THE IMPEDANCE OF A TRANSVERSE WIRE IN A RECTANGULAR WAVE GUIDE*

BY

S. A. SCHELKUNOFF

Bell Telephone Laboratories

The purpose of this paper is to derive approximate formulae for the impedance of a transverse wire carrying uniform current (Fig. 1).

The total impedance Z to the current through the wire may be defined as

$$Z = \frac{V}{I}, \tag{1}$$

where V is the applied voltage and I is the electric current in the wire. The total electromotive force V is the sum[1] of the internal electromotive force V_i and the external electromotive force V_e

$$V = V_i + V_e. \tag{2}$$

Correspondingly, we have an internal impedance Z_i of the wire and the external impedance Z_e. By (1) these two impedances are in series with each other

$$Z = Z_i + Z_e. \tag{3}$$

If the guide is infinitely long on both sides of the wire, the external impedance (above the absolute cut-off frequency) is complex

$$Z_e = R_e + iX_e. \tag{4}$$

The resistance term represents radiation of energy into the guide. If the frequency is within the principal frequency range and if K is the characteristic impedance of the guide to the dominant wave, as seen from the wire,[2] then evidently

$$R_e = \tfrac{1}{2}K. \tag{5}$$

We shall now calculate the impedance of the wire on the assumption that its radius is small. The current in the wire will generate transverse electric waves in which the field is independent of the y-coordinate. The general form of the field (for $z>0$) is

* Received February 8, 1943.

[1] For an explanation, see S. A. Schelkunoff, *Electromagnetic Waves*, D. Van Nostrand Company, Inc., 1943.

[2] And not from a plane current sheet generating a pure dominant wave.

$$H_x(x, z) = \sum_{l=1}^{\infty} H_l \sin \frac{l\pi x}{a} e^{-\Gamma_l z},$$
$$E_y(x, z) = -\sum_{l=1}^{\infty} K_l H_l \sin \frac{l\pi x}{a} e^{-\Gamma_l z}, \tag{6}$$

where Γ_l and K_l are respectively the propagation constant and the specific impedance of a typical $TE_{l,0}$-wave

$$\Gamma_l = \sqrt{\frac{l^2\pi^2}{a^2} - \frac{4\pi^2}{\lambda^2}}, \qquad K_l = \frac{i\omega\mu}{\Gamma_l}. \tag{7}$$

The propagation constant of the dominant $TE_{1,0}$-wave is

$$\Gamma_1 = \sqrt{\frac{\pi^2}{a^2} - \frac{4\pi^2}{\lambda^2}} = \frac{2\pi i}{\lambda}\sqrt{1 - \frac{\lambda^2}{4a^2}}. \tag{8}$$

The dominant wavelength range extends from $\lambda_1 = 2a$ to $\lambda_2 = a$, λ_2 being the cut-off wavelength of $TE_{2,0}$-wave. If $a < \lambda < 2a$, all the propagation constants of secondary waves are real

$$\Gamma_l = \frac{l\pi}{a}\sqrt{1 - \frac{4a^2}{l^2\lambda^2}}, \qquad l > 1. \tag{9}$$

For the specific impedances we obtain

$$K_1 = \eta\left(1 - \frac{\lambda^2}{4a^2}\right)^{1/2}, \qquad \eta = \sqrt{\frac{\mu}{\epsilon}},$$
$$K_l = \frac{i\omega\mu a}{l\pi}\left(1 - \frac{4a^2}{l^2\lambda^2}\right)^{-1/2} = i\eta\frac{2a}{l\lambda}\left(1 - \frac{4a^2}{l^2\lambda^2}\right)^{-1/2}. \tag{10}$$

