, by (6) and (7),

$$u - iv = \frac{dw}{dz} = \frac{dw}{d\zeta} \Big/ \frac{dz}{d\zeta} = \frac{\zeta + \frac{1}{2}e^{i\alpha}}{\zeta + \frac{1}{2}e^{-i\alpha}}$$

and

$$\Phi + i\Psi = \log (\zeta + \tfrac{1}{2}e^{i\alpha}) - \log (\zeta + \tfrac{1}{2}e^{-i\alpha}).$$

This is (in the ζ-plane) the complex potential of a source at $-\frac{1}{2}e^{-i\alpha}$ and a sink at $-\frac{1}{2}e^{i\alpha}$. For small values of α we may approximate this source-sink system by a doublet with a vertical axis.

Thus we replace problem B by

C. *To determine a one-valued analytic function* $\Phi + i\Psi$ *defined on the region exterior to* P_1, *satisfying on* P_1 *the boundary condition*

$$\Psi = - \arctan Y'(x) \tag{8}$$

as well as the condition (2) *and possessing at* L *a singularity which assumes the*

form of a potential of a doublet with a vertical axis when the exterior of P_1 is mapped conformally into that of a circle (by a transformation $f(z)$ with $f'(\infty) > 0$).

The actual solution of this problem is still difficult. Therefore we make use of the fact that P_1 is closely approximated by the slit

$$y = 0, \qquad -1 \leqq x \leqq 1, \qquad\qquad (P_3)$$

and replace the domain of definition of $\Phi + i\Psi$ by the exterior of P_3. Then we obtain the following boundary value problem:

D. *To determine a one-valued analytic function $\Phi + i\Psi$ defined on the region exterior to P_3, satisfying on P_3 condition (8) as well as condition (2) and possessing at -1 a singularity which, in the ζ-plane determined by*

$$z = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right),$$

assumes the form of a potential of a doublet with a vertical axis.

This problem can be easily solved. The presence of the singularity enables us to satisfy both conditions, (2) and (8).

It remains to show that the method described above is identical with the method of the acceleration potential, the latter usually being presented as based upon the assumption

$$\Delta p = 0.$$

By virtue of our hypotheses $p_i - p_\infty$ will be very small as compared to $p_\infty - p_0$ (except at the neighborhood of the leading edge), so that disregarding terms of higher than first order in $(p - p_\infty)/(p_\infty - p_0)$ we have

$$\log (p_0 - p) = \log (p_0 - p_\infty) + \frac{p_\infty - p}{p_0 - p_\infty}$$

and, by (4) and (5),

$$p = -\rho U^2 \Phi + \text{const.}$$

On the basis of the above considerations an estimation of the error (due to replacing the actual problem B first by C and then D) seems to be both desirable and possible.