ON THE MOTION OF A PENDULUM IN A TURBULENT FLUID*

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1. Introduction. In a recent paper, Schumann[1] has investigated the motion of a damped pendulum in a turbulent fluid by considering the effect of the fluid as a continuous fluctuating force. He first considers a damped pendulum which is bombarded by pellets of equal mass mh at equal intervals h of the time, and then treats the case of continuous fluctuations by a limiting process. For this latter case, which "must be regarded as being of far more practical importance," Schumann obtains the very interesting result:[2]

$$r(\xi) = \frac{1}{2\int_0^\infty R(x)e^{-\lambda x}\sin\,(\beta x+\gamma)dx}\left[\int_0^\infty \{R(x+\xi)+R(x-\xi)\}e^{-\lambda x}\sin\,(\beta x+\gamma)dx\right.$$

$$\left. +\int_0^\xi \{R(\xi-x)-R(x-\xi)\}e^{-\lambda x}\sin\,(\beta x+\gamma)dx\right]. \qquad (1.1)$$

The notation is as follows:

$r(\xi) =$ correlation function of the *displacements* of the pendulum at two instants separated by a time interval ξ;

$R(\xi) =$ limiting correlation function of the *velocities* of the impinging pellets;

$\lambda = l + m/M$, M being the mass of the pendulum and l its damping factor;

$\beta^2 = \alpha^2 + l^2 - \lambda^2$, $2\pi/\alpha$ being the (damped) period of the pendulum; $\sin\,\gamma = \beta/\beta_1$;

$\beta_1^2 = \beta^2 + \lambda^2 = \alpha^2 + l^2$.

The analysis used by Schumann is very elegant, but somewhat lengthy. In this article, we shall study the problem from another point of view, and give an alternative derivation of (1.1). This derivation, though unable to cover the case of discontinuous impacts, seems to show the nature of that relation much more clearly.

2. Damped pendulum under the action of a fluctuating force. We shall now investigate the correlation of *displacement* of a damped pendulum in relation to that of the exciting *force*. The notation used in this section should first be regarded as having different (though analogous) interpretations from

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[1] Schumann, T. E. W., Phil. Mag. (7), **33**, 138–150 (1942).

[2] Loc. cit., eq. (57), p. 146.

those used in §1. The identification of the two systems of notation will be made in §3.

Consider the equation of motion of the pendulum

$$\frac{d^2y}{dt^2} + 2\lambda \frac{dy}{dt} + (\beta^2 + \lambda^2)y = P, \tag{2.1}$$

where y is the displacement, λ is the damping factor and $2\pi/\beta$ is the (damped) period of the pendulum, P is the exciting force per unit mass of the pendulum, and t is the time. If the force P is quasi-periodic, and is given by the real part of

$$P = \sum A_n e^{i\omega_n t}, \tag{2.2}$$

where ω_n and A_n are real and complex constants respectively, the steady-state displacement of the frequency $\omega_n/2\pi$ is given by the real part of

$$y_n = a_n e^{i\omega_n t}, \qquad a_n = \frac{A_n}{(\beta^2 + \lambda^2 - \omega_n^2) + 2i\lambda\omega_n}. \tag{2.3}$$

Thus, we have

$$|a_n|^2 = \frac{|A_n|^2}{(\beta^2 + \lambda^2 - \omega_n^2)^2 + 4\lambda^2\omega_n^2}. \tag{2.4}$$

This is the relation between the spectrum of the displacement and that of the force in the case of discrete spectra. It is not difficult to generalize this result to the case of continuous spectra by the considerations of generalized harmonic analysis.[3] We have then

$$f(\omega) = A \frac{F(\omega)}{(\beta^2 + \lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2}, \tag{2.5}$$

where $f(\omega)$ and $F(\omega)$ are the spectra of the displacement and the force respectively,

$$\int_0^\infty f(\omega)d\omega = 1, \qquad \int_0^\infty F(\omega)d\omega = 1, \tag{2.6}$$

and A is a constant of normalization,

$$\frac{1}{A} = \int_0^\infty \frac{F(\omega)d\omega}{(\beta^2 + \lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2}. \tag{2.7}$$

These are the well-known relations in the phenomena of resonance.

The correlations $r(\xi)$ and $R(\xi)$ of the *displacement* and the *force* respectively stand in Fourier transform relations to the spectra[4] (apart from conventional numerical factors):

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[3] Wiener, N., "The Fourier Integral" (Cambridge, 1933), p. 150.