The external electromotive force V_e necessary to support current I through a thin wire of radius r is

$$V_e = -bE_y(d, r), \tag{11}$$

where d is the distance shown in Fig. 1. This equation presupposes that the

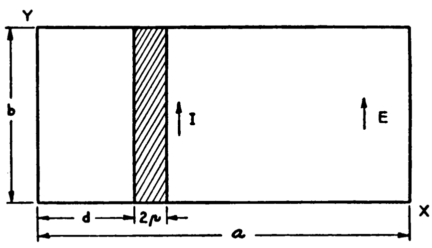

FIG. 1

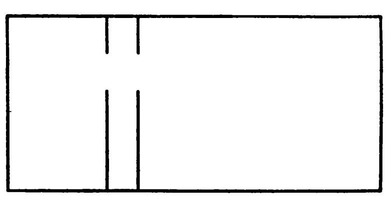

FIG. 2

current is distributed uniformly around the wire. As the radius of the wire increases, the current distribution gradually begins to depart from uniformity. From (1), (6), and (11), we have

$$Z_e = \frac{V_e}{I} = \frac{b}{I}\sum_{l=1}^{\infty} K_l H_l \sin\frac{l\pi d}{a} e^{-\Gamma_l r}. \tag{12}$$

The next step is to calculate the coefficients H_l. We shall assume that the wire is so thin that the field outside the wire could be regarded as nearly equal to that of an infinitely thin electric current filament along the axis of the wire. For an infinitely thin filament, we have

$$H_l = \lim \frac{2}{a}\int_d^{d+2r} \frac{I}{4r}\sin\frac{l\pi x}{a}\,dx \text{ as } r \to 0. \tag{13}$$

Integrating and passing to the limit, we obtain

$$H_l = \frac{I}{a}\sin\frac{l\pi d}{a}. \tag{14}$$

Substituting (14) in (12), we have

$$Z_e = \frac{b}{a}\sum_{l=1}^{\infty} K_l \sin^2\frac{l\pi d}{a} e^{-\Gamma_l r}; \tag{15}$$

and, therefore,

$$\begin{aligned} R_e &\doteq \frac{b}{a} K_1 \sin^2\frac{\pi d}{a}, \\ K = 2R_e &= \frac{2b}{a} K_1 \sin^2\frac{\pi d}{a}, \\ iX_e &= \frac{b}{a}\sum_{l=2}^{\infty} K_l \sin^2\frac{l\pi d}{a} e^{-\Gamma_l r}. \end{aligned} \tag{16}$$

Substituting from (10), we obtain

$$\begin{aligned} K &= \eta\frac{2b}{a}\sin^2\frac{\pi d}{a}\left(1 - \frac{\lambda^2}{4a^2}\right)^{-1/2} \\ X_e &= \eta\frac{2b}{\lambda}\sum_{l=2}^{\infty}\frac{1}{l}\left(1 - \frac{4a^2}{l^2\lambda^2}\right)^{-1/2}\sin^2\frac{l\pi d}{a} e^{-\Gamma_l r}. \end{aligned} \tag{17}$$

Hence, the ratio of the external reactance of the wire to the characteristic impedance of the guide (as seen from the wire) is

$$\frac{X_e}{K} = \frac{a}{\lambda}\sqrt{1 - \frac{\lambda^2}{4a^2}}\ \csc^2\frac{\pi d}{a}\sum_{l=2}^{\infty}\frac{1}{l}\left(1 - \frac{4a^2}{l^2\lambda^2}\right)^{-1/2}\sin^2\frac{l\pi d}{a} e^{-\Gamma_l r}. \tag{18}$$

It is evident that the total inductance of the wire is a series combination of inductances associated with the individual secondary TE waves, generated by the current in the wire.

