A NEW DERIVATION OF MUNK'S FORMULAE*

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Recently M. A. Biot[2] has applied the method of the acceleration potential to some problems of two-dimensional airfoil theory. In this paper this method will be used in order to obtain a short proof of Munk's formulae[3] for the lift and moment of a thin airfoil.

As usual in the theory of thin wing sections we replace the airfoil by its mean camber line supposed to deviate but little from the chord. Studying the plane irrotational flow of an incompressible fluid around this indefinitely thin airfoil, we take its chord as the x-axis of a system of rectangular coordinates x, y, ascribing to the leading and trailing edge the abscissae -1 and $+1$ respectively. Denoting by V the velocity at infinity and by α the angle of attack, supposed to be small, we write the x- and y-components of the velocity vector $\vec{w}$ as $V+u$ and $\alpha V+v$ respectively, where u, v and αV will be small as compared with V. We denote the pressure by p and the density by ρ. Then, by Bernoulli's equation

$$\frac{\rho}{2} w^2 + p = p_0 = \text{const.}$$

Neglecting quantities of the second order, we have

$$\rho(V^2 + 2uV) = -2(p - p_0). \tag{1}$$

The quantity $\Phi = -1/\rho(p-p_0)$ is called the acceleration potential, since the acceleration equals grad Φ.

It is known that $V+u-i(\alpha V+v)$ is an analytic function of $z=x+iy$. Since V is a constant $(\frac{1}{2}V^2+uV)-i(\alpha V^2+vV)$ is also an analytic function of z. The functions $\Phi=(\frac{1}{2}V^2+uV)$, $\Psi=-(\alpha V^2+vV)$ are thus seen to be conjugate harmonic functions.

Let the mean camber line of the airfoil be given by the equation $y=c(x)$, $(-1\leqq x\leqq 1;\ c(1)=c(-1)=0)$. The condition that this be part of a stream line furnishes the condition

$$\frac{\alpha V + v}{V + u} = c'(x) \text{ along } y = c(x), \qquad (-1 \leqq x \leqq 1).$$

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[1] This note has been prepared at the suggestion of Professor W. Prager while the author was a fellow under the Program of Advanced Instruction and Research in Mechanics at Brown University. The author is indebted to Dr. L. Bers for valuable advice.

[2] M. A. Biot, *Some simplified methods in airfoil theory*, Journal of the Aeronautical Sciences **9**, 185–190, (1942).

[3] M. M. Munk, *General theory of wing sections*, N.A.C.A. Techn. Rep. No. 142 (1922).

Neglecting quantities which are small of a higher order than the first, we obtain

$$\Psi = - V^2 c'(x) \text{ along } y = c(x), \qquad (-1 \leqq x \leqq 1). \tag{2}$$

Since v vanishes at infinity we have

$$\Psi(\infty) = - \alpha V^2.$$

As the mean camber line deviates but little from the segment $-1 \leqq x \leqq 1$ of the x-axis, we will not commit an appreciable error by fulfilling the condition (2) along this segment rather than along the mean camber line. We set

$$\Psi = - \alpha V^2 + \Psi_1 + \Psi_2$$

where

$$\Psi_1 = \alpha V^2 \text{ and } \Psi_2 = - V^2 c'(x) \text{ along } -1 \leqq x \leqq 1, \qquad y = 0$$

and

$$\Psi_1(\infty) = \Psi_2(\infty) = 0.$$

Ψ_1 and the conjugate harmonic function Φ_1 have been determined by Biot. From Φ_1 the lift distribution due to the angle of attack can be obtained. In the following we shall set $\alpha = 0$ and thus obtain the lift distribution due to the curvature of the mean camber line. Within the framework of our linear theory these two influences are additive.

In order to solve the boundary value problem for Ψ_2 we map the exterior of the segment of the real axis between $z = -1$ and $z = +1$ onto the exterior of the unit circle in the ζ plane by the conformal transformation

$$z = 1/2(\zeta + 1/\zeta).$$

The line segment $(-1, 1)$ then is transformed into the circumference of the unit circle and we have $x = \cos\theta$ (Fig. 1). Since a conformal transformation takes a harmonic function into a harmonic function, our problem becomes that of finding a harmonic function having the values $- V^2 c'(\cos\theta)$ on the unit circle. If we assume Ψ to be regular on the boundary, the solution is given by the Poisson integral but the resulting function will not vanish at infinity unless

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} c'(\cos\theta)d\theta = 0.$$

Therefore, to satisfy the condition $\Psi(\infty) = 0$ we introduce a singularity corresponding to a source-sink doublet at the leading edge.[4] With the notations of Fig. 1, we obtain

[4] It is natural to assume a singularity at the leading edge since our assumption about u and v being small does not hold here.

$$\Psi(r, \theta) = -2a_0V^2 \frac{\cos \theta_1}{r_1} - \frac{1}{2\pi} V_2 \int_0^{2\pi} [c'(\cos \tau) - a_0] \frac{r^2 - 1}{r^2 + 1 - 2r \cos (\tau - \theta)} d\tau.$$

This function clearly vanishes at infinity and will satisfy the other boundary condition because

$$\frac{\cos \theta_1}{r_1} = \frac{1}{2} \text{ on the unit circle.}$$

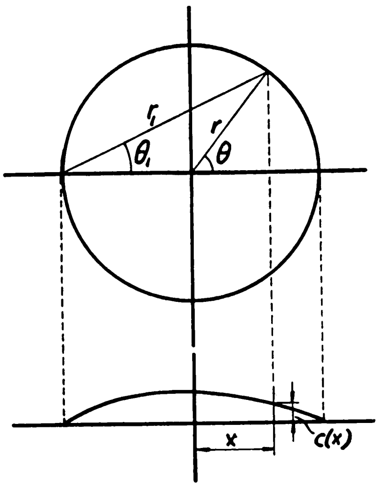

FIG. 1.