[4] Wiener, N., loc. cit., eq. (21.21), p. 161, and discussions on p. 163. To be exact, we should follow Wiener in calling $f(\omega)$ and $F(\omega)$ the spectral densities.

$$f(\omega) = \frac{2}{\pi} \int_0^\infty r(x) \cos \omega x \, dx, \qquad r(\xi) = \int_0^\infty f(\omega) \cos \omega \xi \, d\omega; \qquad (2.8)$$

$$F(\omega) = \frac{2}{\pi} \int_0^\infty R(x) \cos \omega x \, dx, \qquad R(\xi) = \int_0^\infty F(\omega) \cos \omega \xi \, d\omega. \qquad (2.9)$$

From (2.5), (2.8), and (2.9), we have at once

$$r(\xi) = \frac{2A}{\pi} \int_0^\infty \frac{\cos \omega \xi \, d\omega}{(\beta^2 + \lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2} \int_0^\infty R(x) \cos \omega x \, dx. \qquad (2.10)$$

It is not difficult to justify a change of the order of integration, since the correlation functions are expected to go to zero at infinity sufficiently rapidly. The above relation then becomes

$$r(\xi) = \frac{2A}{\pi} \int_0^\infty R(x) dx \int_0^\infty \frac{\cos \omega \xi \cos \omega x}{(\beta^2 + \lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2} \, d\omega. \qquad (2.11)$$

Since we have

$$\cos \omega \xi \cos \omega x = \tfrac{1}{2}\{\cos \omega(\xi + x) + \cos \omega(\xi - x)\}, \qquad (2.12)$$

we can evaluate the integral with respect to ξ in (2.11), if we know

$$I(t) = \int_0^\infty \frac{\cos \omega t}{(\beta^2 + \lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2} \, d\omega. \qquad (2.13)$$

This integral is relatively easy to evaluate. We write

$$I(t) = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{i\omega t}}{(\beta^2 + \lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2} \, d\omega, \qquad (2.14)$$

and consider the corresponding contour integral in the complex ω-plane, the contour being the usual one composed of the real axis and a semi-circle at infinity. The circle is taken in the upper half-plane if $t>0$, and in the lower half-plane if $t<0$. Since (2.13) shows that $I(t)$ is an even function, we shall carry out the calculations for $t>0$ alone.

There is no difficulty in showing that the integral over the semi-circle goes to zero. For, when the imaginary part of ωt is positive, $|e^{i\omega t}|$ is bounded, and $|(\beta^2+\lambda^2-\omega^2)^2+4\lambda^2\omega^2| = O(|\omega|^4)$ for large values of $|\omega|$. The evaluation of (2.14) then reduces to the calculation of the residues of the integrand at the two simple poles $\pm\beta+i\lambda$ ($\lambda>0$) inside the contour. The result can be easily verified to be

$$I(t) = \frac{\pi e^{-\lambda t}}{4\beta\lambda} \sin(\beta t + \gamma), \tag{2.15}$$

where

$$\sin \gamma = \frac{\beta}{(\beta^2 + \lambda^2)^{1/2}}. \tag{2.16}$$

With the help of (2.12), (2.13), and (2.15), the equation (2.11) becomes

$$r(\xi) = \frac{A}{4\beta\lambda}\Bigg[\int_0^\infty \{R(x+\xi) + R(x-\xi)\}e^{-\lambda x}\sin(\beta x + \gamma)dx$$

$$+ \int_0^\xi \{R(\xi - x) - R(x-\xi)\}e^{-\lambda x}\sin(\beta x + \gamma)dx\Bigg]. \tag{2.17}$$

This is Schumann's relation (1.1), if the constants can be identified. There is no difficulty with the normalization coefficient. We have

$$\frac{4\beta\lambda}{A} = 2\int_0^\infty R(x)e^{-\lambda x}\sin(\beta x + \gamma)dx$$

$$= 4\beta\lambda\int_0^\infty \frac{F(\omega)d\omega}{(\beta^2 + \lambda^2 - \omega^2)^2 + 4\lambda^2\omega^2}, \tag{2.18}$$

on putting $\xi = 0$ in (2.17) and recalling (2.7). The second relation is a by-product of our investigation. The limiting case $\lambda \to 0$ reduces to the well-known relation (2.9).

Referring to (2.9), we see that $R(\xi)$ is an even function of ξ, so that the second integral in (2.17) may be dropped.

From the derivation, we see that (2.17) is nothing but the Fourier transform of the well-known resonance relation (2.5).