For the internal impedance of the wire, we have[3]

$$Z_i = \eta_i \frac{bI_0(\sigma_i r)}{2\pi r I_1(\sigma_i r)},$$
$$\sigma_i = \sqrt{i\omega\mu_i(g_i + i\omega\epsilon_i)}, \qquad \eta_i = \sqrt{\frac{i\omega\mu_i}{g_i + i\omega\epsilon_i}}. \tag{19}$$

This is a general expression applicable to dielectric wires as well as to metal wires. In the case of metal wires, we let $\epsilon_i = 0$. Usually, the radii of metal wires will be sufficiently large to make the modified Bessel functions in (19) nearly equal so that approximately

$$Z_i = \frac{b\eta_i}{2\pi r} = \frac{b}{2\pi r}\sqrt{\frac{i\omega\mu_i}{g_i}} = \frac{b}{2\pi r}\sqrt{\frac{\pi f\mu_i}{g_i}}(1 + i). \tag{20}$$

If r/a is small, the series (18) converges slowly. The difficulty may be obviated with the aid of the following device. Let

$$u = \sum_l u_l \tag{21}$$

be a slowly converging series; let

$$v = \sum_l v_l \tag{22}$$

be a series of terms approximating (21) in such a way that the approximation becomes increasingly better as l increases; then

$$u = \sum_l v_l + \sum_l (u_l - v_l) \tag{23}$$

so that (21) can be regarded as the sum of two series, of which the second is more rapidly convergent than the original series. If now the sum of the first series in (23) happens to be known, we have succeeded in replacing the original slowly convergent by a more rapidly convergent series.

We shall apply this device to (18). First we rewrite (18) in the following form

$$\frac{X_e}{K} = \frac{1}{4}\sqrt{\frac{4a^2}{\lambda^2} - 1}\,\csc^2\frac{\pi d}{a}\sum_{l=2}^{\infty}\frac{1 - \cos\dfrac{2l\pi d}{a}}{l\sqrt{1 - \dfrac{4a^2}{l^2\lambda^2}}}\,e^{-\Gamma_l r}; \tag{24}$$

[3] See the book mentioned in footnote 1.

then we note that as l tends to infinity, Γ_l tends to $l\pi/a$ and $\sqrt{1-4a^2/l^2\lambda^2}$ tends to unity. Hence, a typical term of the v-series will be $(1/l)[1-\cos(2l\pi d/a)]e^{-l\pi r/a}$, and (24) may be expressed as

$$\frac{X_e}{K} = \frac{1}{4}\sqrt{\frac{4a^2}{\lambda^2} - 1}\ \csc^2\frac{\pi d}{a}\left[\sum_{l=2}^{\infty} \frac{1 - \cos\dfrac{2l\pi d}{a}}{l} e^{-l\pi r/a} + T\right], \tag{25}$$

$$T = \sum_{l=2}^{\infty} \frac{1 - \cos\dfrac{2l\pi d}{a}}{l}\left(\frac{e^{-\Gamma_l r}}{\sqrt{1 - \dfrac{4a^2}{l^2\lambda^2}}} - e^{-l\pi r/a}\right).$$

It is known that

$$\sum_{l=1}^{\infty} \frac{1}{l} e^{-lp} \cos lq = -\frac{1}{2}\log(1 - 2e^{-p}\cos q + e^{-2p}); \tag{26}$$

therefore,

$$\frac{X_e}{K} = \frac{1}{4}\sqrt{\frac{4a^2}{\lambda^2} - 1}\ \csc^2\frac{\pi d}{a}\left[\frac{1}{2}\log\left(1 - 2e^{-\pi r/a}\cos\frac{2\pi d}{a} + e^{-2\pi r/a}\right)\right.$$
$$\left. - \log(1 - e^{-\pi r/a}) - e^{-\pi r/a}\left(1 - \cos\frac{2\pi d}{a}\right) + T\right]. \tag{27}$$

This can be transformed into

$$\frac{X_e}{K} = \frac{1}{4}\sqrt{\frac{4a^2}{\lambda^2} - 1}\ \csc^2\frac{\pi d}{a}\left[\frac{1}{2}\log\left(\cosh\frac{\pi r}{a} - \cos\frac{2\pi d}{a}\right)\right.$$
$$\left. - \frac{1}{2}\log 2 - \log\sinh\frac{\pi r}{2a} - e^{-\pi r/a}\left(1 - \cos\frac{2\pi d}{a}\right) + T\right]. \tag{28}$$