The Poisson integral used above is only legitimate if

$$\int_0^{2\pi} |\, c'(\cos \tau) \,|\, d\tau = 2 \int_{-1}^{1} \frac{|\, c'(x) \,|}{[1 - x^2]^{1/2}} dx < \infty.$$

This implies a condition on the rapidity with which $c(x)$ tends to zero as x tends to ± 1.

The values of the conjugate function $\Phi(r, \theta)$ on the boundary of the unit circle are given by the formula[5]

$$\Phi(1, \theta) = -2a_0V^2 \frac{\sin \theta_1}{r_1} + \frac{1}{2\pi} \int_0^{*\,2\pi} c'(\cos \tau) \cot \frac{\tau - \theta}{2} d\tau,$$

where $\int^*$ denotes the Cauchy principal value. The total lift L will be given by

[5] J. D. Tamarkin, "Theory of Fourier series," Brown University, 1933, p. 110.

$$
\begin{aligned}
L &= \rho \int_0^{2\pi} \Phi(1, \theta) \sin \theta d\theta \\
&= -2a_0\rho V^2 \int_0^{2\pi} \frac{\sin \theta_1}{r_1} \sin \theta d\theta \\
&\quad + \frac{V^2}{2\pi} \int_0^{2\pi} \sin \theta d0 \int_0^{*\,2\pi} c'(\cos \tau) \cot \frac{\tau - \theta}{2} d\tau.
\end{aligned}
$$

It is easy to calculate:

$$
\int_0^{2\pi} \frac{\sin \theta_1}{r_1} \sin \theta d\theta = \pi.
$$

The second integral is evaluated by making a formal interchange of the order of integration. This interchange can be easily justified. We then have:

$$
\begin{aligned}
&\frac{1}{2\pi} \int_0^{2\pi} \sin \theta d\theta \int_0^{*\,2\pi} c'(\cos \tau) \cot \frac{\tau - \theta}{2} d\tau \\
&\qquad = \int_0^{2\pi} c'(\cos \tau) d\tau \frac{1}{2\pi} \int_0^{*\,2\pi} \sin \theta \cot \frac{\tau - \theta}{2} d\theta
\end{aligned}
$$

and since the function conjugate to $\sin \theta$ is $-\cos \theta$

$$
\frac{1}{2\pi} \int_0^{*\,2\pi} \sin \theta \cot \frac{(\tau - \theta)}{2} d\theta = -\cos \tau.
$$

Using this together with the definition of a_0 we obtain the lift

$$
\begin{aligned}
L &= -\rho V^2 \left[\int_0^{2\pi} c'(\cos \tau) d\tau + \int_0^{2\pi} c'(\cos \tau) \cos \tau d\tau \right] \\
&= -2\rho V^2 \int_{-1}^{1} c'(x) \frac{1 + x}{[1 - x^2]^{1/2}} dx \\
&= -2\rho V^2 \left\{ c(x) \frac{1 + x}{[1 - x^2]^{1/2}} \bigg|_{-1}^{1} - \int_{-1}^{1} c(x) \frac{1 + x}{[1 - x^2]^{3/2}} dx \right. \\
&= 2\rho V^2 \int_{-1}^{1} c(x) \frac{dx}{(1 - x)[1 - x^2]^{1/2}} \cdot
\end{aligned}
$$

We have assumed that $c(x)$ is such that $\lim_{x \to 1} (c(x)(1 - x)^{-1/2} = 0$. The underlined expression is Munk's formula for the total lift, due to the curvature of the wing.

A similar procedure furnishes the moment M of the lift. It is given by

$$
\begin{aligned}
M &= \rho \int_0^{2\pi} \Phi(1, \theta) \cos\theta \sin\theta d\theta \\
&= -2\rho V^2 a_0 \int_0^{2\pi} \frac{\sin\theta_1}{r_1} \cos\theta \sin\theta d\theta \\
&\quad + \frac{\rho V^2}{2\pi} \int_0^{2\pi} \cos\theta \sin\theta d\theta \int_0^{*2\pi} c'(\cos\tau) \cot\frac{\tau-\theta}{2} d\tau.
\end{aligned}
$$

Then

$$
\int_0^{2\pi} \frac{\sin\theta_1}{r_1} \cos\theta \sin\theta d\theta = -\frac{\pi}{2},
$$

and

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \cos\theta \sin\theta d\theta \int_0^{*2\pi} c'(\cos\tau) \cot\frac{\tau-\theta}{2} d\tau \\
= -\frac{1}{2} \int_0^{2\pi} c'(\cos\tau) \cos 2\tau d\tau
\end{aligned}
$$

so that

$$
\begin{aligned}
M &= -\rho V^2 \left[-\frac{1}{2} \int_0^{2\pi} c'(\cos\tau) d\tau + \frac{1}{2} \int_0^{2\pi} c'(\cos\tau) \cos 2\tau d\tau \right. \\
&= -2\rho V^2 \int_{-1}^{1} c'(x) \frac{x^2 - 1}{[1 - x^2]^{1/2}} dx \\
&= 2\rho V^2 \int_{-1}^{1} c(x) \frac{x dx}{[1 - x^2]^{1/2}}
\end{aligned}
$$

which is Munk's formula for the moment, due to the curvature of the wing.[6]

[6] After the manuscript of this paper had been completed (August 1942), H. J. Stewart has published an analysis proceeding along similar lines: *A simplified two-dimensional theory of thin airfoils,* Journal of the Aeronautical Sciences **9,** 452–456 (Oct. 1942).