3. Identification of the results. We shall now identify Schumann's result with ours by showing that in the limiting case, his pendulum has an effective damping factor λ instead of l, and that his correlation of *velocity* of the pellets becomes the correlation of *force*. The equations of motion as given by Schumann are[5]

$$u_r = \frac{M - mh}{M + mh}u_r' + \frac{2mh}{M + mh}v_r, \tag{3.1}$$

at the rth impact at the instant $t = rh$, and

$$\frac{d^2y}{dt^2} + 2l\frac{dy}{dt} + (\alpha^2 + l^2)y = 0, \tag{3.2}$$

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[5] Loc. cit., eqs. (1), (2). It seems that there are some misprints in the original paper.

between successive impacts. In (3.1), u_r' and u_r denote respectively the veloci-
ties of the mass M just before and just after the rth impact, and v_r is the veloc-
ity of the rth pellet.

In the limiting case, the discrete impacts become a continuous force given
by

$$P_1 = \lim_{h \to 0} \frac{u_r - u_r'}{h} M = -2mu + 2mv, \tag{3.3}$$

where u and v are the limiting values for u_r and v_r at the instant $t = rh$. Thus u
is evidently the velocity of the pendulum dy/dt, and v is the velocity of the
infinitesimal impinging pellet in the interval $(t, t+dt)$. The equation of motion
(3.2) then becomes

$$\frac{d^2y}{dt^2} + 2l \frac{dy}{dt} + (\alpha^2 + l^2)y = \frac{1}{M}\left(-2m \frac{dy}{dt} + 2mv\right). \tag{3.4}$$

We see that this limiting case carries an inherent damping factor m/M in
addition to the damping factor l. The term $2mv/M$ is evidently the exciting
force per unit mass, since m is the impinging mass per second, the factor 2
corresponding to the fact that when the impinging mass is infinitesimal com-
pared with the colliding mass, the former rebounds with the colliding speed,—
a fact often used in the kinetic theory of gases.

The identification of Schumann's result with that given in §2 is therefore
complete.

4. **Discussion.** In view of the above derivation, we must be a little careful
in applying Schumann's relation (1.1) to the study of the motion of a pendu-
lum in a turbulent fluid. The correlation function R obtained (by suitable
processes) according to that relation from the correlation function r of the
displacements is that for the hydrodynamic *force* (the part corresponding to
an extra damping being removed). In a turbulent fluid, the connection be-
tween the *velocity* fluctuations of the fluid *before* the introduction of a pendu-
lum and the fluctuating *force* acting upon the pendulum *after* its introduction
is not at once evident. A preliminary careful investigation seems to be neces-
sary before the method suggested by Schumann could be used with advantage.

It is also clear that the spectrum of velocity fluctuations of the pendulum
will be proportional to $\omega^2 f(\omega)$. It is then clear from (2.8) that the correlation
function of the velocity fluctuations will be proportional to $-r''(\xi)$.

There is another point which should be mentioned. The spectrum and
correlation function discussed above refer to those observed at a fixed *spatial*
point, if we assume the pendulum to be never far from its position of equi-
librium. It is not difficult to observe this spectrum with a hot-wire anemome-

ter, as has been done by several observers.[6] The Fourier transform of this spectrum will then give the correlation of the fluctuations at successive instants at the same point in *space*. It is not clear, however, how (as Schumann suggested) the pendulum can be used to observe the correlation function defined with respect to the same *material* point,—the quantity introduced by Taylor[7] in the Lagrangian description of turbulence for the study of the phenomenon of turbulent diffusion.[8]

In conclusion, the author wishes to express his sincere thanks to Professor Theodore von Kármán for suggesting the problem to him and for his invaluable suggestions.

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[6] For example, L. F. G. Simmons and C. Salter, Proc. Roy. Soc. Ser. A, **165,** 73–89 (1938); H. L. Dryden, Proceedings of the Fifth International Congress of Applied Mechanics (Cambridge, Mass., 1938), pp. 362–367; H. Motzfeld, Zeits. f. angew. Math. u. Mech. **18,** 362–365 (1938).

[7] Taylor, G. I., Proc. Lond. Math. Soc., (2), **20,** 196–212, (1921). Clearer statements regarding this point are made in his paper of 1935, Proc. Roy. Soc., A, **151,** 421–478 (1935).

[8] A discussion of the three types of correlations that may be defined in the study of isotropic turbulence has been made by G. Dedebant and P. Wehrle, Comptes Rendus **208,** 625–628 (1939).