An entirely different expression for Z_e can be obtained by the image method. Assuming again that r is sufficiently small and that the wire is not too close to the boundaries of the guide and that consequently there is no "proximity effect," we can immediately obtain

$$Z_e = \frac{1}{4}\eta\beta b\left[\overset{2}{H}_0(\beta r) + 2\sum_{n=1}^{\infty}\overset{2}{H}_0(2n\beta a) - \sum_{n=0}^{\infty}\overset{2}{H}_0(2n\beta a + 2\beta d)\right.$$
$$\left. - \sum_{n=0}^{\infty}\overset{2}{H}_0(2n\beta a + 2\beta a - 2\beta d)\right]. \tag{29}$$

This is a slowly converging series and is useless for direct numerical computations; on the other hand, it may be useful for other purposes. Thus the difference between the external impedances of two wires of different radii is obtained in the following simple form

$$\begin{aligned} Z_e(r_2) - Z_e(r_1) &= \tfrac{1}{4}\eta\beta b\left[\overset{2}{H}_0(\beta r_2) - \overset{2}{H}_0(\beta r_1)\right] \\ &= \tfrac{1}{4}\eta\beta b\left[J_0(\beta r_2) - J_0(\beta r_1)\right] + \tfrac{1}{4}\eta\beta b i\left[N_0(\beta r_1) - N_0(\beta r_2)\right]. \end{aligned} \tag{30}$$

The first term represents the effect[4] of the radius of the wire on the impedance of the guide as seen by the wire. The second term represents the difference between the external reactances of two wires

$$X_e(r_2) - X_e(r_1) = \tfrac{1}{4}\eta\beta b\left[N_0(\beta r_1) - N_0(\beta r_2)\right]. \tag{31}$$

This equation can be used for numerical calculations in conjunction with (28). The slowly converging part of (29) is the mutual impedance between the wire and the wave guide.

An expression for the mutual impedance between two parallel wires in the wave guide can also be obtained by the image method. Thus we have $(d_2 > d_1)$

$$\begin{aligned} Z_{12} = \frac{1}{4}\eta\beta b\Bigg[&\overset{2}{H}_0(\beta d_2 - \beta d_1) + \sum_{n=1}^{\infty}\overset{2}{H}_0(2n\beta a + \beta d_2 - \beta d_1) \\ &+ \sum_{n=1}^{\infty}\overset{2}{H}_0(2n\beta a + \beta d_1 - \beta d_2) - \sum_{n=0}^{\infty}\overset{2}{H}_0(2n\beta a + \beta d_1 + \beta d_2) \\ &- \sum_{n=0}^{\infty}\overset{2}{H}_0(2n\beta a + 2\beta a - \beta d_1 - \beta d_2)\Bigg]. \end{aligned} \tag{32}$$

Next we shall deal briefly with the external impedance of a "split" wire (Fig. 2). Let the electromotive intensity at the surface of the wire and the current in the wire be

$$\begin{aligned} E_y &= -\sum_{m=0}^{\infty} E_m \cos\frac{m\pi y}{b}, \\ I &= \sum_{m=0}^{\infty} I_m \cos\frac{m\pi y}{b} \end{aligned} \tag{33}$$

The complex flow of power is

$$\Psi^* = \frac{1}{2}\overline{Y}_e V_e V_e^* = \frac{1}{2}b\left[E_0^* I_0 + \frac{1}{2}\sum_{m=1}^{\infty} E_m^* I_m\right], \tag{34}$$

where $\overline{Y}_e$ is the external admittance of the split wire and V_e the voltage across this admittance. Since

$$V_e = bE_0, \tag{35}$$

[4] Which was entirely ignored in the derivation of (28).

we obtain from (34)

$$\overline{Y}_e = \frac{I_0}{V_e} + \frac{1}{2}\sum_{m=1}^{\infty} Y_m \frac{E_m E_m^*}{E_0 E_0^*} \tag{36}$$

where

$$Y_m = \frac{I_m}{bE_m}. \tag{37}$$

The first term in (36) is the external admittance corresponding to a uniform current filament (Fig. 1) and, hence, is equal to the reciprocal of either (15) or (29). The second term is the capacitative admittance (assuming that $\lambda > 2b$) between the external surfaces of the two portions of the transverse wire. The impedance diagram is shown in Fig. 3 where the parallel lines

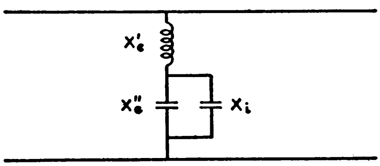

FIG. 3

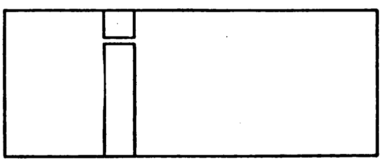

FIG. 4

represent the wave guide, the inductive reactance X_e' is the reactive part of (29), the capacitative reactance X_e'' is the reciprocal of the second term in (36) and X_i is the reactance looking inward from the surface of the gap in the wire. In the case illustrated by Fig. 4 the internal reactance is approximately

$$X_i = \frac{s}{i\omega\epsilon\pi r^2}, \tag{38}$$

where r is the radius of the wire and s is the length of the gap.

The internal reactance of two hollow cylinders as well as the quantity X_e'' will be discussed in a separate paper. Here we shall merely derive general formulae and show that roughly X_e'' is equal to the external capacitative reactance between two sections of a split transverse wire placed across infinite parallel planes.

Each component $I_m \cos(m\pi y/b)$ of the total current I in (33) originates a radial wave of the following type

$$\begin{aligned} H_\phi(\rho) &= \frac{I_m K_1(\gamma_m \rho)}{2\pi r K_1(\gamma_m r)} \cos\frac{m\pi y}{b} \\ E_y(\rho) &= -\frac{\eta\gamma_m I_m K_0(\gamma_m\rho)}{2i\beta r K_1(\gamma_m r)} \cos\frac{m\pi y}{b} \end{aligned} \tag{39}$$

where ρ is the distance from the axis of the current and γ_m is the radial propagation constant of the mth cylindrical wave

$$\gamma_m = \sqrt{\frac{m^2\pi^2}{b^2} - \frac{4\pi^2}{\lambda^2}} = \frac{m\pi}{b}\sqrt{1 - \frac{4b^2}{m^2\lambda^2}}\,. \tag{40}$$

If $\lambda > 2b$, all the radial propagation constants of order m higher than zero are real. This explains why even the nearest image will have but little effect on the admittance Y_m except when the wire is quite close to the walls of the wave guide, or when λ is nearly equal to $2b$. Even when the wire is close to the walls of the guide only the nearest image will have an appreciable effect on Y_m unless λ is nearly equal to $2b$.

The complete expression for the impedance $Z_m = 1/Y_m$ is

$$\begin{aligned} Z_m = \frac{\eta\gamma_m b}{2\pi i\beta r K_1(\gamma_m r)} \Bigg[& K_0(\gamma_m r) + 2\sum_{n=1}^{\infty} K_0(2n\gamma_m a) \\ & - \sum_{n=0}^{\infty} K_0(2n\gamma_m a + 2\gamma_m d) - \sum_{n=0}^{\infty} K_0(2n\gamma_m a + 2\gamma_m a + 2\gamma_m d)\Bigg]. \end{aligned} \tag{41}$$

Equation (29) corresponding to the principal cylindrical wave ($m=0$), is of course, a special case of (41). The propagation constant Γ_0 of the principal wave, however, is pure imaginary and, hence, distant images have a pronounced effect on Z_e.